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一类分数阶修正的不稳定 Schrödinger 方程的新精确解*

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摘要： 研究了分数阶修正的不稳定 Schrödinger 方程 (FMUSE), 该方程描述了光脉冲在非均匀光纤系统中传播的色散、非线性、增益或吸收变化的普适问题. 首先适当地利用广义分数波变换将 FMUSE 转化为常微分方程, 分离实部和虚部并分别令为零, 得到了色散关系. 再利用修改的 (G'/G)-展开法, 求得了一系列带参数的新精确解析解, 其中包括三角函数解、双曲函数解和有理函数解, 并给出了保证解存在的约束条件. 最后当参数取特殊值时得到暗孤波和周期波解.

关　键　词： 修改的 (G'/G)-展开法；　分数阶修正的不稳定 Schrödinger 方程；　精确解

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New Exact Solutions to a Class of Fractional-Order Modified Unstable Schrödinger Equations

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Abstract: The fractional-order modified unstable Schrödinger equation (FMUSE) was studied, which describes the dispersion, nonlinearity, gain or absorption variation of optical pulses propagating in nonuniform fiber systems. First, the generalized fractional wave transform was appropriately used to convert the FMUSE into an ordinary differential equation, and the real and imaginary parts were separated and set as zero respectively, and the dispersion relation was obtained. By means of the modified (G'/G)-expansion method, a series of new exact analytical solutions with parameters were obtained, including trigonometric solutions, hyperbolic solutions and rational solutions, and the constraints ensuring the existence of solutions were given. Finally, the solutions of the dark solitary wave and the periodic wave were obtained with the parameters of special values.

Key words: modified (G'/G)-expansion method; fractional-order modified unstable Schrödinger equation; exact solution

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引　言

分数阶修正的不稳定 Schrödinger 方程 (FMUSE) 是时间-空间均为分数阶导数, 且时间-空间变化是相关联的非线性 Schrödinger 方程. 该方程描述了光脉冲在非均匀光纤系统中的传播, 也是等离子体在非均匀介质中不稳定运动的重要数学模型之一. 所以, 研究它的精确解特别是孤立波解在许多科学及工程领域, 如物理、通讯、核能及医疗技术中有重要应用背景[1].

孤子结构在许多科学领域和工程领域中都有出现, 其中光纤中的孤子是基于多种非线性效应和色散、耗散的共同作用形成. 事实上, 在实际光纤中存在群速色散 (GVD) 和各相之间的精确平衡调制 (SPM). 非线性的增益/损失、时间-空间变化是具有高度关联性而形成的孤波结构[2-4]. FMUSE 是描述这一非线性现象最有意义的数学模型之一, 研究其孤子结构很有意义.

现考虑如下 FMUSE[5]:

$$\mathrm{i}\mathrm{D}_t^{\alpha}q+\mathrm{D}_x^{2\alpha}q+2\lambda|q|^2q-\beta\mathrm{D}_{tx}^{2\alpha}q=0,\qquad 0<\alpha<1,\tag{1}$$

研究孤子在光纤中传播的控制方程是众所周知的. 该 FMUSE 包括线性演化项 GVD 和非线性项. 然而, 最近证明了只有 GVD 的 NLSE 是一个不适定问题. 因此, 本文提出研究 FMUSE 模型将时空色散 (spatial-temporal dispersion, STD) 考虑在内, 使问题具有较好的拟合性.

