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变厚度各向异性斜形薄板的弹性平衡问题的Navier解*

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摘　要

本文首先对变厚度各向异性斜形薄板有关非线性理论的弹性平衡问题进行了讨论，建立了变厚度各向异性斜形薄板的大挠度问题的基本方程，然后采用Navier法，给出了求解的一般途径，并以示例说明求解的具体方法，最后讨论了解的收敛性及本方法的应用范围.

关键词　变厚度　各向异性　斜形薄板　Navier解

一、引　　言

近半个世纪以来，由于工程技术领域的需要，变厚度薄板的弹性平衡稳定问题是研究工作者的重要课题之一，各向异性矩形薄板的静力大挠度问题已由M. K. Prabhakara与C. Y. Chia讨论过[1],[2]，等厚度各向同性斜板的非线性弯曲问题已由J. B. Kennedy与S.Ng讨论过[3]. 变厚度薄板的解析解是一个较困难的问题，关于这方面的研究甚少.

我国叶开沅教授对任意分布荷载下的非均匀变厚度矩形板的弯曲问题，作过深入研究[4]，R. G. Olsson等讨论了板的弯曲刚度和厚度h分别是y的线性函数的小挠度问题[5].

通常，小挠度理论不考虑板的中面伸缩，也就是不计中面内力，而是假定横向荷载完全由板的弯曲承担. 这种假设在大位移的情况下不符合实际情况，只有当板的挠度远小于其厚度时才可能成立. 一般横向荷载由板的弯曲和中面伸缩共同承担，不仅弯曲内力而且中面内力也随荷载增加而增大. 尹思明、阮圣璜在讨论变厚度矩形薄板非线性理论的弹性平衡问题时[6]，考虑了中面内力的影响. 实践证明，按大挠度理论进行计算更符合实际情况.

本文讨论了变厚度各向异性斜形薄板的非线性理论的弹性平衡问题. 建立了变厚度各向异性斜形薄板的大挠度问题的基本方程，并试图通过方程的变形，使方程中的部份偶阶导数项摆脱变系数的困难，从而对变厚度各向异性斜形薄板属大挠度的情形给出一种统一的Navier解[6]式.

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二、变厚度各向异性斜形薄板的基本微分方程

假设平面的几何形状与坐标系如图1所示，其中 x，y，z 为矩形坐标分量，ζ，η 为斜坐标，θ为斜角，变厚度斜板的侧边长为$2a$，$2b$，厚度为$h(\zeta,\eta)$，u^0,v^0，w为板沿x，y和 z 轴方向的中面位移分量．ρ为板的单位面积的质量，ϕ为水条线方向角．由各向异性材料组成的薄的斜板在斜坐标下，应力-应变关系可写为[7]

$$\begin{Bmatrix}\sigma_\zeta\\ \sigma_\eta\\ \sigma_{\zeta\eta}\end{Bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{16}\\ a_{12} & a_{22} & a_{26}\\ a_{16} & a_{26} & a_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_\zeta\\ \varepsilon_\eta\\ \varepsilon_{\zeta\eta}\end{Bmatrix} \qquad (2.1)$$

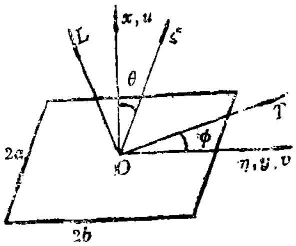


图 1

其中a_{ij}为弹性刚劲，

$$a_{11}=c_{11}/c^3,\quad a_{12}=c_{11}s^2/c^3+c_{12}/c-2sc_{16}/c^2$$

$$a_{16}=c_{16}/c^2-c_{11}s/c^3,\quad a_{66}=c_{66}/c-c_{16}s/c^2-sa_{16}$$

$$a_{26}=c_{26}-sc_{12}/c-s^2a_{16}-2sa_{66}$$

$$a_{22}=cc_{22}+c_{12}s^2/c-2sc_{26}-s^2a_{12}-2sa_{26}$$

$$c=\cos\theta,\quad s=\sin\theta$$

其中c_{ij}是与 x，y 坐标系有关的各向异性平板的材料常数．对于正交各向异性平板，其主方向为(L,T)时，这些常数可以写为

$$\begin{Bmatrix}c_{11}\\ c_{12}\\ c_{22}\\ c_{16}\\ c_{26}\\ c_{66}\end{Bmatrix}=\frac{1}{\mu}\begin{bmatrix}r^4 & 2r^2k^2 & k^4 & 4r^2k^2\\ r^2k^2 & r^4+k^4 & r^2k^2 & -4r^2k^2\\ k^4 & 2r^2k^2 & k^4 & 4r^2k^2\\ kr^3 & rk^3-kr^3 & -rk^3 & 2rk(k^2-r^2)\\ rk^3 & r^3k-rk^3 & -r^3k & 2rk(r^2-k^2)\\ r^2k^2 & -2r^2k^2 & r^2k^2 & (r^2-k^2)^2\end{bmatrix}\begin{Bmatrix}E_L\\ \nu_{LT}E_T\\ E_T\\ \mu G_{LT}\end{Bmatrix} \qquad (2.2)$$

