

定荷载和双面排水条件下非饱和土 一维固结的解析解*

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摘要: 基于多孔介质弹性理论, 结合粒间吸应力表示的有效应力原理, 建立了非饱和土固结的耦合偏微分控制方程. 考虑一维问题, 采用 Laplace 积分变换, 得到了定荷载和双面排水条件下非饱和土固结的解析解答. 通过数值算例, 分析了土体饱和度对超孔隙水压力、有效应力以及土层沉降的影响规律. 结果表明, 土体的初始饱和度越高, 则孔隙水压力消散得越快, 有效应力增加越快.

关键词: 非饱和土; 解析解; 吸应力特征曲线; 孔隙水压力

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引言

非饱和土中孔隙水压力和孔隙气压力随土体变形而变化以及随时间增长而消散的规律属于非饱和土固结理论的重要研究内容. 然而影响非饱和土力学行为的因素多, 相应的研究难度大, 至今尚没有公认研究成熟的非饱和土固结理论. Barden^[1-2] 提出了压实非饱和黏土的一维固结分析, 用 Darcy 定律描述气相和液相的流动, 对于土的不同饱和度, 提出了若干种独立的分析. Fredlund^[3] 提出了用两个偏微分方程求解非饱和土固结过程的孔隙气压力和孔隙水压力. 该方程假定气相是连续的, 将 Darcy 定律和 Fick 定律分别应用于液相和气相的流动. 陈正汉等^[4-5] 以混合物理论为基础, 提出非饱和土有效应力的理论公式, 用公理化方法建立了非饱和土固结的三维理论, 求得一维问题的完全解析解和二维有限元解; 陈正汉团队^[6-8] 后期又论证了双应力状态变量的合理性并加以采用, 建立了多种非饱和土固结模型^[9-11]. Li 等^[12-13] 和杨代泉^[14] 都是用唯象法建立非饱和土的固结理论, 其中 Li 等以多孔介质理论和简化的 Bishop 有效应力公式为基础, 杨代泉则采用双应力状态变量构建了非饱和土的固结理论. 李冰河等^[15] 建立了变荷载情况下的软黏土地基非线性一维固结问题的控制方程, 并编制相应的计算程序, 利用半解析解法进行求解. 秦爱芳等^[16-17] 在 Fredlund 非饱和土固结理论的基础上推导出在瞬时荷载和指数型荷载及单面排水条件下的一维固结解析解.

多孔介质弹性力学方法可用于求解在可变形多孔介质上施加压力时所产生的应力和孔隙流体压力之间关于时间的耦合. Biot 模型^[18-19] 以固相的位移矢量和孔隙水压力作为因变量, 通

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过一对耦合的偏微分方程描述饱和土中的多孔弹性行为,Lo 等^[20-21]对含有两种不混溶的、黏性的、可压缩流体的多孔介质进行研究,得到了非饱和土一维固结的解析解。

本文在 Lo 等^[21]研究成果的基础上,结合 Lu 等^[22-23]提出的非饱和土有效应力原理,建立非饱和土三维固结的偏微分方程组,通过引入加载瞬间计算得出的初始条件,推导得到定荷载和双面排水条件下一维固结问题的解析解。最后以黏土为例,分析初始饱和度对非饱和土一维固结的影响。

1 基本方程

在准静态和不考虑体力的情况下,含有两种不混溶、可压缩、黏性流体的均质多孔弹性介质的动量平衡方程的形式为^[21]

$$R_{11} \left(\frac{\partial \mathbf{u}_a}{\partial t} - \frac{\partial \mathbf{u}_s}{\partial t} \right) = \theta_a \nabla P_a, \quad (1a)$$

$$R_{22} \left(\frac{\partial \mathbf{u}_w}{\partial t} - \frac{\partial \mathbf{u}_s}{\partial t} \right) = \theta_w \nabla P_w, \quad (1b)$$

$$\nabla \cdot \boldsymbol{\sigma} = 0, \quad (1c)$$

式中, \mathbf{u}_a 为气体位移矢量, \mathbf{u}_w 为水的位移矢量, \mathbf{u}_s 为固相位移矢量; θ_a 为气体的体积分数, θ_w 为水的体积分数, $\theta_a = S_a \varphi$, $\theta_w = S_w \varphi$, S_a 和 S_w 分别是气体和水的饱和度, φ 为土体孔隙度; P_a 为孔隙气压力, P_w 为孔隙水压力; $R_{11} = -\theta_a^2 \eta_1 / (k_s k_{ra})$ 为固相和气体的黏性耦合系数, $R_{22} = -\theta_w^2 \eta_2 / (k_s k_{rw})$ 为固相和水的黏性耦合系数, η_1 为气体的动态剪切黏度, η_2 为水的动态剪切黏度, k_s 为多孔介质框架的固有渗透性, k_{ra} 为气体的相对渗透率, k_{rw} 为水的相对渗透率; $\boldsymbol{\sigma}$ 为多孔介质的总应力张量,规定拉力为正,压力为负。

