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带有外部输入项的时间周期 SIR 传染病模型的周期行波解*

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摘要：　研究了一类带有外部输入项的时间周期 SIR 传染病模型周期行波解的存在性和不存在性. 首先, 通过构造辅助系统适当的上下解并定义闭凸锥, 将周期行波解的存在性转化为定义在这个闭凸锥上的非单调算子的不动点问题, 利用 Schauder 不动点定理建立辅助系统周期解的存在性, 并利用 Arzela-Ascoli 定理证明了原模型周期行波解的存在性. 其次, 借助分析技术得到了周期行波解的不存在性.

关　键　词：　周期行波解；　存在性；　辅助系统；　不动点定理

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Periodic Traveling Wave Solutions of Time-Periodic SIR Epidemic Models With External Supplies

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Abstract: The existence and non-existence of periodic traveling wave solutions of a class of time-periodic SIR epidemic models with external supplies were considered. Firstly, the appropriate upper and lower solutions of the auxiliary system were built and a closed convex cone was defined, the existence of periodic traveling waves was transformed into a fixed-point problem of the non-monotonic operator defined on the closed convex cone. The existence of periodic solutions of the auxiliary system was established under the Schauder fixed-point theorem, and the Arzela-Ascoli theorem was used to prove the existence of periodic traveling waves for the original model. Secondly, the non-existence of periodic traveling waves was obtained by analytic techniques.

Key words: periodic traveling wave solution; existence; auxiliary system; fixed-point theorem

引　　言

反应扩散方程的行波解在生态学、传染病学、种群动力学、生物化学等领域有着广泛的应用[1-3], 例如, 传

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染病的传播、种群的入侵等. 目前, 关于经典 Laplace 扩散方程行波解的研究[4-6] 已相对成熟, 特别是 SIR 传染病模型一直是重点研究对象[4-5]. 例如, Wang 等[4] 分别利用 Schauder 不动点定理和双边 Laplace 变换研究了 SIR 传染病模型

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=d_1\Delta S(x,t)-\dfrac{\alpha S(x,t)I(x,t)}{S(x,t)+I(x,t)},\\ \dfrac{\partial I(x,t)}{\partial t}=d_2\Delta I(x,t)+\dfrac{\alpha S(x,t)I(x,t)}{S(x,t)+I(x,t)}-\gamma I(x,t),\\ \dfrac{\partial R(x,t)}{\partial t}=d_3\Delta R(x,t)+\gamma I(x,t)\end{cases}\tag{1}$$

行波解的存在性和不存在性, 其中S,I,R分别表示易感者、感染者和治愈者的密度, $d_i>0\ (i=1,2,3)$表示扩散率, α表示感染率, γ表示恢复 (或治愈) 率. 此后, Wang 等[5] 在模型 (1) 的基础上, 研究了总人口数不变的 SIR 传染病模型行波解的存在性和不存在性.

现实生活中, 人口的增长和传染病的传播会受到时间周期因素的影响. 例如, 麻疹、风疹和腮腺炎等传染病会随季节的变化呈现周期性爆发. 因此, 研究具有时间周期反应扩散方程的行波解更符合客观现实[7-9]. 2018 年, Wang 等[7] 研究了相应于模型 (1) 的具有时间周期的 SIR 传染病模型:

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=d_1\Delta S(x,t)-\dfrac{\alpha(t)S(x,t)I(x,t)}{S(x,t)+I(x,t)},\\ \dfrac{\partial I(x,t)}{\partial t}=d_2\Delta I(x,t)+\dfrac{\alpha(t)S(x,t)I(x,t)}{S(x,t)+I(x.t)}-\gamma(t)I(x,t),\\ \dfrac{\partial R(x,t)}{\partial t}=d_3\Delta R(x,t)+\gamma(t)I(x,t).\end{cases}\tag{2}$$

利用 Schauder 不动点定理和渐近传播速度的性质建立了模型 (2) 周期行波解的存在性和不存在性. 此后, Wu 等[8] 将模型 (2) 的结果推广到具有一般非线性发生率的时间周期 SIR 传染病模型:

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=d_1\Delta S(x,t)-\alpha(t)f(S(x,t),I(x,t)),\\ \dfrac{\partial I(x,t)}{\partial t}=d_2\Delta I(x,t)+\alpha(t)f(S(x,t),I(x,t))-\gamma(t)I(x,t),\\ \dfrac{\partial R(x,t)}{\partial t}=d_3\Delta R(x,t)+\gamma(t)I(x,t).\end{cases}\tag{3}$$

关于具有时间周期反应扩散方程的行波解研究, 还可参见文献 [10-13] 等.