此处E_L与E_T是大小杨氏模量，ν_{LT}与ν_{TL}是Poisson比，G_{LT}是剪切模量，

$$r=\cos\phi,\quad k=\sin\phi,\quad \mu=1-\nu_{LT}\nu_{TL},\quad \nu_{TL}E_L=\nu_{TL}E_T$$

我们取未变形平面的中面与坐标系统的原点的中心相合，应变位移可表示为

$$\varepsilon_\zeta=\varepsilon_\zeta^0-z\frac{\partial^2 w}{\partial\zeta^2},\quad \varepsilon_\eta=\varepsilon_\eta^0-z\frac{\partial^2 w}{\partial\eta^2},\quad \varepsilon_{\zeta\eta}=\varepsilon_{\zeta\eta}^0-2z\frac{\partial^2 w}{\partial\zeta\partial\eta} \qquad (2.3)$$

其中ε_ζ^0，ε_η^0和$\varepsilon_{\zeta\eta}^0$是板在弯曲时中面应变相应的应变分量，

$$\left.\begin{aligned}\varepsilon_\zeta^0&=c\frac{\partial u^0}{\partial\zeta}+s\frac{\partial v^0}{\partial\zeta}+\frac{1}{2}\left(\frac{\partial w}{\partial\zeta}\right)^2\\ \varepsilon_\eta^0&=\frac{\partial v^0}{\partial\eta}+\frac{1}{2}\left(\frac{\partial w}{\partial\eta}\right)^2\\ \varepsilon_{\zeta\eta}^0&=c\frac{\partial u^0}{\partial\eta}+s\frac{\partial v^0}{\partial\eta}+\frac{\partial v^0}{\partial\zeta}+\frac{\partial w}{\partial\zeta}\frac{\partial w}{\partial\eta}\end{aligned}\right\} \qquad (2.4)$$

将方程(2.3)代入方程(2.1)，可得应力合量N_{ij}和应力矩M_{ij}的表示式：

$$N_{ij}=\int_{-h(\zeta,\eta)}^{h(\zeta,\eta)/2}\sigma_{ij}dz,\text{或}\begin{Bmatrix}N_\zeta\\ N_\eta\\ N_{\zeta\eta}\end{Bmatrix}=h(\zeta,\eta)\begin{bmatrix}a_{11} & a_{12} & a_{16}\\ a_{12} & a_{22} & a_{26}\\ a_{16} & a_{26} & a_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_\zeta^0\\ \varepsilon_\eta^0\\ \varepsilon_{\zeta\eta}^0\end{Bmatrix} \qquad (2.5)$$

$$M_{ij}=\int_{-h(\zeta,\eta)/2}^{h(\zeta,\eta)/2}\sigma_{ij}zdz,\ 或\ \begin{Bmatrix}M_\zeta\\M_\eta\\M_{\zeta\eta}\end{Bmatrix}=\frac{-h^3(\zeta,\eta)}{12}\begin{pmatrix}a_{11}&a_{12}&a_{16}\\a_{12}&a_{22}&a_{26}\\a_{16}&a_{26}&a_{66}\end{pmatrix}\begin{Bmatrix}\dfrac{\partial^2 w}{\partial\zeta^2}\\\dfrac{\partial^2 w}{\partial\eta^2}\\2\dfrac{\partial^2 w}{\partial\zeta\partial\eta}\end{Bmatrix} \tag{2.6}$$

由于中面内各点由挠度引起纵向位移，对于大挠度（即非线性理论）的变厚度弹性薄板的弹性平衡问题，必须考虑此项中面位移引起的中面应变和中面应力。

假设ζ，η平面内没有体力，而仅有垂直于板面的横向荷载$q(\zeta,\eta)$，则板元在中面内ζ，η轴向的平衡微分方程为

$$\partial N_\zeta/\partial\zeta+\partial N_{\zeta\eta}/\partial\eta=0,\quad \partial N_{\zeta\eta}/\partial\zeta+\partial N_\eta/\partial\eta=0 \tag{2.7}$$

而板元在与ζ，η平面垂直的方向的平衡方程为

$$q(\zeta,\eta)+N_\zeta\frac{\partial^2 w}{\partial\zeta}+N_\eta\frac{\partial^2 w}{\partial\eta^2}+2N_{\zeta\eta}\frac{\partial^2 w}{\partial\zeta\partial\eta}=L(w) \tag{2.8}$$

