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捕食者食饵均染病的入侵反应扩散捕食系统中扩散的作用*

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摘要： 研究了捕食者食饵均染病的入侵反应扩散捕食系统.利用特征值方法和构造 Lyapunov 函数,获得了入侵扩散对正常数平衡解的影响，当入侵扩散系数充分大时，导致平衡态失稳.进一步，利用拓扑度方法，证明了在一定条件下入侵扩散系数很大，自扩散充分小时，有非常数正平衡解存在.

关 键 词： 入侵反应扩散； 稳定性； 正常数平衡解； 非常数正平衡解

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引 言

由捕食系统和传染病模型结合而成的生态流行病系统,是近些年来生物学一个新的分支. Anderson 和 May[1]首次将生态系统和传染病系统结合起来并提出了食饵染病的食饵捕食系统.在种群捕食过程中,疾病也得以传播.因此,在传染病系统中研究物种的相互作用效果有实际意义.许多学者研究食饵带疾病的捕食模型[2-4],或捕食者染病的捕食食饵模型[5-7].另外,也有捕食者系统均染病的捕食食饵模型[8-10].由于疾病可以在捕食者、食饵群内或群间传播,并且捕食者可选择捕食染病和未染病食饵,从而使这类模型有着丰富的研究内容.

然而,前期的传染病食饵模型,并没有考虑空间分布的不均匀性.即由内竞争与群间躲避和防御等引起的自扩散和交错扩散,这显然不符合实际情形.当捕食者进入食饵密度高的地方捕食时,食饵逃跑或群体防御进攻,从而产生了复杂扩散,相关文章可参考文献[11-14].文献[13]中研究了一类食饵染病的入侵扩散系统：

$$\frac{\partial X}{\partial t}=d_1\Delta X+aX-bX^2-\frac{SX}{1+\theta X},\qquad (t,x)\in(0,T)\times\Omega, \tag{1a}$$

$$\frac{\partial S}{\partial t}=\Delta\left[\left(d_2+\frac{m}{1+X}\right)S\right]-\alpha S+\frac{\beta SX}{1+\theta X}-\gamma SI,\qquad (t,x)\in(0,T)\times\Omega, \tag{1b}$$

$$\frac{\partial I}{\partial t}=d_3\Delta I+\gamma SI-cI,\qquad (t,x)\in(0,T)\times\Omega, \tag{1c}$$

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$$\frac{\partial X}{\partial v}=\frac{\partial S}{\partial v}=\frac{\partial I}{\partial v}=0,\qquad (t,x)\in(0,T)\times\partial\Omega, \tag{1d}$$

$$(X(0,x),S(0,x),I(0,x))=(X_0(x),S_0(x),I_0(x)),\qquad x\in\Omega, \tag{1e}$$

其中 Ω 是 R^N($N>1$, 为正整数)中的有界区域, $\partial\Omega$ 为光滑边界;X 是食饵密度,S 和 I 分别代表易感捕食和感染捕食, $aX-bX^2$ 为 logistic 项;正常数 $\theta,\alpha,\beta,c,\gamma$ 分别代表捕食操作时间,易感捕食者的死亡率,转换率,染病捕食者的死亡率和疾病的传播系数;在扩散项中, d_1,d_2 和 d_3 为自扩散系数, m 代表捕食者朝向食饵的一种趋向,且 m 被称为侵入反应扩散系数.

在此基础上,本文进一步考虑捕食者均染病的入侵扩散捕食系统:

$$\begin{cases}\dfrac{\partial X}{\partial t}=d_1\Delta X+aX-bX^2-\dfrac{SX}{1+\theta X}-kXY, & (x,t)\in\Omega\times(0,T),\\ \dfrac{\partial Y}{\partial t}=d_2\Delta Y+kXY-dY, & (x,t)\in\Omega\times(0,T),\\ \dfrac{\partial S}{\partial t}=\Delta\left(d_3+\dfrac{m}{1+X}\right)S-\alpha S+\dfrac{\beta SX}{1+\theta X}-rSI, & (x,t)\in\Omega\times(0,T),\\ \dfrac{\partial I}{\partial t}=d_4\Delta I+rSI-cI, & (x,t)\in\Omega\times(0,T),\\ \dfrac{\partial X}{\partial v}=\dfrac{\partial Y}{\partial v}=\dfrac{\partial S}{\partial v}=\dfrac{\partial I}{\partial v}=0, & (x,t)\in\partial\Omega\times(0,T),\\ (X(0,x),Y(0,x),S(0,x),I(0,x))=(X_0,Y_0,S_0,I_0), & x\in\Omega,\end{cases} \tag{2}$$