然而, 上述模型并没有考虑外部输入和自然死亡的影响, 对病程较长的传染病, 外部输入和自然死亡是不可忽略的因素. 因此, 考虑外部输入和自然死亡的 SIR 传染病模型的行波解研究, 引起了学者们的广泛关注[14-16]. 例如, Zhou 等[16] 分别利用 Schauder 不动点定理结合 Lyapunov 函数方法、双边 Laplace 变换, 研究了带有外部输入和自然死亡的 SIR 传染病模型

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=d_1\Delta S(x,t)+\Lambda-\dfrac{\alpha S(x,t)I(x,t-\tau)}{S(x,t)+I(x,t-\tau)}-\mu S(x,t),\\ \dfrac{\partial I(x,t)}{\partial t}=d_2\Delta I(x,t)+\dfrac{\alpha S(x,t)I(x,t-\tau)}{S(x,t)+I(x,t-\tau)}-(\mu+\gamma)I(x,t),\\ \dfrac{\partial R(x,t)}{\partial t}=d_3\Delta R(x,t)+\gamma I(x,t)-\mu R(x,t)\end{cases}\tag{4}$$

行波解的存在性和不存在性, 其中, Λ表示外部输入率, μ表示自然死亡率. 注意到, 模型 (4) 并没有考虑时间周期因素的影响, 基于此, 本文研究了带有外部输入和自然死亡的时间周期 SIR 传染病模型

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=d_1\Delta S(x,t)+\Lambda(t)-\alpha(t)f(S(x,t),I(x,t))-\mu(t)S(x,t),\\ \dfrac{\partial I(x,t)}{\partial t}=d_2\Delta I(x,t)+\alpha(t)f(S(x,t),I(x,t))-(\mu(t)+\gamma(t))I(x,t),\\ \dfrac{\partial R(x,t)}{\partial t}=d_3\Delta R(x,t)+\gamma(t)I(x,t)-\mu(t)R(x,t)\end{cases}\tag{5}$$

周期行波解的存在性和不存在性, 其中$\Lambda,\alpha,\gamma,\mu$是关于$t$的严格正 T 周期连续函数, 非线性发生率$f(S,I)$ 满足如下条件:

(A1) 当$S,I\geqslant 0$时, $f(S,I)$是二阶连续可微的; 当$S,I>0$时, $f(0,I)=f(S,0)=0$.

(A2) 当$S>0,I\geqslant 0$时, $\partial_2 f(S,I)>0$, $\partial_{22}f(S,I)\leqslant 0$; 当$S\geqslant 0,I>0$时, $\partial_1 f(S,I)$有界且为正的.

建立模型 (5) 周期行波解的存在性时, 由于模型 (5) 不具有单调性, 常用的单调性方法如比较原理、单调迭代技术[17]、单调半流理论[18] 等失效. 而通过构造适当的上下解并结合 Schauder 不动点定理对非单调方程完全适用. 为此, 首先通过构造适当的上下解定义闭凸锥, 利用 Schauder 不动点定理证明了模型 (5) 周期行波解的存在性. 其次, 借助分析技术证明了模型 (5) 周期行波解的不存在性. 注意到, 模型 (5) 将不含时滞的模型 (4) 行波解的研究推广到了周期情形. 另外, 当$f(S(x,t),I(x,t))=\dfrac{S(x,t)I(x,t)}{S(x,t)+I(x,t)}$时, 模型 (5) 退化为模型 (2). 因此, 本文完善了不考虑外部输入和自然死亡的传染病系统周期行波解的研究结果[7-8], 并将 SIR 传染病系统行波解的研究[16] 推广到了时间周期的 SIR 传染病系统的周期行波解.

1 预备知识

因为系统 (5) 中关于 R 的方程可以解耦, 故仅需考虑以下系统:

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=d_1\Delta S(x,t)+\Lambda(t)-\alpha(t)f(S(x,t),I(x,t))-\mu(t)S(x,t),\\ \dfrac{\partial I(x,t)}{\partial t}=d_2\Delta I(x,t)+\alpha(t)f(S(x,t),I(x,t))-(\mu(t)+\gamma(t))I(x,t).\end{cases}\tag{6}$$

系统 (6) 的时间周期行波解是指形如$(S(x,t),I(x,t))=(\phi(z,t),\psi(z,t))$, 且满足$(\phi(z,t+T),\psi(z,t+T))=(\phi(z,t),\psi(z,t))$, $z\in\mathbb{R},t\in\mathbb{R}$的解, 其中 $z=x+ct$ 是移动坐标, c 是波速, T 是正常数. 则系统 (6) 相应的行波系统为

$$\begin{cases}\phi_t(z,t)=d_1\phi_{zz}(z,t)-c\phi_z(z,t)+\Lambda(t)-\alpha(t)f(\phi(z,t),\psi(z,t))-\mu(t)\phi(z,t),\\ \psi_t(z,t)=d_2\psi_{zz}(z,t)-c\psi_z(z,t)+\alpha(t)f(\phi(z,t),\psi(z,t))-(\mu(t)+\gamma(t))\psi(z,t).\end{cases}\tag{7}$$