对于非饱和土, Lu 等^[22]提出了一类基于粒间吸应力的有效应力方程:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - P_a - \boldsymbol{\sigma}^s, \quad (2)$$

其中, $\boldsymbol{\sigma}'$ 为有效应力, $\boldsymbol{\sigma}^s$ 为土的粒间吸应力^[22]:

$$\boldsymbol{\sigma}^s = \begin{cases} P_a - P_w, & P_a - P_w \leq 0, \\ -\chi(P_a - P_w), & P_a - P_w \geq 0, \end{cases} \quad (3)$$

式中 χ 是与土体饱和度等因素有关的参数。陈正汉等^[4-5]通过理论推导以及实验验证给出了 χ 的表达式为

$$\chi = \frac{K^n}{K^{sn}}, \quad (4)$$

式中 K^n 和 K^{sn} 分别是孔隙率为 n 的多孔介质的体变模量和孔隙率为 sn 的多孔介质的体变模量,其中 n 是土的孔隙率, s 是第一种流体的饱和度。

从式(4)中可以看出,该有效应力参数不仅能反映饱和度的影响,还能反映土体孔隙率的影响。本文为了能够得到解析解答,采用 Lu 等^[23]提出的具有显式表达形式的有效应力参数,即

$$\chi = S_e = \frac{S_w - S_r}{S_s - S_r}, \quad (5)$$

其中 S_s 为完全饱和度, S_r 为残余饱和度。

当 $P_a - P_w \geq 0$ 时,总应力为

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - (1 - S_e)P_a - S_e P_w. \quad (6)$$

将固体骨架和孔隙流体的变形结合起来,得到一般的应力应变关系^[21]:

$$\boldsymbol{\sigma}' = 2G\boldsymbol{e} + \left[\left(a_{11} - \frac{2}{3}G \right) \boldsymbol{e} + a_{12}\boldsymbol{\varepsilon}_a + a_{13}\boldsymbol{\varepsilon}_w \right] \boldsymbol{\delta}, \quad (7)$$

$$-\theta_a P_a = a_{12}e + a_{22}\boldsymbol{\varepsilon}_a + a_{23}\boldsymbol{\varepsilon}_w, \quad (8a)$$

$$-\theta_w P_w = a_{13}e + a_{23}\boldsymbol{\varepsilon}_a + a_{33}\boldsymbol{\varepsilon}_w, \quad (8b)$$

式中 $\boldsymbol{e} = (1/2)(\nabla \boldsymbol{u}_s + \nabla \boldsymbol{u}_s^T)$ 表示固相的应变张量, $e = \nabla \boldsymbol{u}_s$ 是固相的体积应变, $\boldsymbol{\varepsilon}_a = \nabla \boldsymbol{u}_a$ 是气体的体积应变, $\boldsymbol{\varepsilon}_w = \nabla \boldsymbol{u}_w$ 是水的体积应变, G 是多孔介质框架的剪切模量, $\boldsymbol{\delta}$ 是单位张量, $a_{ij}(i, j = 1, 2, 3)$ 是线性弹性系数.关于式(7)和(8)中各个参数的具体确定方法详见文献[24-25].

为了简化公式,令

$$A_1 = \frac{1 - S_e}{\theta_a}, \quad (9a)$$

$$A_2 = \frac{S_e}{\theta_w}, \quad (9b)$$

结合总应力表达式(6)和一般应力应变关系式(7),并将式(9a)、(9b)代入,推导可得

$$\boldsymbol{\sigma} = 2G\boldsymbol{e} + \left[\left(a_{11} - \frac{2}{3}G + A_1 a_{12} + A_2 a_{13} \right) \boldsymbol{e} + (a_{12} + A_1 a_{22} + A_2 a_{23})\boldsymbol{\varepsilon}_a + (a_{13} + A_1 a_{23} + A_2 a_{33})\boldsymbol{\varepsilon}_w \right] \boldsymbol{\delta}. \quad (10)$$

通过联立方程(8a)和(8b),求解可得

$$\boldsymbol{\varepsilon}_a = d_1 \boldsymbol{e} + d_2 P_a + d_3 P_w, \quad (11a)$$

$$\boldsymbol{\varepsilon}_w = d_4 \boldsymbol{e} + d_5 P_a + d_6 P_w, \quad (11b)$$

式中

$$d_1 = \frac{a_{12}a_{33} - a_{13}a_{23}}{a_{23}^2 - a_{22}a_{33}}, \quad d_2 = \frac{\theta_a a_{33}}{a_{23}^2 - a_{22}a_{33}}, \quad d_3 = -\frac{\theta_w a_{23}}{a_{23}^2 - a_{22}a_{33}},$$

$$d_4 = \frac{a_{13}a_{22} - a_{12}a_{23}}{a_{23}^2 - a_{22}a_{33}}, \quad d_5 = -\frac{\theta_a a_{23}}{a_{23}^2 - a_{22}a_{33}}, \quad d_6 = \frac{\theta_w a_{22}}{a_{23}^2 - a_{22}a_{33}}.$$