其中 Y 为染病食饵, d 为染病食饵的死亡率.食饵疾病只在食饵之间传播,初值非负不恒为 0.传播率为 k, 与系统(2)所对应的椭圆系统为

$$\begin{cases}\Delta d_1X+aX-bX^2-\dfrac{SX}{1+\theta X}-kXY=0, & x\in\Omega,\\ d_2\Delta Y+kXY-dY=0, & x\in\Omega,\\ \Delta\left(d_3+\dfrac{m}{1+X}\right)S-\alpha S+\dfrac{\beta SX}{1+\theta X}-rSI=0, & x\in\Omega,\\ d_4\Delta I+rSI-cI=0, & x\in\Omega,\\ \dfrac{\partial X}{\partial v}=\dfrac{\partial Y}{\partial v}=\dfrac{\partial S}{\partial v}=\dfrac{\partial I}{\partial v}=0, & x\in\partial\Omega.\end{cases} \tag{3}$$

1 入侵扩散对稳定性的影响

系统(2)有以下非平凡常数平衡点,分布如下.

当满足条件(H1):$ak>bd$ 时,存在平衡点

$$\boldsymbol{E}_1=\left(\frac{d}{k},\frac{ak-bd}{k},0,0\right);$$

当满足条件(H2):$\beta>\theta\alpha,a(\beta-\theta\alpha)>b\beta$ 时,存在平衡点

$$\boldsymbol{E}_2=\left(\frac{\alpha}{\beta-\theta\alpha},0,\frac{\alpha}{\beta-\theta\alpha}\left(a-b\frac{\alpha}{\beta-\theta\alpha}\right),0\right);$$

当满足条件(H3):$a>\dfrac{c}{r},\dfrac{\beta c}{r+\theta c}>\alpha r$ 时,存在平衡点

$$\boldsymbol{E}_3=\left(\frac{a\theta-b+\sqrt{\Delta}}{2b\theta},0,\frac{c}{r},\frac{1}{r}\left(\frac{\beta c}{r+\theta c}-\alpha r\right)\right),\qquad \Delta=(b-a\theta)^2-4b\theta\left(\frac{c}{r}-a\right);$$

当满足条件(H4)：$\dfrac{\beta X^*}{1+\theta X^*}>\alpha,a-bX^*-\dfrac{S^*}{1+\theta X^*}>0$ 时，存在平衡点

$$\boldsymbol{E}_4=(X^*,Y^*,S^*,I^*),\ X^*=\frac{d}{k},\ Y^*=\frac{1}{k}\left(a-bX^*-\frac{S^*}{1+\theta X^*}\right),$$

$$S^*=\frac{c}{r},\ I^*=\left(\frac{\beta X^*}{1+\theta X^*}-\alpha\right)\frac{1}{r}.$$

本文主要分析正常数平衡点 $\boldsymbol{E}_4$ 的稳定性，记

$$\boldsymbol{\Phi}=\left(d_1X,d_2Y,\left(d_3+\frac{m}{1+X}\right)S,d_4I\right)^{\mathrm{T}},\ \boldsymbol{W}=(X,Y,S,I)^{\mathrm{T}},$$

$$\boldsymbol{G}=\left(aX-bX^2-\frac{SX}{1+\theta X}-kXY,kXY-dY,-\alpha S+\frac{\beta SX}{1+\theta X}-rSI,rSI-cI\right),$$

$$\boldsymbol{\Phi}_{\boldsymbol{W}}=\begin{bmatrix} d_1 & 0 & 0 & 0\\ 0 & d_2 & 0 & 0\\ -\dfrac{mS}{(1+X)^2} & 0 & d_3+\dfrac{m}{1+X} & 0\\ 0 & 0 & 0 & d_4\end{bmatrix},$$

$$\boldsymbol{G}_{\boldsymbol{W}}=\begin{bmatrix} -bX+\dfrac{\theta SX}{(1+\theta X)^2} & -kX & -\dfrac{X}{1+\theta X} & 0\\ kY & 0 & 0 & 0\\ \dfrac{\beta S}{(1+\theta X)^2} & 0 & 0 & -rS\\ 0 & 0 & rI & 0\end{bmatrix}.$$