渐近边界条件为

$$\lim_{z\to-\infty}\phi(z,t)=S_0(t),\ \lim_{z\to-\infty}\psi(z,t)=0,\ \lim_{z\to+\infty}\phi(z,t)=S^*(t),\ \lim_{z\to+\infty}\psi(z,t)=I^*(t),$$

其中$S_0(t)=\int_{-\infty}^{t}\mathrm{e}^{-\int_s^t\mu(\tau)\mathrm{d}\tau}\Lambda(s)\mathrm{d}s$是初始无病状态下易感者的密度, 它是周期方程$\dfrac{\mathrm{d}S(t)}{\mathrm{d}t}=\Lambda(t)-\mu(t)S(t)$的 T 周期解 (可参见文献 [19]). $S^*(t),I^*(t)$分别表示易感者、感染者在疾病暴发后的密度. 将系统 (7) 的第二个方程在无病平衡点$(S_0(t),0)$处线性化得

$$\psi_t(z,t)=d_2\psi_{zz}(z,t)-c\psi_z(z,t)+(\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t))\psi(z,t).\tag{8}$$

令$\bar{H}=1/T\int_0^T H(t)\mathrm{d}t$表示任意的 T 周期函数$H(\cdot)$的平均值, 且

$$\Delta(\lambda,c)=d_2\lambda^2-c\lambda+k_0,\qquad c\in\mathbb{R},\ \lambda\in\mathbb{R},$$

$$k_0=\frac{1}{T}\int_0^T(\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t))\mathrm{d}t.$$

不难验证, 当基本再生数$\Re_0=\dfrac{\int_0^T\alpha(t)\partial_2 f(S_0(t),0)\mathrm{d}t}{\int_0^T(\mu(t)+\gamma(t))\mathrm{d}t}>1$时, $k_0>0$. 进一步地, 记$\lambda_1=\dfrac{c-\sqrt{c^2-4d_2k_0}}{2d_2}$, $\lambda_2=\dfrac{c+\sqrt{c^2-4d_2k_0}}{2d_2}$, 则当$c>c^*=2\sqrt{d_2k_0}$ 时, 对任意的$\lambda\in(\lambda_1,\lambda_2)$, 有$\Delta(\lambda_1,c)=\Delta(\lambda_2,c)=0$, $\Delta(\lambda,c)<0$成立.

由于系统 (7) 中一般非线性发生率的出现, 不易得到感染个体 I 的有界性, 从而很难构造适当的上下解. 为此, 首先引入系统 (7) 的辅助系统:

$$\begin{cases}\phi_t(z,t)=d_1\phi_{zz}(z,t)-c\phi_z(z,t)+\Lambda(t)-\alpha(t)f(\phi(z,t),\psi(z,t))-\mu(t)\phi(z,t),\\ \psi_t(z,t)=d_2\psi_{zz}(z,t)-c\psi_z(z,t)+\alpha(t)f(\phi(z,t),\psi(z,t))-(\mu(t)+\gamma(t))\psi(z,t)-\varepsilon\psi^2(z,t),\end{cases}\tag{9}$$

其中$\varepsilon>0$是一个正常数. 然后通过构造辅助系统 (9) 适当的上下解并结合 Schauder 不动点定理建立辅助系统 (9) 周期解$(\phi_\varepsilon(z,t),\psi_\varepsilon(z,t))$的存在性. 最后, 令$\varepsilon\to 0$, 证明$(\phi_\varepsilon(z,t),\psi_\varepsilon(z,t))$的极限函数$(\phi(z,t),\psi(z,t))$的存在性, 即

系统 (7) 周期解的存在性.

2 周期行波解的存在性

本节中, 首先构造辅助系统 (9) 的上下解并定义闭凸锥, 将系统 (5) 周期行波解的存在性转化为定义在这个闭凸锥上的非单调算子存在不动点的问题, 并借助 Schauder 不动点定理证明了当$\Re_0>1, c>c^*$时, 辅助系统 (9) 周期解的存在性. 其次, 利用 Ascoli-Arzela 定理证明了行波系统 (7) 周期解的存在性, 即系统 (5) 周期行波解的存在性.