对方程 (1) 有 $q=q(x,\ t)$, $q:\Re^2\to\mathbb{C}$, $\dfrac{\partial^\alpha q}{\partial t}=\mathrm{D}_t^{\alpha}q$, $\dfrac{\partial^{2\alpha} q}{\partial x}=\mathrm{D}_x^{2\partial}q$, $\dfrac{\partial^{2\alpha} q}{\partial x}=\mathrm{D}_x^{2\partial}q$, $\dfrac{\partial^{2\alpha} q}{\partial x\partial t}=\mathrm{D}_{tx}^{2\partial}q$. 当 $\alpha=1$ 时, 该方程变成了光纤[5] 中的修正不稳定 Schrödinger 方程. 式 (1) 的第一项为演化项, 这里 $q(x,\ t)$ 是电场的复包络线, t 为迟滞时间, 下标为偏导数, β 为 STD 系数. 该方程定义了稳定介质和不稳定介质在一定程度上的扰动周期, 也得到了拐点波列的不确定性, 并添加了式 (1) 的最后一项, 克服了不适定性[6-8].

近年来, 相关学者用不同的方法建立了分数阶非线性偏微分方程的各种解. 如改进的扩展辅助方程映射方法[9]、有理指数函数法[10]、简单方程法[11]、$G'/(G+G')$-展开法[12]、Adomian 分解方法[13] 等. FMUSE 的解在光学上得到了应用, 例如描述调制波列的某些不稳定性[5]. 本文运用 Shehata 修改的 (G'/G)-展开法并借助文献 [14-15] 获得由 Jumarie[16] 修正的 Riemann-Liouville (RL) 导数意义上的 FMUSE 有用和更一般的解. α 阶修正 RL 导数定义为

$$\mathrm{D}_t^{\alpha}f(t)=\begin{cases}\dfrac{1}{T(1-\beta)}\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t(t-\xi)^{-\alpha}(f(\xi)-f(0))\mathrm{d}\xi, & 0<\alpha<1,\\ (f^n(t))^{(\alpha-n)}, & n\leqslant\alpha<n+1,\ \ n\geqslant 1.\end{cases}\tag{2}$$

1 解 的 构 建

对方程 (1) 做分数阶复变换:

$$q(x,\ t)=u(\eta)\mathrm{e}^{\mathrm{i}\theta},\ \ \eta=K\left(\frac{x^\alpha}{\alpha}-2v\frac{t^\alpha}{\alpha}\right),\ \ \theta=c\frac{x^\alpha}{\alpha}+s\frac{t^\alpha}{\alpha},\tag{3}$$

其中 K 是孤子的宽度, v 是群速度, θ 表示相位分量, c 是相速度, s 表示频率.

将式 (3) 代入式 (1), 分离实部虚部, 并分别令其为 0, 得

$$2\lambda u^3+(\beta cs-c^2-s)u+K^2(1+2v\beta)u''=0\tag{4}$$

及色散关系:

$$v=\frac{K(\beta s-2\mu)}{\beta c-1}.\tag{5}$$

由齐次平衡原理, 平衡方程 (4) 的最高阶导数项与非线性项 ($3n=n+2, n=1$), 假设方程 (4) 的解为

$$u(\xi)=a_{-1}(G'/G)^{-1}+a_0+a_1(G'/G),\tag{6}$$

其中, a_{-1}, a_0, a_1 均为待定常数, G 满足如下的常微分方程:

$$G''+\lambda G'+\mu G=0.\tag{7}$$

将式 (6) 代入方程 (4), 合并$(G'/G)^i(i=-3,-2,\cdots,2,3)$相同幂次项, 并令各幂次系数为 0, 得到关于$a_{-1}$, a_0,a_1和s的方程组:

$$\begin{cases}(G'/G)^{-3}: K^2(1+2\nu\beta)\mu^2a_{-1}+\lambda a_{-1}^3=0,\\ (G'/G)^{-2}: K^2(1+2\nu\beta)\lambda\mu a_{-1}+2\lambda a_{-1}^2a_0=0,\\ (G'/G)^{-1}: 2K^2(1+2\nu\beta)\mu a_{-1}+K^2(1+2\nu\beta)\lambda^2a_{-1}+6\lambda(a_{-1}a_0^2+a_{-1}^2a_1)+(\beta cs-c^2-s)a_{-1}=0,\\ (G'/G)^0: 12\lambda a_{-1}a_0a_1+2\lambda a_0^3+(\beta cs-c^2-s)a_0+K^2(1+2\nu\beta)\lambda\mu a_1=0,\\ (G'/G)^1: 2K^2(1+2\nu\beta)\mu a_1+K^2(1+2\nu\beta)\lambda^2a_1+6\lambda(a_{-1}a_1^2+a_0^2a_1)+(\beta cs-c^2-s)a_1=0,\\ (G'/G)^2: K^2(1+2\nu\beta)\lambda a_1+2\lambda a_0a_1^2=0,\\ (G'/G)^3: K^2(1+2\nu\beta)a_1+\lambda a_1^3=0.\end{cases}\tag{8}$$

解方程组 (8) 得a_{-1}, a_0, a_1和s的结果为

$$\begin{cases}a_{-1}=0,\ a_0=\pm\sqrt{\dfrac{c^2+s-\beta cs}{2\lambda}},\ a_1=0\,;\\ a_{-1}=0,\ a_0=\pm\sqrt{\dfrac{2K^2(1+2\nu\beta)\mu+K^2(1+2\nu\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}},\ a_1=\pm\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}},\\ \pm K^2(1+2\nu\beta)\lambda\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}=(c^2+s-\beta cs)a_0-2\lambda a_0^3;\\ a_{-1}=\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}},\ a_0=\pm\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}},\ a_1=0,\\ c^2+s-\beta cs=2K^2(1+2\nu\beta)\mu-\dfrac{1}{2}K^2(1+2\nu\beta)\lambda^2;\\ a_{-1}=\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}},\ a_0=\pm\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}},\ a_1=\pm\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}},\\ c^2+s-\beta cs=-4K^2(1+2\nu\beta)\mu-\dfrac{1}{2}K^2(1+2\nu\beta)\lambda^2;\\ a_{-1}=\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}},\ a_0=\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}},\ a_1=\pm\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}},\\ c^2+s-\beta cs=8K^2(1+2\nu\beta)\mu-\dfrac{1}{2}K^2(1+2\nu\beta)\lambda^2.\end{cases}\tag{9}$$

现令$\Delta=\lambda^2-4\mu$.

将结果 (9) 分别代入式 (6),得到式 (1) 五种情形的孤波解.

情形 1

$$\begin{cases}u_1=\pm\sqrt{\dfrac{c^2+s-\beta cs}{2\lambda}},\\ q_1=\pm\sqrt{\dfrac{c^2+s-\beta cs}{2\lambda}}\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right],\end{cases}\tag{10}$$

其中s是任意实数.

情形 2

① 当$\Delta>0$时, 方程 (1) 有如下的双曲函数形式解:

$$
\begin{cases}
u_{2.1}=\pm\sqrt{\dfrac{2K^2(1+2v\beta)\mu+K^2(1+2v\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}}\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\pm \\
\quad \dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\left(\dfrac{A_1\sinh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)+A_2\cosh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)}{A_1\cosh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)+A_2\sinh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)}\right), \\
q_{2.1}=\left[\pm\sqrt{\dfrac{2K^2(1+2v\beta)\mu+K^2(1+2v\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}}\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\pm\right. \\
\quad \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\left(\dfrac{A_1\sinh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)+A_2\cosh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)}{A_1\cosh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)+A_2\sinh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)}\right)\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\
\pm K^2(1+2v\beta)\lambda\mu\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}=(c^2+s-\beta cs)a_0-2\lambda a_0^3,
\end{cases}
\tag{11}
$$

其中$\eta=K\left(\dfrac{x^\alpha}{\alpha}-2v\dfrac{t^\alpha}{\alpha}\right)$, A_1, A_2是常数.