对式(1a)和(1b)左右两边同时取散度,并将式(11a)和(11b)代入其中,得到耦合扩散方程:

$$d_2 \frac{\partial P_a}{\partial t} + d_3 \frac{\partial P_w}{\partial t} + (d_1 - 1) \frac{\partial e}{\partial t} = \frac{\theta_a}{R_{11}} \nabla^2 P_a, \quad (12a)$$

$$d_5 \frac{\partial P_a}{\partial t} + d_6 \frac{\partial P_w}{\partial t} + (d_4 - 1) \frac{\partial e}{\partial t} = \frac{\theta_w}{R_{22}} \nabla^2 P_w. \quad (12b)$$

然后将式(11a)和(11b)代到方程(10)中消除 $\boldsymbol{\varepsilon}_a$ 和 $\boldsymbol{\varepsilon}_w$, 得总应力表达式:

$$\boldsymbol{\sigma} = 2G\boldsymbol{e} + (H_1 e + H_2 P_a + H_3 P_w) \boldsymbol{\delta}, \quad (13)$$

其中系数分别表示为

$$H_1 = a_{11} - 2G/3 + A_1 a_{12} + A_2 a_{13} + (a_{12} + A_1 a_{22} + A_2 a_{23})d_1 + (a_{13} + A_1 a_{23} + A_2 a_{33})d_4,$$

$$H_2 = (a_{12} + A_1 a_{22} + A_2 a_{23})d_2 + (a_{13} + A_1 a_{23} + A_2 a_{33})d_5,$$

$$H_3 = (a_{12} + A_1 a_{22} + A_2 a_{23})d_3 + (a_{13} + A_1 a_{23} + A_2 a_{33})d_6.$$

最后结合方程(1c)和方程(13),推导出一般的应力平衡方程:

$$G \nabla^2 u_s + (H_1 + G) \nabla e + H_2 \nabla P_a + H_3 \nabla P_w = \mathbf{0}. \quad (14)$$

方程(12a)、(12b)、(14)是3个耦合的偏微分方程,可用来求解非饱和土的三维固结.

2 一维固结的控制方程

图1为本文的计算模型,在无限均布荷载 P^* 不随时间变化的情况下,仅对 z 方向进行考虑,则 $e_{xx} = e_{yy} = 0, e = e_{zz} = \partial w / \partial z$,于是方程(12a)、(12b)、(13)简化为

$$d_2 \frac{\partial P_a}{\partial t} + d_3 \frac{\partial P_w}{\partial t} + (d_1 - 1) \frac{\partial}{\partial t} \left(\frac{\partial w}{\partial z} \right) = \frac{\theta_a}{R_{11}} \frac{\partial^2 P_a}{\partial z^2}, \quad (15a)$$

$$d_5 \frac{\partial P_a}{\partial t} + d_6 \frac{\partial P_w}{\partial t} + (d_4 - 1) \frac{\partial}{\partial t} \left(\frac{\partial w}{\partial z} \right) = \frac{\theta_w}{R_{22}} \frac{\partial^2 P_w}{\partial z^2}, \quad (15b)$$

$$-P^* = (2G + H_1) \frac{\partial w}{\partial z} + H_2 P_a + H_3 P_w, \quad (16)$$

式中 w 表示固相位移矢量沿 z 方向的分量,由方程(16)解出

$$\frac{\partial w}{\partial z} = -\frac{P^*}{2G + H_1} - \frac{H_2}{2G + H_1} P_a - \frac{H_3}{2G + H_1} P_w. \quad (17)$$

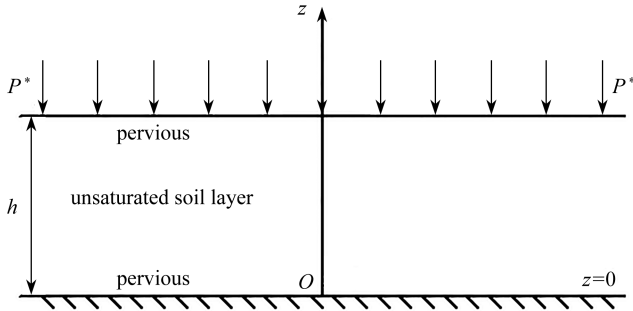


图1 双面排水的非饱和土层固结的计算模型

Fig. 1 The computational model for consolidation of unsaturated soil layer with double-side drainage

将方程(17)代入方程(15a)、(15b),得控制方程:

$$q_1 \frac{\partial P_a}{\partial t} + q_2 \frac{\partial P_w}{\partial t} = b_1 \frac{\partial^2 P_a}{\partial z^2}, \quad (18a)$$

$$q_3 \frac{\partial P_a}{\partial t} + q_4 \frac{\partial P_w}{\partial t} = b_2 \frac{\partial^2 P_w}{\partial z^2}, \quad (18b)$$

其中系数 q_1, q_2, q_3, q_4 和 b_1, b_2 分别为

$$q_1 = d_2 - \frac{H_2(d_1 - 1)}{2G + H_1}, \quad q_2 = d_3 - \frac{H_3(d_1 - 1)}{2G + H_1}, \quad q_3 = d_5 - \frac{H_2(d_4 - 1)}{2G + H_1},$$

$$q_4 = d_6 - \frac{H_3(d_4 - 1)}{2G + H_1}, \quad b_1 = \frac{\theta_a}{R_{11}}, \quad b_2 = \frac{\theta_w}{R_{22}}.$$

在方程(18b)中, $b_2/q_4 = c_v$ 是孔隙水压力的扩散系数,在固结理论中称为固结系数.