2.1 上下解构造

假设$\Re_0>1$, $c>c^*$, 令$K(t)=\mathrm{e}^{\int_0^t[d_2\lambda_1^2-c\lambda_1+(\alpha(s)\partial_2 f(S_0(s),0)-\mu(s)-\gamma(s))]\mathrm{d}s}$. 容易验证当$t\geqslant 0$时, $K(t+T)=K(t)$. 为了方便, 记

$$\phi^+(z,t)=S_0(t),\ \phi^-(z,t)=\max\{S_0(t)(1-M_1\mathrm{e}^{\varpi_1 z}),0\},$$

$$\psi^+(z,t)=\min\{K(t)\mathrm{e}^{\lambda_1 z},A\},\ \psi^-(z,t)=\max\{K(t)\mathrm{e}^{\lambda_1 z}(1-M_2\mathrm{e}^{\varpi_2 z}),0\},$$

其中 A 是足够大的正常数, 且满足$\alpha(t)f(S_0(t),A)-(\mu(t)+\gamma(t))A-\varepsilon A^2<0$. M_i 和$\varpi_i(i=1,2)$都是正常数, 其定义见下面的引理 3 和引理 4.

引理 1　函数$\phi^+(z,t)=S_0(t)$满足不等式

$$\phi_t^+(z,t)\geqslant d_1\phi_{zz}^+(z,t)-c\phi_z^+(z,t)+\Lambda(t)-\alpha(t)f(\phi^+(z,t),\psi^-(z,t))-\mu(t)\phi^+(z,t). \tag{10}$$

证明　因为$\phi_t^+(z,t)=\Lambda(t)-\mu(t)S_0(t)$, 所以

$$d_1\phi_{zz}^+(z,t)-c\phi_z^+(z,t)+\Lambda(t)-\alpha(t)f(\phi^+(z,t),\psi^-(z,t))-\mu(t)\phi^+(z,t)=$$
$$\Lambda(t)-\alpha(t)f(\phi^+(z,t),\psi^-(z,t))-\mu(t)\phi^+(z,t)\leqslant\Lambda(t)-\mu(t)\phi^+(z,t)=\phi_t^+(z,t).$$

引理 2　当$z\neq z_0:=\lambda_1^{-1}\ln\dfrac{A}{K(t)}$时, 函数$\psi^+(z,t)$满足不等式

$$\psi_t^+(z,t)\geqslant d_2\psi_{zz}^+(z,t)-c\psi_z^+(z,t)+\alpha(t)f(\phi^+(z,t),\psi^+(z,t))-(\mu(t)+\gamma(t))\psi^+(z,t)-\varepsilon[\psi^+(z,t)]^2. \tag{11}$$

证明　若$z>z_0$, 则$\psi^+(z,t)=A$, 不等式 (11) 显然成立. 若$z<z_0$, 则$\psi^+(z,t)=K(t)\mathrm{e}^{\lambda_1 z}$, 所以

$$\psi_t^+(z,t)=K'(t)\mathrm{e}^{\lambda_1 z}=[d_2\lambda_1^2-c\lambda_1+\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]K(t)\mathrm{e}^{\lambda_1 z}=$$
$$d_2\psi_{zz}^+(z,t)-c\psi_z^+(z,t)+\alpha(t)\partial_2 f(S_0(t),0)\psi^+(z,t)-(\mu(t)+\gamma(t))\psi^+(z,t).$$

由条件 (A2) 可知$f(\phi^+(z,t),\psi^+(z,t))\leqslant\partial_2 f(S_0(t),0)\psi^+(z,t)$, 故

$$\psi_t^+(z,t)=d_2\psi_{zz}^+(z,t)-c\psi_z^+(z,t)+\alpha(t)\partial_2 f(S_0(t),0)\psi^+(z,t)-(\mu(t)+\gamma(t))\psi^+(z,t)\geqslant$$
$$d_2\psi_{zz}^+(z,t)-c\psi_z^+(z,t)+\alpha(t)f(\phi^+(z,t),\psi^+(z,t))-(\mu(t)+\gamma(t))\psi^+(z,t)-\varepsilon[\psi^+(z,t)]^2.$$

引理 3　假设存在充分小的正数$\varpi_1<\min\left\{\lambda_1,\dfrac{c}{d_1}\right\}$和充分大的正数$M_1>1$, 则对任意的$z\neq z_1:=\dfrac{1}{\varpi_1}\ln\dfrac{1}{M_1}$, 函数$\phi^-(z,t)$满足不等式

$$\phi_t^-(z,t)\leqslant d_1\phi_{zz}^-(z,t)-c\phi_z^-(z,t)+\Lambda(t)-\alpha(t)f(\phi^-(z,t),\psi^+(z,t))-\mu(t)\phi^-(z,t). \tag{12}$$