特别地, 若取式 (11) 中的$A_1\neq 0$, $A_2=0$, 得到方程 (1) 的暗孤波解:

$$
\begin{cases}
q_{2.1.1}=\left[\pm\sqrt{\dfrac{2K^2(1+2v\beta)\mu+K^2(1+2v\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}}\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\pm\right. \\
\quad \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\tanh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\
\pm K^2(1+2v\beta)\lambda\mu\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}=(c^2+s-\beta cs)a_0-2\lambda {a_0}^3.
\end{cases}
\tag{12}
$$

当参数$\lambda=3$, $\mu=2$, $K=1$, $v=1$, $\beta=-1$, $c=1$, $s=-15$, $\alpha=1/2$时, $q_{2.1.1}$的图像如图 1 所示.

② 当$\varDelta<0$时, 方程 (1) 有如下的三角函数形式解:

$$
\begin{cases}
u_{2.2}=\pm\sqrt{\dfrac{2K^2(1+2v\beta)\mu+K^2(1+2v\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}}\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\mp \\
\quad \dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\left(\dfrac{-A_1\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}{A_1\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}\right), \\
q_{2.2}=\left[\pm\sqrt{\dfrac{2K^2(1+2v\beta)\mu+K^2(1+2v\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}}\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\mp\right. \\
\quad \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\left(\dfrac{-A_1\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}{A_1\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}\right)\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\
\pm K^2(1+2v\beta)\lambda\mu\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}=(c^2+s-\beta cs)a_0-2\lambda a_0^3,
\end{cases}
\tag{13}
$$

其中$\eta=K\left(\dfrac{x^\alpha}{\alpha}-2v\dfrac{t^\alpha}{\alpha}\right)$, A_1, A_2是常数.

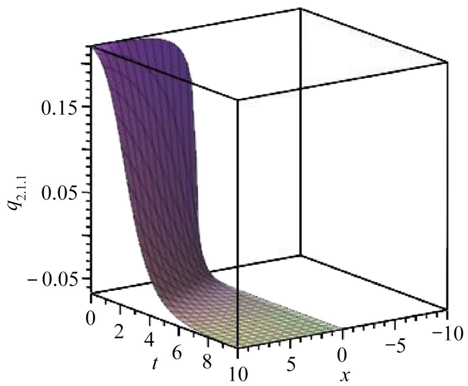

图 1　$\lambda=3, \mu=2, K=1, v=1, \beta=-1, c=1, s=-15, \alpha=1/2$时, $q_{2.1.1}$的图像

Fig. 1　The graphic corresponding to $q_{2.1.1}$ ($\lambda=3, \mu=2, K=1, v=1, \beta=-1, c=1, s=-15, \alpha=1/2$)

又若取式 (13) 中的$A_1\neq 0$, $A_2=0$或$A_1=0$, $A_2\neq 0$, 分别得到方程 (1) 的周期孤波解:

$$\begin{cases} q_{2.2.1}=\left[\pm\sqrt{\dfrac{2K^2(1+2v\beta)\mu+K^2(1+2v\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}}\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\mp\right. \\ \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\tan\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\ \pm K^2(1+2v\beta)\lambda\mu\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}=(c^2+s-\beta cs)a_0-2\lambda a_0^3, \end{cases} \tag{14}$$

$$\begin{cases} q_{2.2.2}=\left[\pm\sqrt{\dfrac{2K^2(1+2v\beta)\mu+K^2(1+2v\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}}\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\mp\right. \\ \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\cot\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\ \pm K^2(1+2v\beta)\lambda\mu\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}=(c^2+s-\beta cs)a_0-2\lambda a_0^3. \end{cases} \tag{15}$$

当参数$\lambda=2, \mu=2, K=1, v=1, \beta=-1, c=1, s=-5, \alpha=1/2$时, $q_{2.2.1}$的图像如图 2 所示. 当参数$\lambda=2, \mu=2, K=1, v=1, \beta=-1, c=1, s=-5, \alpha=1/2$时, $q_{2.2.2}$的图像如图 3 所示.