在点 $\boldsymbol{E}_4$ 处，系统(2)的线性化系统为

$$\boldsymbol{W}_t=\boldsymbol{\Phi}_{\boldsymbol{W}}(\boldsymbol{E}_4)\Delta\boldsymbol{W}+\boldsymbol{G}_{\boldsymbol{W}}(\boldsymbol{E}_4)\boldsymbol{W}.$$

对应的特征方程记为

$$f(\lambda)=\lambda^4+A_1\lambda^3+A_2\lambda^2+A_3\lambda+A_4,$$

$$A_1=bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}+d_1\mu_k+\mu_kd_2+\mu_kd_4+\mu_k\left(d_3+\frac{m}{1+X^*}\right),$$

$$A_2=\mu_kd_2\left[bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}+d_1\mu_k\right]+k^2X^*Y^*+\mu_k^2d_4\left(d_3+\frac{m}{1+X^*}\right)+$$
$$r^2S^*I^*+\left[\frac{mS^*\mu_k}{(1+X^*)^2}+\frac{\beta S^*}{(1+\theta X^*)^2}\right]\frac{X^*}{1+\theta X^*}+$$
$$\left[bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}+kY^*+d_1\mu_k+\mu_kd_2\right]\left[\mu_kd_4+\mu_k\left(d_3+\frac{m}{1+X^*}\right)\right],$$

$$A_3=\left[bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}+d_1\mu_k+\mu_kd_2\right]\left[\mu_k^2d_4\left(d_3+\frac{m}{1+X^*}\right)+r^2S^*I^*\right]+$$
$$\left[\mu_kd_2\left(bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}+d_1\mu_k\right)+k^2X^*Y^*\right]\left[d_4\mu_k+\left(d_3+\frac{m}{1+X^*}\right)\mu_k\right]+$$

$$\left[\frac{mS^*\mu_k}{(1+X^*)^2}+\frac{\beta S^*}{(1+\theta X^*)^2}\right]\frac{X^*}{1+\theta X^*}(d_2\mu_k+d_4\mu_k),$$

$$A_4=\left[\mu_k d_2\left(bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}+d_1\mu_k\right)+k^2X^*Y^*\right]\times$$

$$\left[\mu_k^2 d_4\left(d_3+\frac{m}{1+X^*}\right)+r^2S^*I^*\right]+$$

$$\left[\frac{mS^*\mu_k}{(1+X^*)^2}+\frac{\beta S^*}{(1+\theta X^*)^2}\right]\frac{X^*}{1+\theta X^*}d_2d_4\mu_k^2,$$

其中 $0=\mu_0\leqslant\mu_1\leqslant\mu_2\leqslant\cdots\leqslant\mu_k\leqslant\cdots$, 为算子 Δ 满足齐次 Neumann 边值条件的特征值.

先证明 $m=0$ 时的情形,记

$$r=\frac{b(1+\theta X^*)^2}{\theta S^*},\ R=\frac{b(1+\theta X^*)}{\theta S^*}.$$

定理 1 若 $m=0$ 且 $r>1$, 那么正平衡点 $\boldsymbol{E}_4$ 是局部渐近稳定的.

证 当 $m=0$ 时,为了方便仍然记 $(X^*,Y^*,S^*,I^*)=(X,Y,S,I)$, 此时特征方程为

$$f_0(\lambda)=\lambda^4+A_{10}\lambda^3+A_{20}\lambda^2+A_{30}\lambda+A_{40},$$

其中

$$A_{10}=bX-\frac{\theta SX}{(1+\theta X)^2}+\mu_kd_1+\mu_kd_2+\mu_kd_3+\mu_kd_4,$$

$$A_{20}=\mu_kd_2\left(bX-\frac{\theta SX}{(1+\theta X)^2}+\mu_kd_1\right)+k^2XY+\mu_k^2d_3d_4+r^2SI+\frac{\beta SX}{(1+\theta X)^3},$$

$$A_{30}=\left(bX-\frac{\theta SX}{(1+\theta X)^2}+\mu_kd_1+\mu_kd_2\right)(\mu_k^2d_3d_4+r^2SI)+$$

$$\left[\mu_kd_2\left(bX-\frac{\theta SX}{(1+\theta X)^2}+\mu_kd_1\right)+k^2XY\right](d_3\mu_k+d_4\mu_k)+$$

$$\frac{\beta SX}{(1+\theta X)^3}(d_2\mu_k+d_4\mu_k),$$

$$A_{40}=\left[\mu_kd_2\left(bX-\frac{\theta SX}{(1+\theta X)^2}+\mu_kd_1\right)+k^2XY\right](\mu_k^2d_3d_4+r^2SI)+$$