证明　若$z>z_1$, 则$\phi^-(z,t)=0$, 不等式 (12) 显然成立. 若$z<z_1$, 则$\phi^-(z,t)=S_0(t)(1-M_1\mathrm{e}^{\varpi_1 z})$, 结合$\psi^+(z,t)$的定义及条件 (A2), 不等式 (12) 等价于

$$[\Lambda(t)-\mu(t)S_0(t)](1-M_1\mathrm{e}^{\varpi_1 z})\leqslant-d_1S_0(t)M_1\varpi_1^2\mathrm{e}^{\varpi_1 z}+cS_0(t)M_1\varpi_1\mathrm{e}^{\varpi_1 z}+\Lambda(t)-$$
$$\alpha(t)f(S_0(t)(1-M_1\mathrm{e}^{\varpi_1 z}),K(t)\mathrm{e}^{\lambda_1 z})-\mu(t)S_0(t)(1-M_1\mathrm{e}^{\varpi_1 z}).$$

由条件 (A1) 和 (A2) 及$\varpi_1<\lambda_1$可知

$$f(S_0(t)(1-M_1\mathrm{e}^{\varpi_1 z}),K(t)\mathrm{e}^{\lambda_1 z})<f(S_0(t),K(t)\mathrm{e}^{\lambda_1 z}),$$

$$\frac{f(S_0(t),K(t)\mathrm{e}^{\lambda_1 z})}{\mathrm{e}^{\varpi_1 z}}\leqslant\frac{\partial_2 f(S_0(t),0)K(t)\mathrm{e}^{\lambda_1 z}}{\mathrm{e}^{\varpi_1 z}}\leqslant\partial_2 f(S_0(t),0)K(t),$$

所以

$$-d_1S_0(t)M_1\varpi_1^2\mathrm{e}^{\varpi_1 z}+cS_0(t)M_1\varpi_1\mathrm{e}^{\varpi_1 z}+\Lambda(t)-\alpha(t)f(S_0(t)(1-M_1\mathrm{e}^{\varpi_1 z}),K(t)\mathrm{e}^{\lambda_1 z})-\mu(t)S_0(t)(1-M_1\mathrm{e}^{\varpi_1 z})=$$

$$\left[S_0(t)M_1\varpi_1(c-d_1\varpi_1)-\frac{\alpha(t)f(S_0(t)(1-M_1\mathrm{e}^{\varpi_1 z}),K(t)\mathrm{e}^{\lambda_1 z})}{\mathrm{e}^{\varpi_1 z}}+\mu(t)S_0(t)M_1\right]\mathrm{e}^{\varpi_1 z}+\Lambda(t)-\mu(t)S_0(t)\geqslant$$
$$[S_0(t)M_1\varpi_1(c-d_1\varpi_1)-\alpha(t)\partial_2 f(S_0(t),0)K(t)+\mu(t)S_0(t)M_1]\mathrm{e}^{\varpi_1 z}+\Lambda(t)-\mu(t)S_0(t).$$

为了得到不等式 (12), 只需证明

$$[S_0(t)M_1\varpi_1(c-d_1\varpi_1)-\alpha(t)\partial_2 f(S_0(t),0)K(t)+\mu(t)S_0(t)M_1]\mathrm{e}^{\varpi_1 z}+\Lambda(t)-\mu(t)S_0(t)\geqslant$$
$$[\Lambda(t)-\mu(t)S_0(t)](1-M_1\mathrm{e}^{\varpi_1 z}).$$

因为$\mathrm{e}^{\varpi_1 z}>0$, 故只需验证$S_0(t)M_1\varpi_1(c-d_1\varpi_1)\geqslant\alpha(t)\partial_2 f(S_0(t),0)K(t)-\Lambda(t)M_1$, 注意到$\alpha(t),\Lambda(t)$和$K(t)$都是正的$T$周期函数, 取$0<\varpi_1<\min\left\{\lambda_1,\dfrac{c}{d_1}\right\}$,$M_1>1$充分大时, 上述不等式成立, 这意味着不等式 (12) 成立. 证毕.