此处

$$\begin{aligned}L(w)=&\frac{h^3(\zeta,\eta)}{12}\left[a_{11}\frac{\partial^4 w}{\partial\zeta^4}+4a_{16}\frac{\partial^4 w}{\partial\zeta^3\partial\eta}+2(a_{12}+2a_{66})\frac{\partial^4 w}{\partial\zeta^2\partial\eta^2}+4a_{26}\frac{\partial^4 w}{\partial\zeta\partial\eta^3}\right.\\&\left.+a_{22}\frac{\partial^4 w}{\partial\eta^4}\right]+\frac{h^2(\zeta,\eta)}{2}\left[\left(a_{11}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{16}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial w^3}{\partial\zeta^3}\right.\\&+\left(3a_{16}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+\overline{a_{12}+2a_{66}}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\zeta^2\partial\eta}+\left(a_{12}+2a_{66}\frac{\partial h(\zeta,\eta)}{\partial\zeta}\right.\\&\left.+3a_{26}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\zeta\partial\eta^2}+\left.\left(a_{26}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{22}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\eta^3}\right]\\&+\frac{h(\zeta,\eta)}{4}\left[(h_1(\zeta,\eta)a_{11}+h_2(\zeta,\eta)a_{12}+h_3(\zeta,\eta)a_{16})\frac{\partial^2 w}{\partial\zeta^2}\right.\\&+(h_1(\zeta,\eta)a_{12}+h_2(\zeta,\eta)a_{22}+h_3(\zeta,\eta)a_{26})\frac{\partial^2 w}{\partial\eta^2}+2(h_1(\zeta,\eta)a_{16}\\&\left.+h_2(\zeta,\eta)a_{26}+h_3(\zeta,\eta)a_{66})\frac{\partial^2 w}{\partial\zeta\partial\eta}\right]\end{aligned}$$

其中

$$h_1(\zeta,\eta)=2\left(\frac{\partial h(\zeta,\eta)}{\partial\zeta}\right)^2+h(\zeta,\eta)\frac{\partial^2 h(\zeta,\eta)}{\partial\zeta^2}$$

$$h_2(\zeta,\eta)=2\left(\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)^2+h(\zeta,\eta)\frac{\partial^2 h(\zeta,\eta)}{\partial\eta^2}$$

$$h_3(\zeta,\eta)=4\frac{\partial h(\zeta,\eta)}{\partial\zeta}\frac{\partial h(\zeta,\eta)}{\partial\eta}+2h(\zeta,\eta)\frac{\partial^2 h(\zeta,\eta)}{\partial\zeta\partial\eta}$$

引进应力函数F

$$N_{\zeta}=h(\zeta,\eta)\frac{\partial^2 F}{\partial\eta^2},\quad N_{\eta}=h(\zeta,\eta)\frac{\partial^2 F}{\partial\zeta^2},\quad N_{\zeta\eta}=-h(\zeta,\eta)\frac{\partial^2 F}{\partial\zeta\partial\eta} \tag{2.9}$$

并将式(2.9)代入式(2.8)，在横向运动的方程可变为

$$L(w)-q(\zeta,\eta)=h(\zeta,\eta)\left(\frac{\partial^2 F}{\partial\eta^2}\frac{\partial^2 w}{\partial\zeta^2}+\frac{\partial^2 F}{\partial\zeta^2}\frac{\partial^2 w}{\partial\eta^2}-2\frac{\partial^2 F}{\partial\zeta\partial\eta}\frac{\partial^2 w}{\partial\zeta\partial\eta}\right) \tag{2.10}$$

由方程组(2.4)，取其二阶导数，则可得形变协调方程

$$\frac{\partial^2\varepsilon_{\zeta}^0}{\partial\eta^2}+\frac{\partial^2\varepsilon_{\eta}^0}{\partial\zeta^2}-\frac{\partial^2\varepsilon_{\zeta\eta}^0}{\partial\zeta\partial\eta}=\left(\frac{\partial^2 w}{\partial\zeta\partial\eta}\right)^2-\frac{\partial^2 w}{\partial\zeta^2}\frac{\partial^2 w}{\partial\eta^2} \tag{2.11}$$

由方程组(2.5)，相容性条件(2.9)可以简化为

$$A_{22}\frac{\partial^4 F}{\partial\zeta^4}-2A_{26}\frac{\partial^4 F}{\partial\zeta^3\partial\eta}+(2A_{12}+A_{66})\frac{\partial^4 F}{\partial\zeta^2\partial\eta^2}-2A_{16}\frac{\partial^4 F}{\partial\zeta\partial\eta^3}+A_{11}\frac{\partial^4 F}{\partial\eta^4}$$

$$=\left(\frac{\partial^2 w}{\partial\zeta\partial\eta}\right)^2-\frac{\partial^2 w}{\partial\zeta^2}\frac{\partial^2 w}{\partial\eta^2} \tag{2.12}$$

此处

$$\left.\begin{aligned}
&A_{16}=e_0a_{12}/(e_0e_3-e_1e_2), && A_{12}=(a_{12}-e_3A_{16})/e_2\\
&A_{11}=(1-a_{12}A_{12}-a_{16}A_{16})/a_{11}, && A_{26}=(e_2A_{12}-a_{12})/e_0\\
&A_{22}=(1-a_{12}A_{12}-a_{26}A_{26})/a_{22}, && A_{66}=(1-a_{16}A_{16}-a_{26}A_{26})/a_{66}\\
&e_0=a_{16}a_{22}-a_{26}a_{12}, && e_1=a_{26}a_{16}-a_{66}a_{12}\\
&e_2=(a_{12})^2-a_{11}a_{22}, && e_3=a_{12}a_{16}-a_{11}a_{26}
\end{aligned}\right\} \tag{2.13}$$

非线性偏微分方程组 (2.10)、(2.12) 即为变厚度各向异性弹性斜板的大挠度问题的基本方程，它为变系数的高阶非线性偏微分方程组，除了有些近似解法外，一般无法从微分方程直接求解.