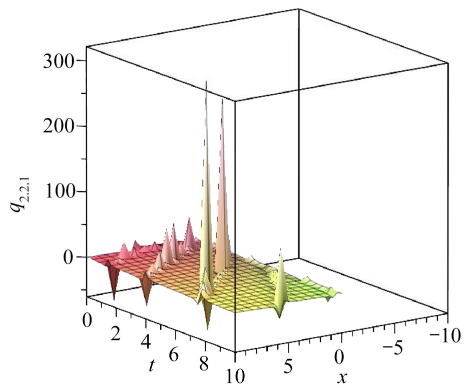

图 2　$\lambda=2, \mu=2, K=1, v=1, \beta=-1, c=1, s=-5, \alpha=1/2$时, $q_{2.2.1}$的图像

Fig. 2　The graphic corresponding to $q_{2.2.1}$ ($\lambda=2, \mu=2, K=1, v=1, \beta=-1, c=1, s=-5, \alpha=1/2$)

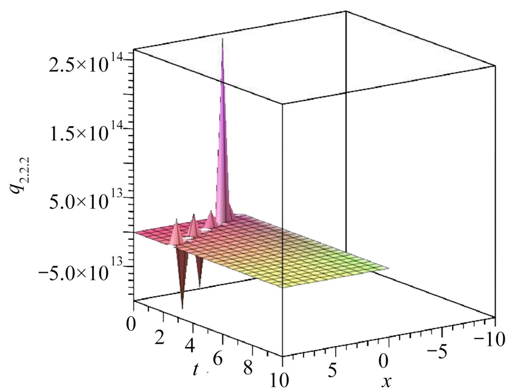

图 3　$\lambda=2, \mu=2, K=1, v=1, \beta=-1, c=1, s=-5, \alpha=1/2$时, $q_{2.2.2}$的图像

Fig. 3　The graphic corresponding to $q_{2.2.2}$ ($\lambda=2, \mu=2, K=1, v=1, \beta=-1, c=1, s=-5, \alpha=1/2$)

③ 当$\varDelta=0$时, 方程 (1) 有如下的有理函数形式解:

$$\begin{cases} u_{2.3}=\pm\sqrt{\dfrac{2K^2(1+2\nu\beta)\mu+K^2(1+2\nu\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}}\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\pm \\ \quad \dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\dfrac{A_1}{A_1\varsigma+A_2}, \\ q_{2.3}=\left[\pm\sqrt{\dfrac{2K^2(1+2\nu\beta)\mu+K^2(1+2\nu\beta)\lambda^2+\beta cs-c^2-s}{6\lambda}}\mp\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\pm\right. \\ \quad \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\dfrac{A_1}{A_1\varsigma+A_2}\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\ \pm K^2(1+2\nu\beta)\lambda\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}=(c^2+s-\beta cs)a_0-2\lambda a_0^3. \end{cases} \tag{16}$$

情形 3

① 当$\varDelta>0$时, 方程 (1) 有如下的双曲函数形式解:

$$\begin{cases} u_{3.1}=\pm\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{\sqrt{\varDelta}}{2}\dfrac{A_1\sinh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)+A_2\cosh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)}{A_1\cosh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)+A_2\sinh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)}\right)^{-1}, \\ q_{3.1}=\left[\pm\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{\sqrt{\varDelta}}{2}\dfrac{A_1\sinh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)+A_2\cosh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)}{A_1\cosh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)+A_2\sinh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)}\right)^{-1}\right]\times \\ \quad \exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\ c^2+s-\beta cs=2K^2(1+2\nu\beta)\mu-\dfrac{1}{2}K^2(1+2\nu\beta)\lambda^2, \end{cases} \tag{17}$$

其中$\eta=K\left(\dfrac{x^\alpha}{\alpha}-2\nu\dfrac{t^\alpha}{\alpha}\right)$, A_1, A_2是常数.