3 初始条件和边界条件

在施加均布荷载 P^* ($P^* = 0.1 \text{ MPa}$) 的瞬间, 既来不及排气也来不及排水, 则水和气体的含量都保持不变^[5]. 结合方程式 (18) 可得

$$\int_0^{0^+} b_1 \frac{\partial^2 P_a}{\partial z^2} dt = \int_0^{0^+} b_2 \frac{\partial^2 P_w}{\partial z^2} dt = 0, \quad (19)$$

式中积分限 0 和 0^+ 分别代表开始加载的时刻和加载完成的时刻.

由方程 (15) 得

$$(d_1 - 1) \frac{\partial w(z, 0^+)}{\partial z} + d_2 P_a(z, 0^+) + d_3 P_w(z, 0^+) = 0, \quad (20a)$$

$$(d_4 - 1) \frac{\partial w(z, 0^+)}{\partial z} + d_5 P_a(z, 0^+) + d_6 P_w(z, 0^+) = 0. \quad (20b)$$

将式 (17) 代入到方程 (20a) 和 (20b), 解得初始条件:

$$P_a(z, 0^+) = r_1 P^*, \quad (21a)$$

$$P_w(z, 0^+) = r_2 P^*, \quad (21b)$$

其中

$$r_1 = \frac{(1 - d_4)q_2 - (1 - d_1)q_4}{(2G + H_1)(q_1q_4 - q_2q_3)}, \quad (22a)$$

$$r_2 = \frac{(1 - d_1)q_3 - (1 - d_4)q_1}{(2G + H_1)(q_1q_4 - q_2q_3)}. \quad (22b)$$

图 1 中, 土层厚度为 h , 水平向无限大, 在 $z = 0$, $z = h$ 的这两个界面是渗透面, 则有边界条件:

$$P_a(0, t) = P_w(0, t) = 0, \quad (23a)$$

$$P_a(h, t) = P_w(h, t) = 0. \quad (23b)$$

4 方程求解

对于 $P_a(z, t)$ 和 $P_w(z, t)$, 假设它们取级数形式, $\lambda_n = n\pi/h$, 则

$$P_a(z, t) = \sum_{n=0}^{\infty} P_{an}(t) \sin(\lambda_n z), \quad (24a)$$

$$P_w(z, t) = \sum_{n=0}^{\infty} P_{wn}(t) \sin(\lambda_n z). \quad (24b)$$

结合方程 (24a)、(24b) 和 (18a)、(18b) 得

$$\sum_{n=0}^{\infty} [q_1 P'_{an}(t) + q_2 P'_{wn}(t) + b_1 \lambda_n^2 P_{an}(t)] \sin(\lambda_n z) = 0, \quad (25a)$$

$$\sum_{n=0}^{\infty} [q_3 P'_{an}(t) + q_4 P'_{wn}(t) + b_2 \lambda_n^2 P_{wn}(t)] \sin(\lambda_n z) = 0. \quad (25b)$$

由于 \sin 函数的正交性, 得

$$q_1 P'_{an}(t) + q_2 P'_{wn}(t) + b_1 \lambda_n^2 P_{an}(t) = 0, \quad (26a)$$

$$q_3 P'_{an}(t) + q_4 P'_{wn}(t) + b_2 \lambda_n^2 P_{wn}(t) = 0. \quad (26b)$$

对方程 (26a) 和 (26b) 进行 Laplace 变换得

$$(sq_1 + b_1\lambda_n^2)P_{an}(s) + sq_2P_{wn}(s) - q_1P_{an}(0) - q_2P_{wn}(0) = 0, \quad (27a)$$

$$(sq_4 + b_2\lambda_n^2)P_{wn}(s) + sq_3P_{an}(s) - q_3P_{an}(0) - q_4P_{wn}(0) = 0. \quad (27b)$$

利用分离变量法,对于方程(24a)、(24b),当 $t = 0$ 时

$$P_a(z, 0^+) = \sum_{n=0}^{\infty} P_{an}(0) \sin(\lambda_n z) = r_1 P^*, \quad (28)$$