$$\frac{\beta SX}{(1+\theta X)^3}d_2d_4\mu_k^2.$$

容易得到,当 $r>1$ 时, $f_0(0)=A_{40}>0$,并且 $f'_0(\lambda)=4\lambda^3+3A_{10}\lambda^2+2A_{20}\lambda+A_{30}>0$.这说明 $f_0(\lambda)=0$ 无正实根,下面说明 $f_0(\lambda)=0$ 无纯虚根.假设 $\lambda=\mathrm{i}\omega\ (\omega>0)$ 是特征方程的特征根,代入方程,则有

$$(\omega^{40}-A_{20}\omega^2+A_{40})+\mathrm{i}(A_{30}\omega-A_{10}\omega^3)=0.$$

由上式可得

$$\omega^4-A_{20}\omega^2+A_{40}=0,\ A_{30}\omega-A_{10}\omega^3=0,$$

所以有

$$A_{10}A_{20}A_{30}-A_{30}^2-A_{10}^2A_{40}=0.$$

通过计算可以得出矛盾.事实上,只要取常数项进行比较.$A_{10}A_{20}A_{30}-A_{30}{}^2$ 按 μ_k 降幂排列,常数项为

$$\left(k^2XY + \frac{\beta SX}{(1+\theta X)^3}\right) r^2 SI\left(bX - \frac{\theta SX}{(1+\theta X)^2}\right)^2,$$

而 $A_{10}{}^2A_{40}$ 按 μ_k 降幂排列,常数项为

$$k^2XYr^2SI\left(bX - \frac{\theta SX}{(1+\theta X)^2}\right)^2.$$

从而特征方程不存在虚根.由文献[15]中定理 3.3.1 知,特征方程的根均有负实部.即当 $r > 1$ 时, $\boldsymbol{E}_4$ 具有局部渐近稳定性.

定理 2 若 $m = 0$ 且 $R > 1$,那么正平衡点 $\boldsymbol{E}_4$ 是全局渐近稳定的.

证 定义 Lyapunov 函数:

$$V(t) = \int_\Omega \left[X - X^* - X^* \ln\frac{X}{X^*} + Y - Y^* - Y^* \ln\frac{Y}{Y^*} + h\left(S - S^* - S^*\ln\frac{S}{S^*}\right) + h\left(I - I^* - I^*\ln\frac{I}{I^*}\right)\right]\mathrm{d}x,$$

其中 $h = (1+\theta X)/\beta$.则

$$\begin{aligned}V'(t) = &-\int_\Omega\left(\frac{d_1X^*}{X^2}|\nabla X|^2 + \frac{d_2Y^*}{Y^2}|\nabla Y|^2 + \frac{d_3S^*h}{S^2}|\nabla S|^2 + \frac{d_4I^*h}{I^2}|\nabla I|^2\right)\mathrm{d}x + \\ &\int_\Omega\left[(X-X^*)\left(a - bX - \frac{S}{1+\theta X} - kY\right) + (Y-Y^*)(kX-d) + \right. \\ &\left. h(S-S^*)\left(-\alpha + \frac{\beta X}{1+\theta X} - rI\right) + h(I-I^*)(rS-c)\right]\mathrm{d}x \leqslant \\ &-b\int_\Omega(X-X^*)^2\mathrm{d}x - \int_\Omega(X-X^*)\left(\frac{S}{1+\theta X} - \frac{S^*}{1+\theta X^*}\right)\mathrm{d}x - \\ &k\int_\Omega(X-X^*)(Y-Y^*)\mathrm{d}x + k\int_\Omega(Y-Y^*)(X-X^*)\mathrm{d}x + \\ &h\int_\Omega(S-S^*)\left[\left(\frac{\beta X}{1+\theta X} - \frac{\beta X^*}{1+\theta X^*}\right) - r(I-I^*)\right]\mathrm{d}x + \\ &hr\int_\Omega(I-I^*)(S-S^*)\mathrm{d}x \leqslant -\int_\Omega\left(b - \frac{\theta S^*}{1+\theta X^*}\right)(X-X^*)^2\mathrm{d}x.\end{aligned}$$

故当 $R > 1$ 时, $V(t) \leqslant 0$, 从而正平衡点是全局渐近稳定的.

定理 3 若满足

$$bX^* - \frac{\theta S^*X^*}{(1+\theta X^*)^2} + \frac{S^*X^*}{(1+\theta X^*)(1+X^*)} < 0,$$

则存在正常数 m_0,当 $m > m_0$,那么正平衡点 $\boldsymbol{E}_4$ 是不稳定的.