引理 4 假设存在充分小的正数$\varpi_2<\min\{\varpi_1,\lambda_2-\lambda_1\}$和充分大的正数$M_2>1$满足$\dfrac{1}{\varpi_2}\ln\dfrac{1}{M_2}<\dfrac{1}{\varpi_1}\ln\dfrac{1}{M_1}$, 则当$z\neq z_2:=\dfrac{1}{\varpi_2}\ln\dfrac{1}{M_2}$时, 函数$\psi^-(z,t)$满足不等式

$$\psi^-_t(z,t)\leqslant d_2\psi^-_{zz}(z,t)-c\psi^-_z(z,t)+\alpha(t)f(\phi^-(z,t),\psi^-(z,t))-(\mu(t)+\gamma(t))\psi^-(z,t)-\varepsilon[\psi^-(z,t)]^2. \tag{13}$$

证明 由$\dfrac{1}{\varpi_2}\ln\dfrac{1}{M_2}<\dfrac{1}{\varpi_1}\ln\dfrac{1}{M_1}$可知, $z_2<z_1<0$. 若$z>z_2$, 则$\psi^-(z,t)=0$, 不等式 (13) 显然成立. 若$z<z_2$, 则$\psi^-(z,t)=K(t)\mathrm{e}^{\lambda_1 z}(1-M_2\mathrm{e}^{\varpi_2 z})$, 不等式 (13) 等价于

$$\psi^-_t(z,t)-d_2\psi^-_{zz}(z,t)+c\psi^-_z(z,t)-[\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]\psi^-(z,t)\leqslant$$
$$\alpha(t)[f(\phi^-(z,t),\psi^-(z,t))-\partial_2 f(S_0(t),0)\psi^-(z,t)]-\varepsilon[\psi^-(z,t)]^2. \tag{14}$$

根据$K(t)$和$\psi^-(z,t)$的表达式可得

$$\begin{aligned}&\psi^-_t(z,t)-d_2\psi^-_{zz}(z,t)+c\psi^-_z(z,t)-[\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]\psi^-(z,t)=\\&\quad\mathrm{e}^{\lambda_1 z}\{K'(t)-d_2\lambda_1^2K(t)+c\lambda_1K(t)-[\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]K(t)\}-\\&\quad M_2\mathrm{e}^{(\lambda_1+\varpi_2)z}\{K'(t)-[d_2\lambda_1^2K(t)+d_2\varpi_2\lambda_1K(t)+d_2(\lambda_1+\varpi_2)\varpi_2K(t)]+c\lambda_1K(t)+\\&\quad c\varpi_2K(t)-[\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]K(t)\}=\\&\quad -M_2\mathrm{e}^{(\lambda_1+\varpi_2)z}\{[d_2\lambda_1^2-c\lambda_1+\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]K(t)-d_2(\lambda_1+\varpi_2)^2K(t)+\\&\quad c(\lambda_1+\varpi_2)K(t)-[\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]K(t)\}=\\&\quad -M_2\mathrm{e}^{(\lambda_1+\varpi_2)z}K(t)\{d_2\lambda_1^2-c\lambda_1-[d_2(\lambda_1+\varpi_2)^2-c(\lambda_1+\varpi_2)]\}=\\&\quad M_2\mathrm{e}^{(\lambda_1+\varpi_2)z}K(t)\Delta(\lambda_1+\varpi_2,c).\end{aligned}$$

故不等式 (13) 等价于

$$-M_2\mathrm{e}^{(\lambda_1+\varpi_2)z}K(t)\Delta(\lambda_1+\varpi_2,c)\geqslant\alpha(t)[\partial_2 f(S_0(t),0)\psi^-(z,t)-f(\phi^-(z,t),\psi^-(z,t))]+\varepsilon[\psi^-(z,t)]^2. \tag{15}$$

进一步, 式 (15) 化为

$$-M_2\mathrm{e}^{(\lambda_1+\varpi_2)z}K(t)\Delta(\lambda_1+\varpi_2,c)\geqslant\left(\frac{\alpha(t)[\partial_2 f(S_0(t),0)\psi^-(z,t)-f(\phi^-(z,t),\psi^-(z,t))]}{[\psi^-(z,t)]^2}+\varepsilon\right)K^2(t)\mathrm{e}^{2\lambda_1 z}(1-M_2\mathrm{e}^{\varpi_2 z})^2.$$

因为$0<1-M_2\mathrm{e}^{\varpi_2 z}<1$, 则只需验证

$$-M_2\Delta(\lambda_1+\varpi_2,c)\geqslant\left(\frac{\alpha(t)[\partial_2 f(S_0(t),0)\psi^-(z,t)-f(\phi^-(z,t),\psi^-(z,t))]}{[\psi^-(z,t)]^2}+\varepsilon\right)\mathrm{e}^{(\lambda_1-\varpi_2)z}K(t).$$