则有

$$P_{an}(0) = \frac{2}{h} \int_0^h r_1 P^* \sin(\lambda_n z) dz. \quad (29)$$

积分可得

$$P_{an}(0) = \frac{2}{h\lambda_n} r_1 P^* [1 - \cos(n\pi)]. \quad (30)$$

同理可得

$$P_{wn}(0) = \frac{2}{h\lambda_n} r_2 P^* [1 - \cos(n\pi)]. \quad (31)$$

结合方程(30)、(31)和(27a)、(27b),得到 Laplace 域上的表达式:

$$P_{an}(s) = \frac{(s + q_1 b_2 \lambda_n^2 / \Delta) P_{an}(0) + (q_2 b_2 \lambda_n^2 / \Delta) P_{wn}(0)}{s^2 + [(q_1 b_2 \lambda_n^2 + q_4 b_1 \lambda_n^2) / \Delta] s + b_1 b_2 \lambda_n^4 / \Delta}, \quad (32a)$$

$$P_{wn}(s) = \frac{(q_3 b_1 \lambda_n^2 / \Delta) P_{an}(0) + (s + q_4 b_1 \lambda_n^2 / \Delta) P_{wn}(0)}{s^2 + [(q_1 b_2 \lambda_n^2 + q_4 b_1 \lambda_n^2) / \Delta] s + b_1 b_2 \lambda_n^4 / \Delta}, \quad (32b)$$

其中

$$\Delta = q_1 q_4 - q_2 q_3. \quad (33)$$

然后进行 Laplace 逆变换,利用部分分式展开法,对于式(32a),假设

$$P_{an}(s) = \frac{D_1}{s + C_1} + \frac{D_2}{s + C_2} = \frac{s(D_1 + D_2) + D_1 C_2 + D_2 C_1}{s^2 + s(C_1 + C_2) + C_1 C_2}, \quad (34)$$

得到以下方程组:

$$\begin{cases} C_1 + C_2 = \frac{q_1 b_2 \lambda_n^2 + q_4 b_1 \lambda_n^2}{\Delta}, \\ C_1 C_2 = \frac{b_1 b_2 \lambda_n^4}{\Delta}, \\ D_1 + D_2 = P_{an}(0), \\ D_1 C_2 + D_2 C_1 = \frac{q_1 b_2 \lambda_n^2}{\Delta} P_{an}(0) + \frac{q_2 b_2 \lambda_n^2}{\Delta} P_{wn}(0). \end{cases} \quad (35)$$

解方程组(35)得

$$C_1 = \frac{\lambda_n^2}{2} \left[\frac{q_1 b_2 + q_4 b_1}{\Delta} + \sqrt{\frac{1}{\Delta^2} (q_1 b_2 + q_4 b_1)^2 - \frac{4b_1 b_2}{\Delta}} \right], \quad (36)$$

$$C_2 = \frac{\lambda_n^2}{2} \left[\frac{q_1 b_2 + q_4 b_1}{\Delta} - \sqrt{\frac{1}{\Delta^2} (q_1 b_2 + q_4 b_1)^2 - \frac{4b_1 b_2}{\Delta}} \right], \quad (37)$$

$$D_1 = \frac{1}{C_1 - C_2} \left[P_{an}(0) \left(C_1 - \frac{q_1 b_2 \lambda_n^2}{\Delta} \right) - \frac{q_2 b_2 \lambda_n^2}{\Delta} P_{wn}(0) \right], \quad (38)$$

$$D_2 = \frac{1}{C_1 - C_2} \left[P_{an}(0) \left(\frac{q_1 b_2 \lambda_n^2}{\Delta} - C_2 \right) + \frac{q_2 b_2 \lambda_n^2}{\Delta} P_{wn}(0) \right]. \quad (39)$$

同理,对于式(32b),令

$$P_{wn}(s) = \frac{D_3}{s + C_1} + \frac{D_4}{s + C_2}, \quad (40)$$

可得

$$D_3 = \frac{1}{C_1 - C_2} \left[P_{wn}(0) \left(C_1 - \frac{q_4 b_1 \lambda_n^2}{\Delta} \right) - \frac{q_3 b_1 \lambda_n^2}{\Delta} P_{an}(0) \right], \quad (41)$$

$$D_4 = \frac{1}{C_1 - C_2} \left[P_{wn}(0) \left(\frac{q_4 b_1 \lambda_n^2}{\Delta} - C_2 \right) + \frac{q_3 b_1 \lambda_n^2}{\Delta} P_{an}(0) \right]. \quad (42)$$

由此,可以推导出 $P_{an}(t)$ 和 $P_{wn}(t)$ 在时间域上的表达式:

$$P_{an}(t) = D_1 e^{-C_1 t} + D_2 e^{-C_2 t}, \quad (43a)$$

$$P_{wn}(t) = D_3 e^{-C_1 t} + D_4 e^{-C_2 t}. \quad (43b)$$

最后,结合式(16)、(24a)、(24b)和(43a)、(43b),求得与时间相关的总沉降:

$$S(t) = - \int_0^h \frac{\partial w}{\partial z} dz = \frac{P^* h}{2G + H_1} + \frac{H_2}{2G + H_1} \sum_{n=0}^{\infty} \frac{D_1 e^{-C_1 t} + D_2 e^{-C_2 t}}{\lambda_n} [1 - \cos(n\pi)] + \frac{H_3}{2G + H_1} \sum_{n=0}^{\infty} \frac{D_3 e^{-C_1 t} + D_4 e^{-C_2 t}}{\lambda_n} [1 - \cos(n\pi)]. \quad (44)$$