证 为了方便,仍然记 $(X^*,Y^*,S^*,I^*) = (X,Y,S,I)$.由于

$$\begin{aligned}&\det \boldsymbol{G}_{\boldsymbol{W}}(\boldsymbol{E}_4) = \\ &\left\{\mu_kd_2\left[\mu_kd_1 + bX - \frac{\theta SX}{(1+\theta X)^2}\right] + k^2XY\right\}\left[\left(d_3 + \frac{m}{1+X}\right)d_4\mu_k^2 + r^2SI\right] + \\ &\frac{X}{1+\theta X}d_2d_4\mu_k^2\left[\frac{mS\mu_k}{(1+X)^2} + \frac{\beta S}{(1+\theta X)^2}\right],\end{aligned}$$

取极限

$$\lim_{m\to\infty}\det \boldsymbol{G}_W(\boldsymbol{E}_4)=$$
$$\frac{d_4}{1+X}\mu_k^2\left\{\mu_k d_2\left[\mu_k d_1+bX-\frac{\theta SX}{(1+\theta X)^2}\right]+k^2XY\right\}+$$
$$\frac{X}{1+\theta X}d_2d_4\mu_k^2\frac{mS\mu_k}{(1+X)^2}=\frac{d_1d_2d_4}{1+X}\mu_k^4+$$
$$\left[\left(bX-\frac{\theta SX}{1+\theta X}\right)\frac{d_2d_4}{1+X}+\frac{SX}{(1+\theta X)}\frac{d_2d_4}{(1+X)^2}\right]\mu_k^3+\frac{k^2XYd_4}{1+X}\mu_k^2=$$
$$\mu_k^2(B_1\mu_k^2+B_2\mu_k+B_0),$$

其中

$$B_0=\frac{d_4k^2XY}{1+X},\ B_1=\frac{d_1d_2d_4}{1+X},$$
$$B_2=\left(bX-\frac{\theta SX}{(1+\theta X)^2}\right)\frac{d_2d_4}{1+X}+\frac{SX}{(1+\theta X)^2}\frac{d_2d_4}{(1+X)^2}.$$

设 $\mu_1(m),\mu_2(m),\mu_3(m),\mu_4(m)$ 为 $\det \boldsymbol{G}_W(\boldsymbol{E}_4)=0$ 的 4 个根,且 $\mu_1(m)\leqslant\mu_2(m)\leqslant\mu_3(m)\leqslant\mu_4(m)$, 则

$$\mu_1(m)\cdot\mu_2(m)\cdot\mu_3(m)\cdot\mu_4(m)=k^2XY\cdot r^2XY>0,$$
$$\lim_{m\to\infty}\mu_1(m)=\lim_{m\to\infty}\mu_2(m)=0,\ \lim_{m\to\infty}\mu_3(m)>0,\ \lim_{m\to\infty}\mu_4(m)>0.$$

由定理条件可知, $B_2<0$.由连续性可知,存在常数 m_0,当 $m>m_0$ 时,有

(i) $-\infty<\mu_1(m)<\mu_2(m)<0<\mu_3(m)<\mu_4(m)<+\infty$;

(ii) $0<\mu_1(m)<\mu_2(m)<\mu_3(m)<\mu_4(m)<+\infty$;

因此,存在 $\mu_k\in(\mu_3(m),\mu_4(m))$,使得 $\det \boldsymbol{G}_W(\boldsymbol{E}_4)<0$.这说明此时至少有一正的特征根.故正平衡点 $\boldsymbol{E}_4$ 不稳定.

2 非常数正平衡解的存在性

引理 1 存在正常数 A, 使得系统(3)的解满足

$$A^{-1}\leqslant(X,Y,S,I)\leqslant A.$$

证 由极值原理易知

$$0\leqslant X\leqslant\frac{a}{b}.$$

将第一个方程和第二个方程相加,同样由极值原理,有

$$0\leqslant Y\leqslant\frac{a}{d}X\leqslant\frac{a^2}{db},$$

记

$$\phi(x)=d_1\beta X+d_2Y+\left(d_3+\frac{m}{1+X}\right)S+d_4I.$$

设 $\phi(x_0)=\max\phi(x)$, 由极值原理

$$S(x_0)\leqslant\frac{a^2}{\alpha b},\ I(x_0)\leqslant\frac{a^2}{bc},$$
$$\left(d_3+\frac{m}{1+X}\right)S\leqslant\phi(x_0)\leqslant\frac{d_1\beta a}{b}+\frac{d_2a^2}{db}+\frac{(d_3+m)a^2}{\alpha b}+\frac{d_4a^2}{bc},$$