由条件 (A2) 知, $\dfrac{f(S,I)}{I}$在$[0,+\infty)\times(0,+\infty)$上关于$I$是单调递减的, 所以对任意的$\epsilon\in(0,\max_{t\in[0,T]}\partial_2 f(S_0(t),0))$, 存在$\sigma_0>0$, 使得当$S_0(t)-\sigma_0<S<S_0(t)$, $0<I<\sigma_0$时, $\dfrac{f(S,I)}{I}\geqslant\partial_2 f(S_0(t),0)-\epsilon$. 又因为$z\to-\infty$时, $\psi^-\to0$, $\phi^-\to S_0(t)$, 选取$\hat{M}_2>0$, 使得当$z<-\hat{M}_2$时, 有$0<\psi^-<\sigma_0$, $S_0(t)-\sigma_0\leqslant\phi^-\leqslant S_0(t)$. 令$M_2>\max\left\{M_1^{\frac{\varpi_2}{\varpi_1}},\mathrm{e}^{\varpi_2\hat{M}_2}\right\}$, 则

$$\partial_2 f(S_0(t),0)\psi^--f(\phi^-,\psi^-)=\left(\partial_2 f(S_0(t),0)-\frac{f(\phi^-,\psi^-)}{\psi^-}\right)\psi^-\leqslant\left(\frac{\partial_2 f(S_0(t),0)-\dfrac{f(\phi^-,\psi^-)}{\psi^-}+\psi^-}{2}\right)^2\leqslant$$
$$(\partial_2 f(S_0(t),0)-(\partial_2 f(S_0(t),0)-\epsilon)+\psi^-)^2=(\epsilon+\psi^-)^2.$$

综上, $\left(\dfrac{\alpha(t)[\partial_2 f(S_0(t),0)\psi^-(z,t)-f(\phi^-(z,t),\psi^-(z,t))]}{[\psi^-(z,t)]^2}+\varepsilon\right)$是有界的, 又因为$\varpi_2<\min\{\varpi_1,\lambda_2-\lambda_1\}$, 且$0<\varpi_1<\lambda_1$, 则$\lambda_1-\varpi_2>0$, 又$z<\dfrac{1}{\varpi_2}\ln\dfrac{1}{M_2}$, 则只需验证

$$-M_2\Delta(\lambda_1+\varpi_2,c)\geqslant\left(\frac{\alpha(t)[\partial_2 f(S_0(t),0)\psi^-(z,t)-f(\phi^-(z,t),\psi^-(z,t))]}{[\psi^-(z,t)]^2}+\varepsilon\right)M_2^{\frac{-(\lambda_1-\varpi_2)}{\varpi_2}}K(t).$$

易知$\lambda_1+\varpi_2\in(\lambda_1,\lambda_2)$,从而$\Delta(\lambda_1+\varpi_2,c)<0$. 当$M_2>\max\left\{M_1^{\frac{\varpi_2}{\varpi_1}},\mathrm{e}^{\varpi_2\hat{M}_2}\right\}$充分大时, 上述不等式成立. 证毕.

2.2 不动点问题

令$X=B_{\mathrm{BUC}}(\mathbb{R},\mathbb{R})$是从$\mathbb{R}$映到$\mathbb{R}$上一致连续有界函数构成的具有上确界范数$\|\cdot\|_X$的 Banach 空间, 令$X^+=\{u\in X:u(x)\geqslant0,x\in\mathbb{R}\}$. 根据文献 [20] 中的定理 1.5, $d\Delta$在 X 上可以生成一个强连续解析半群$T(t)$, 且$T(t)X^+\subset X^+$, $t\geqslant0$, 此外,

$$(T(t)u)(x)=\frac{1}{\sqrt{4\pi dt}}\int_{\mathbb{R}}\mathrm{e}^{-\frac{(x-y)^2}{4dt}}u(y)\mathrm{d}y,\qquad t>0,\ x\in\mathbb{R},\ u(\cdot)\in X. \tag{16}$$

对任意给定的正常数μ, 定义函数空间

$$B_\mu(\mathbb{R}\times[0,T],\mathbb{R}^2)=\left\{u=(u_1,u_2)\ \middle|\ \begin{array}{l}u_i\in B_{\mathrm{BUC}}(\mathbb{R}\times[0,T],\mathbb{R}),\\ \sup\limits_{t\in[0,T],x\in\mathbb{R}}\mathrm{e}^{-\mu|x|}|u_i(x,t)|<\infty,\\ u_i(x,0)=u_i(x,T),x\in\mathbb{R},i=1,2\end{array}\right\},$$

且其上的范数为$\|u\|_\mu=\max\left\{\sup_{t\in[0,T],x\in\mathbb{R}}\mathrm{e}^{-\mu|x|}|u_1(x,t)|,\sup_{t\in[0,T],x\in\mathbb{R}}\mathrm{e}^{-\mu|x|}|u_2(x,t)|\right\}$.