特别地, 若取式 (17) 中的$A_1\neq0$, $A_2=0$, 可得到方程 (1) 的暗孤波解:

$$\begin{cases} q_{3.1.1}=\left[\pm\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{\sqrt{\varDelta}}{2}\tanh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)\right)^{-1}\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\ \pm K^2(1+2\nu\beta)\lambda\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}=(c^2+s-\beta cs)a_0-2\lambda {a_0}^3. \end{cases}$$

② 当$\varDelta<0$时, 方程 (1) 有如下的三角函数形式解:

$$\begin{cases} u_{3.2}=\pm\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{\sqrt{-\varDelta}}{2}\dfrac{-A_1\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}{A_1\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}\right)^{-1}, \\ q_{3.2}=\left[\pm\dfrac{\lambda}{2}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{\sqrt{-\varDelta}}{2}\dfrac{-A_1\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}{A_1\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}\right)^{-1}\right]\times \\ \quad \exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\ c^2+s-\beta cs=2K^2(1+2\nu\beta)\mu-\dfrac{1}{2}K^2(1+2\nu\beta)\lambda^2, \end{cases} \tag{18}$$

其中$\eta = K\left(\frac{x^\alpha}{\alpha} - 2v\frac{t^\alpha}{\alpha}\right)$, A_1, A_2是常数.

又若取式 (18) 中的$A_1 \neq 0$, $A_2 = 0$或$A_1 = 0$, $A_2 \neq 0$, 可分别得到方程 (1) 的周期孤波解:

$$\begin{cases} q_{3.2.1} = \left[\pm\frac{\lambda}{2}\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}} \pm \mu\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}}\left(-\frac{\lambda}{2} - \frac{\sqrt{-\Delta}}{2}\tan\left(\frac{\sqrt{-\Delta}}{2}\eta\right)\right)^{-1}\right]\exp\left[\mathrm{i}\left(c\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha}\right)\right], \\ c^2 + s - \beta cs = 2K^2(1+2v\beta)\mu - \frac{1}{2}K^2(1+2v\beta)\lambda^2, \end{cases} \tag{19}$$

$$\begin{cases} q_{3.2.2} = \left[\pm\frac{\lambda}{2}\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}} \pm \mu\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}}\left(-\frac{\lambda}{2} - \frac{\sqrt{-\Delta}}{2}\cot\left(\frac{\sqrt{-\Delta}}{2}\eta\right)\right)^{-1}\right]\exp\left[\mathrm{i}\left(c\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha}\right)\right], \\ c^2 + s - \beta cs = 2K^2(1+2v\beta)\mu - \frac{1}{2}K^2(1+2v\beta)\lambda^2. \end{cases} \tag{20}$$

③ 当$\Delta = 0$时, 方程 (1) 有如下的有理函数形式解:

$$\begin{cases} u_{3.3} = \pm\frac{\lambda}{2}\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}} \pm \mu\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}}\left(\frac{A_1}{A_1\zeta + A_2}\right), \\ q_{3.3} = \left[\pm\frac{\lambda}{2}\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}} \pm \mu\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}}\left(\frac{A_1}{A_1\zeta + A_2}\right)\right]\exp\left[\mathrm{i}\left(c\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha}\right)\right], \\ c^2 + s - \beta cs = 2K^2(1+2v\beta)\mu - \frac{1}{2}K^2(1+2v\beta)\lambda^2. \end{cases} \tag{21}$$