5 参数分析

由于黏性耦合系数 R_{11} 和 R_{22} 依赖于相对渗透率,因此,水的饱和度跟线性弹性系数的变化会影响毛细管压力的变化,其线性弹性系数为^[20]

$$a_{11} = \frac{K_s}{N_3} \{ (1 - \varphi) N_1 [K_1 K_2 + K_1 N_2 S_a + K_2 N_2 (1 - S_a)] +$$

$$K_b K_s \varphi [K_1 (1 - S_a) + K_2 S_1 + N_2] \},$$

$$a_{12} = a_{21} = \frac{N_1 K_s K_1 S_a (K_2 + N_2)}{N_3},$$

$$a_{13} = a_{31} = \frac{N_1 K_s \varphi K_2 (1 - S_a) (K_1 + N_2)}{N_3},$$

$$a_{22} = \frac{K_1 S_a^2 \varphi}{N_3} \left[K_s^2 \varphi \left(K_2 + \frac{N_2}{S_a} \right) + \frac{K_2 N_1 N_2}{S_a} (1 - S_a) \right],$$

$$a_{23} = a_{32} = - \frac{K_1 K_2 \varphi S_a (1 - S_a) (N_1 N_2 - \varphi K_s^2)}{N_3},$$

$$a_{33} = \frac{K_2 (1 - S_a)^2 \varphi}{N_3} \left[K_s^2 \varphi \left(K_1 + \frac{N_2}{1 - S_a} \right) + \frac{K_1 N_1 N_2 S_a}{1 - S_a} \right],$$

式中 S_a 是气体饱和度, $S_a + S_w = 1$, K_b 是土体体积模量, K_1 是空气体积模量, K_2 是水的体积模量, K_s 是固相体积模量,参数 N_1, N_2, N_3 定义为

$$N_1 \equiv K_s (1 - \varphi) - K_b, \quad (45)$$

$$N_2 \equiv \frac{dp_c}{dS_a} S_a(1 - S_a), \quad (46)$$

$$N_3 \equiv N_1[K_1N_2S_a + K_1K_2 + K_2N_2(1 - S_a)] + K_s^2[K_1(1 - S_a) + N_2 + K_2S_a], \quad (47)$$

其中 p_c 表示毛细管压力,利用 VG 模型的关系式:

$$S_e = [1 + (\zeta h_c)^n]^{-m}, \quad (48)$$

式中 n, m, ζ 均是拟合参数, $m = 1 - 1/n$.

基质势与毛细管压力关系为^[20]

$$h_c = \frac{P_c}{\rho_w g}, \quad (49)$$

式中 ρ_w 是水的密度, g 是重力加速度,结合式(48)、(49)和(46),得 N_2 的表达式:

$$N_2 = \frac{\rho_w g}{mn\zeta} \frac{S_a S_w}{S_s - S_r} \left[\left(\frac{1 - S_r}{S_s - S_r} - \frac{S_a}{S_s - S_r} \right)^{-1/m} - 1 \right]^{1/n-1} \left(\frac{1 - S_r}{S_s - S_r} - \frac{S_a}{S_s - S_r} \right)^{-(1/m)+1}. \quad (50)$$

考虑空气和水的相对渗透率^[26]:

$$k_{ra} = (1 - S_e)^\eta [1 - (S_e)^{1/m}]^{2m}, \quad (51a)$$

$$k_{rw} = S_e^\eta \{1 - [1 - (S_e)^{1/m}]^m\}^2, \quad (51b)$$

式中 η 是孔隙连通性参数,表 1 列出了文献[24-25]所得黏土的计算参数.

当含水量很高时,空气变得不连续,在土体孔隙中以气泡形式分布;当含水量很低时,孔隙水紧紧接触于固体颗粒的周围,在固结过程中不能形成连续相,因此土体中的水和空气只有在水的饱和度介于高低界限之间时,才能用连续介质力学公式.由于黏土的饱和度一般在 0.6~1 之间,所以在求解方程(24a)、(24b)和(44)时,饱和度 S_w 分别取 0.7, 0.8, 0.9.