三、边 界 条 件

各向异性斜形薄板的简支边界条件为

$$\left.\begin{aligned}
&\zeta=\pm a\text{时},\ w=M_{\zeta}\cos\theta=0\Rightarrow\partial^2 w/\partial\zeta^2=0\\
&N_{\zeta}=h(\zeta,\eta)\partial^2 F/\partial\eta^2=0\Rightarrow\partial^2 F/\partial\eta^2=0,\ v^0=\partial^2 F/\partial\zeta^2=0\\
&\eta=\pm b\text{时},\ w=M_{\eta}\cos\theta=0\Rightarrow\partial^2 w/\partial\eta^2=0\\
&N_{\eta}=h(\zeta,\eta)\partial^2 F/\partial\zeta^2=0\Rightarrow\partial^2 F/\partial\zeta^2=0,\ u^0=\partial^2 F/\partial\eta^2=0
\end{aligned}\right\} \tag{3.1}$$

在整个边界Γ上，上述边界条件也可统一表达为

$$w=0,\quad \nabla^2 w=0,\quad F=0,\quad \nabla^2 F=0$$

四、变厚度各向异性斜形薄板的弹性平衡问题的Navier解式

我们应用 Navier 法探求变厚度各向异性斜形薄板的弹性平衡问题的解，设非线性偏微分方程组(2.10)、(2.12)、(3.1)的解为如下的双重三角级数形式：

$$\left.\begin{aligned} w(\zeta,\eta)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w_{mn}\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\zeta+b)\\ F(\zeta,\eta)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F_{mn}\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b)\end{aligned}\right\}\tag{4.1}$$

显然(4.1)式满足边界条件(3.1)式.

将方程(2.10)式，(2.12)式变形为

$$\begin{aligned} &a_{11}\frac{\partial^4 w}{\partial\zeta^4}+2(a_{12}+2a_{66})\frac{\partial^4 w}{\partial\zeta^2\partial\eta^2}+a_{22}\frac{\partial^4 w}{\partial\eta^4}\\ &=\frac{12}{h^2(\zeta,\eta)}\left(\frac{\partial^2 F}{\partial\eta^2}\frac{\partial^2 w}{\partial\zeta^2}+\frac{\partial^2 F}{\partial\zeta^2}\frac{\partial^2 w}{\partial\eta^2}-2\frac{\partial^2 F}{\partial\zeta\partial\eta}\frac{\partial^2 w}{\partial\zeta\partial\eta}\right)-4a_{16}\frac{\partial^4 w}{\partial\zeta^3\partial\eta}\\ &\quad-4a_{26}\frac{\partial^4 w}{\partial\zeta\partial\eta^3}-\frac{6}{h(\zeta,\eta)}\left[\left(a_{11}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{16}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\zeta^3}\right.\\ &\quad+\left(3a_{26}\frac{\partial h(\zeta,\eta)}{\partial\eta}+\overline{2a_{66}+a_{12}}\frac{\partial h(\zeta,\eta)}{\partial\zeta}\right)\frac{\partial^3 w}{\partial\zeta\partial\eta^2}+\left(3a_{16}\frac{\partial h(\zeta,\eta)}{\partial\zeta}\right.\\ &\quad\left.\left.+\overline{a_{12}+2a_{66}}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\zeta^2\partial\eta}+\left(a_{26}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{22}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\eta^3}\right]\\ &\quad-\frac{3}{h^2(\zeta,\eta)}\left[(h_1(\zeta,\eta)a_{11}+h_2(\zeta,\eta)a_{12}+h_3(\zeta,\eta)a_{16})\frac{\partial^2 w}{\partial\zeta^2}\right.\\ &\quad+(h_1(\zeta,\eta)a_{12}+h_2(\zeta,\eta)a_{22}+h_3(\zeta,\eta)a_{26})\frac{\partial^2 w}{\partial\eta^2}+2(h_1(\zeta,\eta)a_{16}+h_2(\zeta,\eta)a_{26}\\ &\quad\left.+h_3(\zeta,\eta)a_{66})\frac{\partial^2 w}{\partial\zeta\partial\eta}\right]+\frac{12q(\zeta,\eta)}{h^3(\zeta,\eta)}\end{aligned}\tag{4.2}$$

$$\begin{aligned} &A_{22}\frac{\partial^4 F}{\partial\zeta^4}+(2A_{12}+A_{66})\frac{\partial^4 F}{\partial\zeta^2\partial\eta^2}+A_{11}\frac{\partial^4 F}{\partial\eta^4}\\ &=2A_{26}\frac{\partial^4 F}{\partial\zeta^3\partial\eta}+2A_{16}\frac{\partial^4 F}{\partial\zeta\partial\eta^3}+\left(\frac{\partial^2 w}{\partial\zeta\partial\eta}\right)^2-\frac{\partial^2 w}{\partial\zeta^2}\frac{\partial^2 w}{\partial\eta^2}\end{aligned}\tag{4.3}$$