从而有

$$S \leqslant \frac{1}{d_3}\left(\frac{d_1\beta a}{b} + \frac{d_2a^2}{db} + \frac{(d_3+m)a^2}{\alpha b} + \frac{d_4a^2}{bc}\right) = C_1,$$

$$I(x) \leqslant \frac{1}{d_3}\phi(x) \leqslant \frac{C_1}{d_3},$$

取

$$A = \max_{x\in\bar{\Omega}}\left\{\frac{a}{b},\frac{a^2}{db},C_1,\frac{C_1}{d_3}\right\}.$$

下面证明 X,Y,S,I 下界，记

$$\phi_1(x) = \left(d_3 + \frac{m}{1+X}\right)S$$

满足方程

$$-\Delta\phi_1(x) = c(x)\phi_1(x),$$

其中

$$\|\phi(x)\|_\infty = \left\|\frac{-\alpha + \dfrac{\beta X}{1+\theta X} - rI}{d_3 + \dfrac{m}{1+X}}\right\|_\infty \leqslant \frac{\alpha\beta}{d_3b}.$$

由 Harnack[16]不等式可得

$$\max_{x\in\bar{\Omega}}\phi_1(x) \leqslant A_1\min_{x\in\bar{\Omega}}\phi_1(x),$$

从而

$$\max_{x\in\bar{\Omega}}S(x) \leqslant A_2\min_{x\in\bar{\Omega}}S(x).$$

同理有

$$\max_{x\in\bar{\Omega}}X(x) \leqslant A_3\min_{x\in\bar{\Omega}}X(x);\ \max_{x\in\bar{\Omega}}Y(x) \leqslant A_4\min_{x\in\bar{\Omega}}Y(x);\ \max_{x\in\bar{\Omega}}I(x) \leqslant A_5\min_{x\in\bar{\Omega}}I(x).$$

将方程组(3)中第二个和第四个方程积分，由散度定理可得

$$\int_\Omega(kXY - dY)\mathrm{d}x = 0,$$

因此存在 $x_1\in\Omega$，使 $X(x_1) = d/k$，结合 Harnack 不等式可得

$$\min_{x\in\bar{\Omega}}X(x) \geqslant \frac{d}{A_3k}.$$

用同样的方法可证

$$\min_{x\in\bar{\Omega}}S(x) \geqslant \frac{c}{A_2r}.$$

对于 $I(x)$ 下界利用反证法，假设存在序列 $\{d_{i,k},m_k\}_{k=1}^{+\infty}(i=1,2,3,4)$，使得 $k\to+\infty$ 时，

$$\min_{x\in\bar{\Omega}}I_k(x)\to 0,\ (X_k,Y_k,S_k)\to(X^*,Y^*,S^*).$$

由 Harnack 不等式可知，$\max_{x\in\bar{\Omega}}I_k(x)\to 0$. 则 $I_k(x)\to 0$ 在 Ω 一致成立. 直接将方程组(3)中第三个方程序列积分，由散度定理可得

$$\int_\Omega\left(-\alpha S_k + \frac{\beta S_kX_k}{1+\theta X_k} - rS_kI_k\right)\mathrm{d}x = 0.$$

令 $k \to +\infty$ 并由积分中值定理,存在 $x_2 \in \Omega$, 使得

$$-\alpha + \frac{\beta X(x_5)}{1+\theta X(x_5)} = 0,$$

这与条件(H4)矛盾.

利用同样方法可证明 Y 的下界,引理证毕.

记系统(3)为

$$-\Delta \boldsymbol{\Phi}(\boldsymbol{W}) = \boldsymbol{G}(\boldsymbol{W}),\ B = \{x \in \Omega \mid A^{-1} \leqslant X, Y, S, I \leqslant A\},$$

将其记为

$$\boldsymbol{F}(\boldsymbol{W}) = \boldsymbol{W} - (\mathrm{I}-\Delta)^{-1}\{[\boldsymbol{\Phi}(\boldsymbol{W})(\boldsymbol{W})]^{-1}[\boldsymbol{G}(\boldsymbol{W}) + \nabla \boldsymbol{W} \cdot \boldsymbol{\Phi}_{WW}(\boldsymbol{W}) \cdot \nabla \boldsymbol{W}] + \boldsymbol{W}\} = \boldsymbol{0},$$