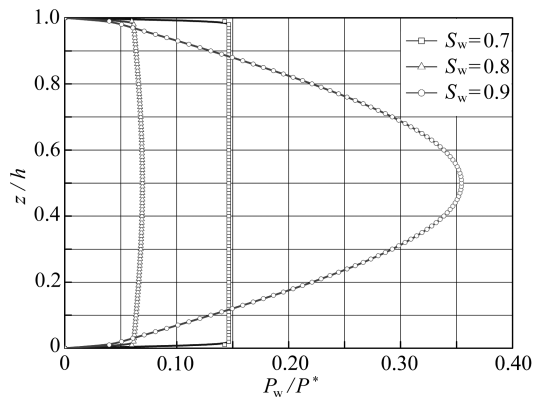
表 1 黏土的基本物理力学参数^[24-25]

Table 1 Basic physical and mechanical parameters of the clay^[24-25]

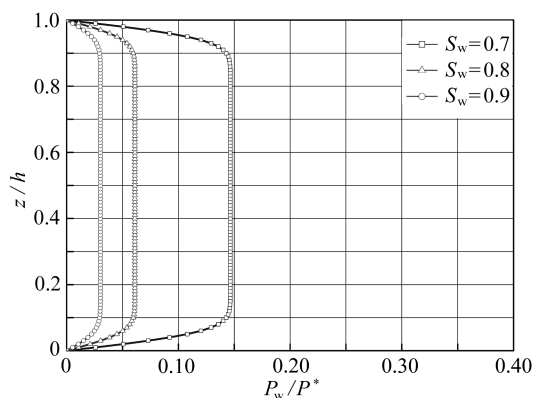
parameter	value	parameter	value
porosity φ	0.475	fitting parameter $\zeta / (\text{m}^{-1})$	1.168
fitting parameter n	1.165	intrinsic permeability k_s / m^2	1.699×10^{-14}
shear modulus G / Pa	2.4×10^6	bulk modulus of air K_1 / MPa	0.145
bulk modulus of water K_2 / GPa	2.25	bulk modulus of solid K_s / GPa	35
bulk modulus K_b / Pa	4.5×10^6	material density of water $\rho_w / (\text{kg} \cdot \text{m}^{-3})$	997
viscosity of air $\eta_1 / (\text{N} \cdot \text{s} \cdot \text{m}^{-2})$	1.8×10^{-5}	viscosity of water $\eta_2 / (\text{N} \cdot \text{s} \cdot \text{m}^{-2})$	0.001
gravity acceleration $g / (\text{m} \cdot \text{s}^{-2})$	9.8	fitting parameter η	1.165

定义无量纲时间 $T = (c_v/h^2)t$, 当饱和度 S_w 取 0.7, 0.8, 0.9 和土体厚度 $h = 1 \text{ m}$ 时,固结系数分别为 $c_v = 2.36 \times 10^{-9}$, 7.7×10^{-9} , $2.9 \times 10^{-8} \text{ m}^2/\text{s}$. 定义无量纲深度为 z/h , 无量纲孔隙水压力 P_w/P^* , 无量纲有效应力 σ'/P^* , 分别取无量纲时间 $T = 10^{-5}$, 10^{-3} , 10^{-1} .

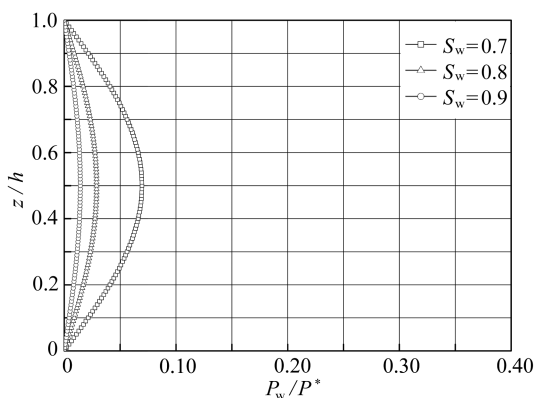
从图 2 的曲线变化可以反映出,越接近排水边界孔隙水压力越小,越靠近土层中心孔隙水压力越大,在土层中间的位置孔隙水压力达到最大值,并且最大值随着时间的发展而逐渐减小.相应地,孔隙水压力随着固结的完成逐渐减小,但其减小程度随时间的变化在深度方向上不一致,从图 2(b)和图 2(c)可以看出,在孔隙水压力稳定后 ($T > 10^{-3}$), 接近边界处的孔隙水压力消散得快,越往中间孔隙水压力消散得越慢.



(a) $T = 10^{-5}$



(b) $T = 10^{-3}$

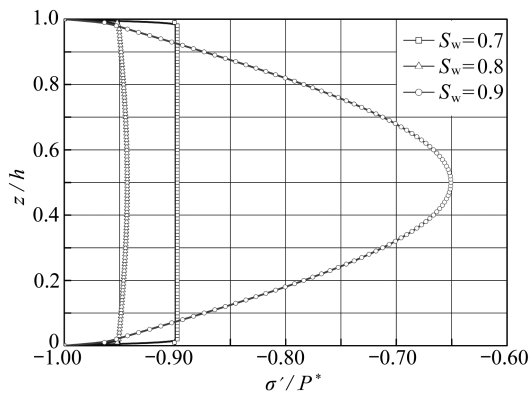


(c) $T = 10^{-1}$

图 2 不同饱和度下无量纲孔隙水压力随无量纲深度的变化

Fig. 2 Variation of the dimensionless pore water pressure with the dimensionless depth for different saturations

当孔隙水压力逐渐趋于稳定后,其在较高的初始饱和度时消散得更快,这是因为湿润的土壤具有较高的水力传导性,所以水通过渗透性的边界流动得更快.在很短的时间 $T = 10^{-5}$ 时,图 2(a) 显示出了最高的孔隙水压力,此时的有效应力最低,有效应力的变化趋势可由图 3 看出.