将方程(4.2)式和(4.3)式右端分解为同样的双重三角级数形式

$$\begin{aligned} &\frac{12}{h^2(\zeta,\eta)}\left(\frac{\partial^2 F}{\partial\eta^2}\frac{\partial^2 w}{\partial\zeta^2}+\frac{\partial^2 F}{\partial\zeta^2}\frac{\partial^2 w}{\partial\eta^2}-2\frac{\partial^2 F}{\partial\zeta\partial\eta}\frac{\partial^2 w}{\partial\zeta\partial\eta}\right)-4a_{16}\frac{\partial^4 w}{\partial\zeta^3\partial\eta}\\ &\quad-4a_{26}\frac{\partial^4 w}{\partial\zeta\partial\eta^3}-\frac{6}{h(\zeta,\eta)}\left[\left(a_{11}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{16}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\zeta^3}\right.\\ &\quad+\left(3a_{26}\frac{\partial h(\zeta,\eta)}{\partial\eta}+\overline{a_{12}+2a_{66}}\frac{\partial h(\zeta,\eta)}{\partial\zeta}\right)\frac{\partial^3 w}{\partial\zeta\partial\eta^2}+\left(3a_{16}\frac{\partial h(\zeta,\eta)}{\partial\zeta}\right.\\ &\quad\left.\left.+\overline{a_{12}+2a_{66}}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\zeta^2\partial\eta}+\left(a_{26}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{22}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\eta^3}\right]\\ &\quad-\frac{3}{h^2(\zeta,\eta)}\left[(h_1(\zeta,\eta)a_{11}+h_2(\zeta,\eta)a_{12}+h_3(\zeta,\eta)a_{16})\frac{\partial^2 w}{\partial\zeta^2}+(h_1(\zeta,\eta)a_{12}\right.\end{aligned}$$

$$+h_2(\zeta,\eta)a_{22}+h_3(\zeta,\eta)a_{26})\frac{\partial^2 w}{\partial\eta^2}+2(h_1(\zeta,\eta)a_{16}+h_2(\zeta,\eta)a_{26}+h_3(\zeta,\eta)a_{66})\frac{\partial^2 w}{\partial\zeta\partial\eta}\Big]$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b) \tag{4.4}$$

$$2A_{26}\frac{\partial^4 F}{\partial\zeta^3\partial\eta}+2A_{16}\frac{\partial^4 F}{\partial\eta\partial\eta^3}+\left(\frac{\partial^3 w}{\partial\zeta\partial\eta}\right)^2-\frac{\partial^2 w}{\partial\zeta^2}\frac{\partial^2 w}{\partial\eta^2}$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b) \tag{4.5}$$

$$\frac{12}{h^3(\zeta,\eta)}q(\zeta,\eta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b) \tag{4.6}$$

在(4.4)、(4.5)、(4.6)式中，A_{mn}，B_{mn}，q_{mn}为双重三角级数的Fourier系数，它们由下列各式确定：

$$A_{mn}=\frac{1}{ab}\int_{-a}^{a}\int_{-b}^{b}\Big\{\frac{12}{h^2(\zeta,\eta)}\left(\frac{\partial^2 F}{\partial\eta^2}\frac{\partial^2 w}{\partial\zeta^2}+\frac{\partial^2 F}{\partial\zeta^2}\frac{\partial^2 w}{\partial\eta^2}-2\frac{\partial^2 F}{\partial\zeta\partial\eta}\frac{\partial^2 w}{\partial\zeta\partial\eta}\right)$$

$$-4a_{16}\frac{\partial^4 w}{\partial\zeta^3\partial\eta}-4a_{26}\frac{\partial^4 w}{\partial\zeta\partial\eta^3}-\frac{6}{h(\zeta,\eta)}\Big[\Big(a_{11}\frac{\partial h(\zeta,\eta)}{\partial\zeta}$$

$$+a_{16}\frac{\partial h(\zeta,\eta)}{\partial\eta}\Big)\frac{\partial^3 w}{\partial\zeta^3}+\left(3a_{16}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{12}+2a_{66}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\zeta^2\partial\eta}$$

$$+\left(a_{12}+2a_{66}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+3a_{26}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{\partial^3 w}{\partial\zeta\partial\eta^2}+\Big(a_{26}\frac{\partial h(\zeta,\eta)}{\partial\zeta}$$

$$+a_{22}\frac{\partial h(\zeta,\eta)}{\partial\eta}\Big)\frac{\partial^3 w}{\partial\eta^3}\Big]-\frac{3}{h^2(\zeta,\eta)}\Big[(h_1(\zeta,\eta)a_{11}+h_2(\zeta,\eta)a_{12}$$

$$+h_3(\zeta,\eta)a_{16})\frac{\partial^2 w}{\partial\zeta^2}+(h_1(\zeta,\eta)a_{12}+h_2(\zeta,\eta)a_{22}+h_3(\zeta,\eta)a_{26})\frac{\partial^2 w}{\partial\eta^2}$$

$$+2(h_1(\zeta,\eta)a_{16}+h_2(\zeta,\eta)a_{26}+h_3(\zeta,\eta)a_{66})\frac{\partial^2 w}{\partial\zeta\partial\eta}\Big]\Big\}\sin\frac{m\pi}{2a}(\zeta$$

$$+a)\sin\frac{n\pi}{2b}(\eta+b)d\zeta d\eta$$

$$=\psi_1^{(mn)}w_{mn}+\psi_2^{(mn)}F_{mn}w_{mn} \tag{4.7}$$

$$B_{mn}=\frac{1}{ab}\int_{-a}^{a}\int_{-b}^{b}\Big[2A_{26}\frac{\partial^4 F}{\partial\zeta^3\partial\eta}+2A_{16}\frac{\partial^4 F}{\partial\zeta\partial\eta^3}+\left(\frac{\partial^2 w}{\partial\zeta\partial\eta}\right)^2$$