其中 $(\mathrm{I}-\Delta)^{-1}$ 是 $\mathrm{I}-\Delta$ 的逆,易证对所有 $\boldsymbol{W}$, $\det \boldsymbol{\Phi}_W > 0$.则 $\boldsymbol{\Phi}_W^{-1}$ 存在,由于算子 $\boldsymbol{F}(\cdot)$ 是恒等算子 I 的一个紧干扰算子.对 $\forall \boldsymbol{W} \in \partial B$ 有 $\boldsymbol{F}(\boldsymbol{W}) \neq \boldsymbol{0}$, 则有 Leray-Schauder 拓扑度 $\deg(\boldsymbol{F}(\cdot), 0, B)$.直接计算得

$$\boldsymbol{D}_W \boldsymbol{F}(\boldsymbol{E}_4) = \boldsymbol{I} - (\mathrm{I}-\Delta)^{-1}\{[\boldsymbol{\Phi}_W(\boldsymbol{E}_4)]^{-1}\boldsymbol{G}_W(\boldsymbol{E}_4) + \boldsymbol{I}\},$$

记

$$H(\mu) = \det\{\mu \boldsymbol{I} - (\boldsymbol{\Phi}_W(\boldsymbol{E}_4))^{-1}\boldsymbol{G}_W(\boldsymbol{E}_4)\}.$$

由 Leray-Schauder 定理,有以下引理.

引理 2 对所有正整数 i 都有 $H(\mu_i) \neq 0$, 则

$$\operatorname{index}(\boldsymbol{F}(\cdot), \boldsymbol{E}_4) = (-1)^m, \qquad m = \sum_{i \geqslant 0, H(\mu_i) < 0} \dim \boldsymbol{E}(\mu_i).$$

证 由于 $\det\{[\boldsymbol{\Phi}(\boldsymbol{W})(\boldsymbol{E}_4)]^{-1}\} > 0$,故只需考虑 $\det[\mu \boldsymbol{\Phi}_W(\boldsymbol{E}_4) - \boldsymbol{G}_W(\boldsymbol{E}_4)]$ 的符号来判断 $H(\mu)$ 的符号.

定理 4 设满足条件(H4)以及

$$bX^* - \frac{\theta S^* X^*}{(1+\theta X^*)^2} + \frac{S^* X^*}{(1+\theta X^*)(1+X^*)} < 0,$$

$$4d_1\left[r^2 S^* I^* d_1 d_2 + k^2 X^* Y^* d_3 d_4 + \frac{\beta S^* X^*}{(1+\theta X^*)^3} d_2 d_4\right] > \left(\frac{d_1 d_2{}^2 r^2 S^* I^*}{k^2 X^* Y^*} + d_2 d_3 d_4\right)\left(bX^* - \frac{\theta S^* X^*}{(1+\theta X^*)^2}\right)^2,$$

并且 $\mu_3(m) \in (\mu_{i-1}, \mu_i)$, $\mu_4(m) \in (\mu_{j-1}, \mu_j)$, $j-i$ 为奇数.那么存在正常数 M,当 $m > M$ 时,系统(2)至少有一个非常数正平衡解.

证 利用反证法.假设结论不成立,对 $t \in [0,1]$, 定义

$$\boldsymbol{\Phi}(t, \boldsymbol{W}) = \left(d_1 X, d_2 Y, d_3 S + \frac{tmS}{1+X}, d_4 I\right).$$

考虑如下方程:

$$\begin{cases} -\Delta \boldsymbol{\Phi}(t, \boldsymbol{W}) = \boldsymbol{G}(\boldsymbol{W}), & x \in \Omega, \\ \dfrac{\partial \boldsymbol{W}}{\partial n} = \boldsymbol{0}, & x \in \partial \Omega. \end{cases} \tag{4}$$

记

$$\boldsymbol{F}(t, \boldsymbol{W}) =$$

$$\boldsymbol{W}-(\mathrm{I}-\Delta)^{-1}\{[\boldsymbol{\Phi}_{\boldsymbol{W}}(t,\boldsymbol{W})]^{-1}[\boldsymbol{G}(\boldsymbol{W})+\nabla\boldsymbol{W}\cdot\boldsymbol{\Phi}_{\boldsymbol{WW}}(t,\boldsymbol{W})\cdot\nabla\boldsymbol{W}]+\boldsymbol{W}\}=\mathbf{0}.$$

当 $t=1$ 时,系统(4)即是系统(3),并且系统(4)只有唯一正常数解 $\boldsymbol{E}_4$.由定理条件及引理2可知 $\mathrm{index}(\boldsymbol{F}(1,\cdot),\boldsymbol{E}_4)=(-1)^{\sigma_n}=-1$.