(a) $T = 10^{-5}$

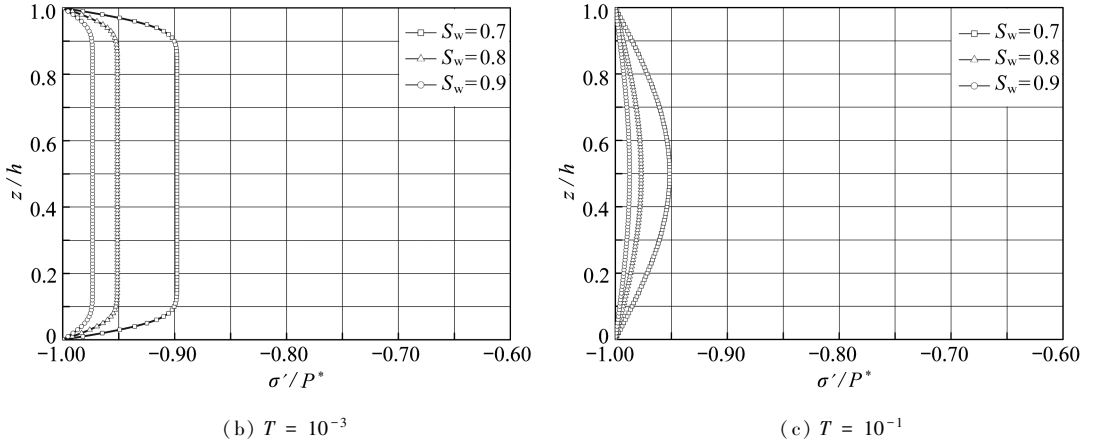


图3 不同饱和度下无量纲有效应力随无量纲深度的变化

Fig. 3 Variation of the dimensionless effective stress with the dimensionless depth for different saturations

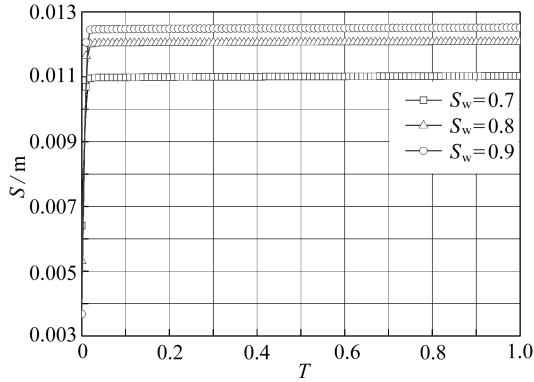


图4 饱和度对双面排水条件下非饱和黏土总沉降的影响

Fig. 4 Effect of the saturation on the total settlement of the unsaturated clay under double-side drainage

当受到外部荷载时,孔隙水压力最高时的饱和度为 $S_w = 0.9$,其次是 $S_w = 0.7$ 和 $S_w = 0.8$ 。因此,虽然湿土的孔隙水压力可以消散得更快,但它具有最高的孔隙水压力和最低的有效应力。

最终,由于孔隙压力的减小而产生更大的有效应力,也增加了相应的固结。从图4可以看出,在受压瞬间 ($T = 0$ 时) 初始压力完全由流体来承受,因为饱和度0.7的黏土含空气较多所以导致瞬间沉降量较大。此外,黏土因初始饱和度不同造成达到稳定后的总沉降量不同,饱和度为0.9时最大,0.8时次之,而0.7时最小。

6 结 论

1) 结合多孔介质弹性理论,通过理论分析得到了在定荷载和双面排水条件下孔隙气压力、孔隙水压力和总沉降的解析解。

2) 在固结过程中,孔隙水压力的消散速度随初始饱和度的增大而增大;土体在受压瞬间,初始饱和度越小瞬间沉降量越大,而与之相反,总沉降量随着土体饱和度的增加而增加。

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Analytical Solution for 1D Consolidation of Unsaturated Soil Under Constant Load and Double-Side Drainage

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Abstract: Based on the elastic theory for porous media and the principle of the effective stress expressed with intergranular suction stresses, the coupled partial differential governing equation for unsaturated soil consolidation was established. For the 1D consolidation problem, the analytical solution for the consolidation of unsaturated soil under constant load and double-side drainage was obtained through the Laplace integral transform. The effects of soil saturation on the excessive pore water pressure, the effective stress and the soil settlement were analyzed with numerical examples. The results show that, the higher the initial saturation of the soil is, the faster the pore water pressure will dissipate and the faster the effective stress will increase.

Key words: unsaturated soil; analytical solution; suction stress characteristic curve; pore water pressure

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