$$-\frac{\partial^2 w}{\partial\zeta^2}\frac{\partial^2 w}{\partial\eta^2}\Big]\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b)d\zeta d\eta$$

$$=\psi_3^{(mn)}F_{mn}+\psi_4^{(mn)}w_{mn}^2 \tag{4.8}$$

$$q_{mn}=\frac{12}{ab}\int_{-a}^{a}\int_{-b}^{b}\frac{q(\zeta,\eta)}{h^3(\zeta,\eta)}\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b)d\zeta d\eta \tag{4.9}$$

式中

$$\psi_1^{(mn)}=\frac{mn\pi^4}{16a^2b^2}\int_{-a}^{a}\int_{-b}^{b}\left(a_{16}\frac{m^2}{a^2}+a_{26}\frac{n^2}{b^2}\right)\sin\frac{m\pi}{a}(\zeta+a)\sin\frac{n\pi}{b}(\eta+b)d\zeta d\eta$$

$$-\frac{6}{ab}\int_{-a}^{a}\int_{-b}^{b}\frac{1}{16h(\zeta,\eta)}\left[\left(a_{11}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{16}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{m^3\pi^3}{a^3}\sin\frac{m\pi}{a}(\zeta+a)\right.$$

$$\cdot\sin^2\frac{n\pi}{2b}(\eta+b)+\left(3a_{16}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{12}+2a_{66}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)\frac{m^2n\pi^3}{a^2b}$$

$$\cdot\sin^2\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{b}(\eta+b)+\left(a_{12}+2a_{66}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+3a_{26}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)$$

$$\cdot\frac{mn^2\pi^3}{ab^2}\sin\frac{m\pi}{a}(\zeta+a)\sin^2\frac{n\pi}{2b}(\eta+b)+\left(a_{26}\frac{\partial h(\zeta,\eta)}{\partial\zeta}+a_{12}\frac{\partial h(\zeta,\eta)}{\partial\eta}\right)$$

$$\left.\cdot\frac{n^3\pi^3}{b^3}\sin^2\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2a}(\eta+b)\right]d\zeta d\eta+\frac{3}{ab}\int_{-a}^{a}\int_{-b}^{b}\frac{1}{4h^2(\zeta,\eta)}$$

$$\cdot\left[\left(h_1(\zeta,\eta)a_{11}+h_2(\zeta,\eta)a_{12}+h_3(\zeta,\eta)a_{16}\right)\frac{m^2\pi^2}{a^2}\sin^2\frac{m\pi}{2a}(\zeta+a)\sin^2\frac{n\pi}{2b}\right.$$

$$\cdot(\eta+b)+((h_1(\zeta,\eta)a_{12}+h_2(\zeta,\eta)a_{22}+h_3(\zeta,\eta)a_{26})\frac{n^2\pi^2}{b^2}\sin^2\frac{m\pi}{2a}(\zeta+a)$$

$$\cdot\sin^2\frac{n\pi}{2b}(\eta+b)-(h_1(\zeta,\eta)a_{16}+h_2(\zeta,\eta)a_{26}+h_3(\zeta,\eta)a_{66})$$

$$\left.\cdot\frac{mn\pi^2}{2ab}\sin\frac{m\pi}{a}(\zeta+a)\sin\frac{n\pi}{b}(\eta+b)\right]d\zeta d\eta \tag{4.10}$$

$$\psi_2^{(mn)}=\frac{3m^2n^2\pi^4}{2a^3b^3}\int_{-a}^{a}\int_{-b}^{b}\frac{1}{h^2(\zeta,\eta)}\left[\sin^3\frac{m\pi}{2a}(\zeta+a)\sin^3\frac{n\pi}{2b}(\eta+b)\right.$$

$$\left.-\cos^2\frac{m\pi}{2a}(\zeta+a)\cos^2\frac{n\pi}{2b}(\eta+b)\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b)\right]d\zeta d\eta \tag{4.11}$$

$$\psi_3^{(mn)}=\frac{mn\pi^4}{32a^2b^2}\int_{-a}^{a}\int_{-b}^{b}\left(A_{26}\frac{m^2}{a^2}+A_{16}\frac{n^2}{b^2}\right)\sin\frac{m\pi}{a}(\zeta+a)\sin\frac{n\pi}{b}(\eta+b)d\zeta d\eta \tag{4.12}$$

$$\psi_4^{(mn)}=\frac{m^2n^2\pi^4}{16a^3b^3}\int_{-a}^{a}\int_{-b}^{b}\left[\sin\frac{m\pi}{2a}(\zeta+a)\cos^2\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b)\right.$$

$$\left.\cdot\cos^2\frac{n\pi}{2b}(\eta+b)d\zeta d\eta-\sin^3\frac{m\pi}{2a}(\zeta+a)\sin^3\frac{n\pi}{2b}(\eta+b)\right]d\zeta d\eta \tag{4.13}$$