情形 4

① 当$\Delta > 0$时, 方程 (1) 有如下的双曲函数形式解:

$$\begin{cases} u_{4.1} = \pm\mu\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}}\left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\frac{A_1\sinh\left(\frac{\sqrt{\Delta}}{2}\eta\right) + A_2\cosh\left(\frac{\sqrt{\Delta}}{2}\eta\right)}{A_1\cosh\left(\frac{\sqrt{\Delta}}{2}\eta\right) + A_2\sinh\left(\frac{\sqrt{\Delta}}{2}\eta\right)}\right)^{-1} \pm \\ \quad \frac{\Delta}{4}\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}}\frac{A_1\sinh\left(\frac{\sqrt{\Delta}}{2}\eta\right) + A_2\cosh\left(\frac{\sqrt{\Delta}}{2}\eta\right)}{A_1\cosh\left(\frac{\sqrt{\Delta}}{2}\eta\right) + A_2\sinh\left(\frac{\sqrt{\Delta}}{2}\eta\right)}, \\ q_{4.1} = \left[\pm\mu\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}}\left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\frac{A_1\sinh\left(\frac{\sqrt{\Delta}}{2}\eta\right) + A_2\cosh\left(\frac{\sqrt{\Delta}}{2}\eta\right)}{A_1\cosh\left(\frac{\sqrt{\Delta}}{2}\eta\right) + A_2\sinh\left(\frac{\sqrt{\Delta}}{2}\eta\right)}\right)^{-1} \pm \right. \\ \quad \left. \frac{\Delta}{4}\sqrt{-\frac{K^2(1+2v\beta)}{\lambda}}\frac{A_1\sinh\left(\frac{\sqrt{\Delta}}{2}\eta\right) + A_2\cosh\left(\frac{\sqrt{\Delta}}{2}\eta\right)}{A_1\cosh\left(\frac{\sqrt{\Delta}}{2}\eta\right) + A_2\sinh\left(\frac{\sqrt{\Delta}}{2}\eta\right)}\right]\exp\left[\mathrm{i}\left(c\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha}\right)\right], \\ c^2 + s - \beta cs = -4K^2(1+2v\beta)\mu - \frac{1}{2}K^2(1+2v\beta)\lambda^2, \end{cases} \tag{22}$$

其中$\eta = K\left(\frac{x^\alpha}{\alpha} - 2v\frac{t^\alpha}{\alpha}\right)$, A_1, A_2是常数.

特别地, 若取式 (22) 中的$A_1 \neq 0$, $A_2 = 0$, 可得到方程 (1) 的暗孤波解:

$$\begin{cases} q_{4.1.1}=\left[\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{\sqrt{\varDelta}}{2}\tanh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)\right)^{-1}\pm\right.\\ \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\tanh\left(\dfrac{\sqrt{\varDelta}}{2}\eta\right)\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right],\\ c^2+s-\beta cs=-4K^2(1+2\nu\beta)\mu-\dfrac{1}{2}K^2(1+2\nu\beta)\lambda^2. \end{cases} \tag{23}$$

② 当$\varDelta<0$时, 方程 (1) 有如下的三角函数形式解:

$$\begin{cases} u_{4.2}=\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{\sqrt{-\varDelta}}{2}\dfrac{-A_1\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}{A_1\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}\right)^{-1}\mp\\ \dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\dfrac{-A_1\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}{A_1\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)},\\ q_{4.2}=\left[\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{\sqrt{-\varDelta}}{2}\dfrac{-A_1\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}{A_1\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}\right)^{-1}\mp\right.\\ \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\dfrac{-A_1\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}{A_1\cos\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)+A_2\sin\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)}\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right],\\ c^2+s-\beta cs=-4K^2(1+2\nu\beta)\mu-\dfrac{1}{2}K^2(1+2\nu\beta)\lambda^2. \end{cases} \tag{24}$$