另外,考虑 $m=0$ 时,

$$f(\lambda)=\lambda^4+A_{10}\lambda^3+A_{20}\lambda^2+A_{30}\lambda+A_{40},$$

$$A_{40}=\left[\mu d_2\left(bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}+\mu d_1\right)+k^2X^*Y^*\right](\mu_k^2d_3d_4+r^2S^*I^*)+\frac{\beta S^*X^*}{(1+\theta X^*)^3}d_2d_4\mu^2=$$

$$B_{10}\mu^4+B_{20}\mu^3+B_{30}\mu^2+B_{40}\mu+B_{50},$$

其中

$$B_{10}=d_1d_2d_3d_4,\ B_{20}=d_2d_3d_4\left(bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}\right),$$

$$B_{30}=r^2S^*I^*d_1d_2+k^2X^*Y^*d_3d_4+\frac{\beta S^*X^*}{(1+\theta X^*)^3}d_2d_4,$$

$$B_{40}=r^2S^*I^*d_2\left[bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}\right],\ B_{50}=k^2X^*Y^*r^2S^*I^*.$$

当满足条件(H5):$4B_{10}B_{30}>B_{20}^2$, $4\left(B_{30}-\frac{B_{20}^2}{4B_{10}}\right)B_{50}>B_{40}^2$ 时,对 $\forall u>0,A_{40}>0$.通过计算可得,当满足条件

$$4d_1\left[r^2S^*I^*d_1d_2+k^2X^*Y^*d_3d_4+\frac{\beta S^*X^*}{(1+\theta X^*)^3}d_2d_4\right]>\left(\frac{d_1d_2{}^2r^2S^*I^*}{k^2X^*Y^*}+d_2d_3d_4\right)\left(bX^*-\frac{\theta S^*X^*}{(1+\theta X^*)^2}\right)^2$$

时,恒有 $H(\mu)>0$.因此对任意 $\mu>0$, $\mathrm{index}(\boldsymbol{F}(0,\cdot),\boldsymbol{E}_4)=(-1)^0=1$.

因为由假设,在 B 中系统(4)只有唯一正常数解 $\boldsymbol{E}_4$.因此有

$$\deg(\boldsymbol{F}(0,\cdot),0,B)=\mathrm{index}(\boldsymbol{F}(0,\cdot),\boldsymbol{E}_4),$$

$$\deg(\boldsymbol{F}(1,\cdot),0,B)=\mathrm{index}(\boldsymbol{F}(1,\cdot),\boldsymbol{E}_4).$$

这与拓扑度的同伦不变性 $\deg(\boldsymbol{F}(0,\cdot),0,B)=\deg(\boldsymbol{F}(1,\cdot),0,B)$ 相矛盾,这说明在 B 内系统(3)一定有不同于 $\boldsymbol{E}_4$ 的解.证毕.

3 结　　论

由文中定理1~3的结论,首先讨论没有入侵扩散时,正常数平衡点的稳定性,得到结论如下:当扩散系数 $m=0$ 且满足阈值 $r>1$ 时,平衡点 $\boldsymbol{E}_4$ 是局部渐近稳定的;当扩散系数 $m=0$ 且满足阈值 $R>1$ 时,平衡点 $\boldsymbol{E}_4$ 是全局渐近稳定的.而当入侵存在,扩散系数 m 充分大且满足一定条件时,平衡点 $\boldsymbol{E}_4$ 是不稳定的.这就说明入侵会导致平衡态失稳.进一步地,由定理4的结论可知,当自扩散系数 $d_i(i=1,2,3,4)$ 固定且满足一定条件,扩散系数 m 足够大时,能进一步产生其他非常数正平衡解,此时捕食者和食饵能共存.

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Influence of Diffusion on an Invasion-Diffusion Prey-Predator Model With Disease Infection in Both Populations

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Abstract: An invasion-diffusion prey-predator epidemic system with disease infection in both populations was studied. The influence of invasion diffusion on the equilibrium solutions of positive constants was obtained through analysis of the eigenvalue and construction of the Lyapunov function. Furthermore, with the topological method, it was proved that the coefficient of invasion diffusion will be big enough while the self-diffusion coefficient is sufficiently small, then there exists a positive non-constant equilibrium solution.

Key words: invasion diffusion; stability; positive equilibrium solution; non-constant positive equilibrium solution

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