将(4.1)中的第一式，(4.4)式，(4.6)式代入方程(4.2)得

$$\frac{\pi^4}{16}\left[a_{11}\frac{m^4}{a^4}+2(a_{12}+2a_{66})\frac{m^2n^2}{a^2b^2}+a_{22}\frac{n^4}{b^4}\right]w_{mn}\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b)$$

$$=(A_{mn}+q_{mn})\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b) \tag{4.14}$$

由此

$$w_{mn}=\psi_5^{(mn)}(A_{mn}+q_{mn}) \tag{4.15}$$

式中

$$\psi_5^{(mn)}=\frac{1}{\pi^4[a_{11}m^4/a^4+2(a_{12}+2a_{66})m^2n^2/a^2b^2+a_{22}n^4/b^4]/16} \tag{4.16}$$

将式(4.7)代入式(4.15)得

$$w_{mn}=\psi_5^{(mn)}(\psi_1^{(mn)}w_{mn}+\psi_2^{(mn)}F_{mn}w_{mn}+q_{mn}) \tag{4.17}$$

我们再把(4.1)式中第二式，(4.5)式代入方程(4.3)得

$$\frac{\pi^4}{16}\left[A_{22}\frac{m^4}{a^4}+(2A_{12}+A_{66})\frac{m^2n^2}{a^2b^2}+A_{11}\frac{n^4}{b^4}\right]F_{mn}\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b)$$

$$=B_{mn}\sin\frac{m\pi}{2a}(\zeta+a)\sin\frac{n\pi}{2b}(\eta+b)$$

由此，

$$F_{mn}=\psi_6^{(mn)}B_{mn} \tag{4.18}$$

式中

$$\psi_6^{(mn)}=\frac{1}{\pi^4[A_{22}m^4/a^4+(2A_{12}+A_{66})m^2n^2/a^2b^2+A_{11}n^4/b^4]/16} \tag{4.19}$$

将式(4.8)代入(4.18)式得

$$F_{mn}=\psi_6^{(mn)}(\psi_3^{(mn)}F_{mn}+\psi_4^{(mn)}w_{mn}^2)$$

或

$$F_{mn}=\psi_7^{(mn)}w_{mn}^2 \tag{4.20}$$

式中

$$\psi_7^{(mn)}=\psi_4^{(mn)}\psi_6^{(mn)}/(1-\psi_3^{(mn)}\psi_6^{(mn)}) \tag{4.21}$$

将式(4.20)代入式(4.17)得

$$\psi_8^{(mn)}w_{mn}^3+\psi_9^{(mn)}w_{mn}+\psi_{10}^{(mn)}=0 \qquad (m,\ n=1,3,5,\cdots) \tag{4.22}$$

其中

$$\psi_8^{(mn)}=\psi_2^{(mn)}\psi_5^{(mn)}\psi_7^{(mn)},\quad \psi_9^{(mn)}=\psi_5^{(mn)}\psi_1^{(mn)}-1,\quad \psi_{10}^{(mn)}=\psi_5^{(mn)}q_{mn} \tag{4.23}$$

方程(4.22)为一个三次非线性方程组，给脚标 m, n 以不同的自然数奇数值，便得到无穷个关于w_{mn}的三次代数方程组，解此方程组，可得w_{mn}，再把w_{mn}代入式(4.21)得F_{mn}，由(4.1)式可求得解$w(\zeta,\eta)$，$F(\zeta,\eta)$。最后，将$w(\zeta,\eta)$，$F(\zeta,\eta)$代入(2.6)、(2.9)可得弯曲内力和中面应力。

五、计 算 示 例

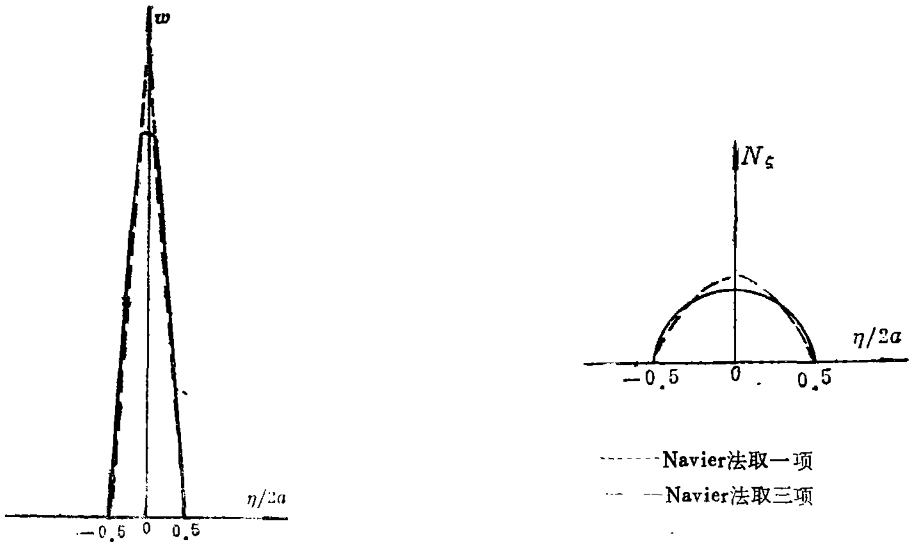
设变厚度弹性正交各向异性薄板为斜板。其边长$a=b=1$；两边夹角$\theta=30°$；水条线方向角$\phi=30°$；板关于主方向(L,T)的材料常数为E_L，E_T，ν_{LT}，ν_{TL}，G_{LT}，其大小杨氏模量之比$E_L/E_T=17.7$，$G_{LT}/E_T=0.35$，其中G_{LT}为剪切模量；Poisson比$\nu_{LT}=0.26$。板受静水压力的作用，压力$q=[\lambda(\zeta+a)+1]^3$；板的厚度用h表示，在直线$\zeta=0$上，厚度用$h_0=1$表示，取$h=[\lambda(\zeta+1)+1]$，其中$\lambda=1$。试求沿板中线$\eta=0$处挠度和内力。