又若取式 (24) 中的$A_1\neq0,\ A_2=0$或$A_1=0,\ A_2\neq0$, 可分别得到方程 (1) 的周期孤波解:

$$\begin{cases} q_{4.2.1}=\left[\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}-\dfrac{\sqrt{-\varDelta}}{2}\tan\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)\right)^{-1}\pm\right.\\ \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\tan\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right],\\ c^2+s-\beta cs=-4K^2(1+2\nu\beta)\mu-\dfrac{1}{2}K^2(1+2\nu\beta)\lambda^2, \end{cases} \tag{25}$$

$$\begin{cases} q_{4.2.1}=\left[\pm\mu\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}-\dfrac{\sqrt{-\varDelta}}{2}\cot\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)\right)^{-1}\pm\right.\\ \left.\dfrac{\varDelta}{4}\sqrt{-\dfrac{K^2(1+2\nu\beta)}{\lambda}}\cot\left(\dfrac{\sqrt{-\varDelta}}{2}\eta\right)\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right],\\ c^2+s-\beta cs=-4K^2(1+2\nu\beta)\mu-\dfrac{1}{2}K^2(1+2\nu\beta)\lambda^2. \end{cases} \tag{26}$$

当参数$\lambda=2,\ \mu=2,\ K=1,\ \nu=1,\ \beta=-1,\ c=1,\ \alpha=1/2$时, $q_{4.2.1}$解的图像如图 4 所示.

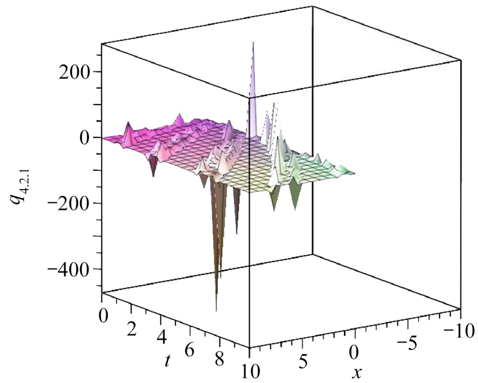

图 4　$\lambda=2, \mu=2, K=1, v=1, \beta=-1, c=1, \alpha=1/2$时, $q_{4.2.1}$的图像

Fig. 4　The graphic corresponding to $q_{4.2.1}(\lambda=2, \mu=2, K=1, v=1, \beta=-1, c=1, \alpha=1/2)$

③ 当$\Delta=0$时, 方程 (1) 有如下的有理函数形式解:

$$\begin{cases} u_{4.3}=\pm\mu\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{A_1}{A_1\zeta+A_2}\right)^{-1}\pm\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\left(\dfrac{A_1}{A_1\zeta+A_2}\right), \\ q_{4.3}=\left[\pm\mu\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\left(-\dfrac{\lambda}{2}+\dfrac{A_1}{A_1\zeta+A_2}\right)^{-1}\pm\sqrt{-\dfrac{K^2(1+2v\beta)}{\lambda}}\left(\dfrac{A_1}{A_1\zeta+A_2}\right)\right]\exp\left[\mathrm{i}\left(c\dfrac{x^\alpha}{\alpha}+s\dfrac{t^\alpha}{\alpha}\right)\right], \\ c^2+s-\beta cs=-4K^2(1+2v\beta)\mu-\dfrac{1}{2}K^2(1+2v\beta)\lambda^2. \end{cases} \tag{27}$$

情形 5　可仿照情形 4 给出.

2　结　　论

本文研究了 FMUSE, 先对方程进行分数阶复变换转化为常微分方程, 再分离实部和虚部并分别令其为零, 得到了色散关系. 对 Riccati 方程, 利用修改的(G'/G)-展开法, 构建了一系列带参数的精确行波通解, 其中包括有理函数解、三角函数解和双曲函数解. 将所得结果与文献 [1] 中的解进行比较, $q_{2.2}$, $q_{2.2.1}$, $q_{2.2.2}$, $q_{3.2}$, $q_{3.2.1}$, $q_{3.2.2}$, $q_{4.2}$, $q_{4.2.1}$, $q_{4.2.2}$是本文求得的新解. 通过绘图软件, 给出典型参数下代表性孤波解图像, 这有助于直观了解孤波传输图案和应用上用不同参数作物理控制.

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