我们首先利用本文中(2.1)，(2.2)式及E_L，E_T，ν_{TL}，ν_{LT}，G_{LT}，可算得方程(2.10)与(2.12)的系数；再利用公式(4.7)～(4.23)，求得w_{mn}；将求得的w_{mn}回代到(4.1)式第一

表 1

项目 $\frac{\eta}{2a}$ \ m n	$w(0,\ \eta)$			$N_\zeta(0,\eta)$		
	$m=1$ $n=1$	$m=1,3$ $n=1,3$	$m=1,3,5$ $n=1,3,5$	$m=1$ $n=1$	$m=1,3$ $n=1,3$	$m=1,3,5$ $n=1,3,5$
−0.25	5.287248463	6.053549477	5.966135115	0.411883539	0.463262409	0.461902619
0	7.477299342	6.07821693	6.17313652	0.582491354	0.509767534	0.511689393
0.25	5.287248463	6.053549477	5.966135115	0.411883539	0.463262409	0.461902619

表中数值分别乘以q_0a^4/D_0h_0，$q_0^2a^6Eh_0/D_0^2$.

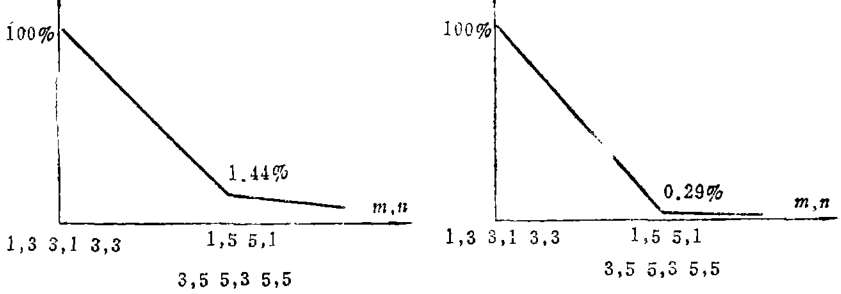


(a) (b)

图 2

式，从而求得 $w(\zeta,\eta)$；最后利用内力公式进而求得中面应力.

从本例的分析可见应用Navier法解变厚度斜形薄板的弹性平衡问题时，挠度和内力的收敛速度是较快的，我们假设展开式第二项，m，$n=1$，3时的数值为100%. 中间截面$\eta=\pm1/2$时的挠度w及中面力N_ζ的级数收敛情形如图3，同一截面的弯矩M_ζ，M_η和扭矩$M_{\zeta\eta}$运用公式(2.6)可以计算. 运用本方法亦可解决其它非线性变系数的高阶偏微分方程组的各类定解问题.



(a) (b)

图 3

参 考 文 献

[1] Prabhakara, M. K. and C. Y. Chia, Postbuckling of angle-ply and anisotropic plates, *Ingenieur-Archiv.*, 45 (1976), 131—140.

[2] Prabhakara, M. K. and C. Y. Chia, Nonlinear bending of symmetrically laminated and homogeneous anisotropic plates, *Rozprawy Lnzynierskie*, 25 (1977), 559—570.

[3] Kennedy, J. B. and S. Ng, Linear and nonlinear analyses of skewed plates, *ASME Journal of Applied Mechanics*, 34 (1967), 271—277.

[4] 叶开沅等，非均匀变厚度弹性力学的若干问题的一般解，兰州大学学报（自然科学版，力学专号），1 (1979).

[5] Timoshenko, S. and S. Woinowsky-Krieger, *Theory of Plates and Shells*, second edition, McGrow-Hill Book Comp. Inc. (1959).

[6] 尹思明、阮圣璜，变厚度矩形薄板的线性和非线性理论的弹性平衡问题的Navier解，应用数学和力学，6，6 (1985), 519—530.

[7] Chia, C. Y. and M. Sathtamoorthy, Nonlinear vibration of anisotropic skew plates, *Fibre Science and Technology*, 13 (1980), 81—95.

Navier Solution for the Elastic Equilibrium Problems of Anisotropic Skew Thin Plate with Variable Thickness in Nonlinear Theories

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Abstract

This paper discusses the elastic equilibrium problems of anisotropic skew thin plate of variable thickness simply supported on all four sides in nonlinear theories, and uses the Navier method to seek an approach to the problem, and to illustrate the solution with the examples. In conclusion, the mention is made of the scope of application and the convergency of the solution.

Key words variable thickness, anisotropic, skew thin plate, Navier solution