定义集合

$$D=\left\{(\tilde\phi,\tilde\psi)(z,t)\in B_\mu(\mathbb{R}\times[0,T],\mathbb{R}^2)\ \middle|\ \begin{array}{l}\phi^-(z,t)\leqslant\tilde\phi(z,t)\leqslant\phi^+(z,t)\\ \psi^-(z,t)\leqslant\tilde\psi(z,t)\leqslant\psi^+(z,t)\end{array}\right\}.$$

对任意给定的$(\tilde\phi(z,t),\tilde\psi(z,t))\in D$, 定义

$$h_1[\tilde\phi,\tilde\psi](z,t)=\beta_1\tilde\phi(z,t)+\Lambda(t)-\alpha(t)f(\tilde\phi,\tilde\psi)(z,t)-\mu(t)\tilde\phi(z,t),$$

$$h_2[\tilde\phi,\tilde\psi](z,t)=\beta_2\tilde\psi(z,t)+\alpha(t)f(\tilde\phi,\tilde\psi)(z,t)-(\mu(t)+\gamma(t))\tilde\psi(z,t)-\varepsilon\tilde\psi^2(z,t),$$

其中$\beta_1,\beta_2>0$, 并且满足$\beta_1>\hat\alpha\sup\partial_1 f(S,I)+\hat\mu$, 这里

$$S\in\left[0,\max_{t\in[0,T]}S_0(t)\right],I\in[0,A],\ \hat\alpha=\max_{t\in[0,T]}\alpha(t)\ \hat\mu=\max_{t\in[0,T]}\mu(t).\beta_2>\max_{t\in[0,T]}[\mu(t)+\gamma(t)]+2\varepsilon A.$$

考虑下面的抛物初值问题:

$$\begin{cases}\phi_t(z,t)-d_1\phi_{zz}(z,t)+c\phi_z(z,t)+\beta_1\phi(z,t)=h_1[\tilde\phi,\tilde\psi](z,t), & 0<t\leqslant T,\ z\in\mathbb{R},\\ \psi_t(z,t)-d_2\psi_{zz}(z,t)+c\psi_z(z,t)+\beta_2\psi(z,t)=h_2[\tilde\phi,\tilde\psi](z,t), & 0<t\leqslant T,\ z\in\mathbb{R},\\ \phi(z,0)=\phi_0(z),\ \psi(z,0)=\psi_0(z), & z\in\mathbb{R}.\end{cases} \tag{17}$$

系统 (17) 化为积分系统:

$$\begin{cases}\phi(t)=T_1(t)\phi_0+\displaystyle\int_0^t T_1(t-s)h_1[\tilde\phi,\tilde\psi](s)\mathrm{d}s,\\ \psi(t)=T_2(t)\psi_0+\displaystyle\int_0^t T_2(t-s)h_2[\tilde\phi,\tilde\psi](s)\mathrm{d}s,\end{cases} \tag{18}$$

其中$(\phi(t),\psi(t))=(\phi(z,t),\psi(z,t))$,$T_i(t)$, $i=1,2$是由$A_i:D_{\mathrm{domain}}(A_i)\to C(\mathbb{R})$生成的解析半群, A_i定义为$A_iu=d_iu_{zz}-cu_z-\beta_iu$, $i=1,2$, 且

$$D_{\mathrm{domain}}(A_i)=\left\{u\in\bigcap_{p\geqslant1}W^{2,p}_{\mathrm{loc}}(\mathbb{R}):A_iu\in C(\mathbb{R})\right\},\qquad i=1,2.$$

此外, 由式 (16) 知

$$(T_i(t)u)(\xi)=\mathrm{e}^{-\beta_i t}\frac{1}{\sqrt{4\pi d_i t}}\int_{\mathbb{R}}\mathrm{e}^{-\frac{(\xi-ct-x)^2}{4d_it}}u(x)\mathrm{d}x,\qquad t>0,\ \xi\in\mathbb{R},\ u(\cdot)\in X. \tag{19}$$

称积分系统 (18) 的解为系统 (17) 的温和解.

接下来证明, 对任意$(\tilde{\phi}(z,t),\tilde{\psi}(z,t))\in D$, 存在唯一的$(\phi^*(z,t),\psi^*(z,t))\in D$满足

$$\begin{cases}\phi^*(t)=T_1(t)\phi_0^*+\int_0^t T_1(t-s)h_1[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s,\\ \psi^*(t)=T_2(t)\psi_0^*+\int_0^t T_2(t-s)h_2[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s.\end{cases}\tag{20}$$

给定$\eta>0$, 定义函数空间

$$\tilde{B}_\eta(\mathbb{R},\mathbb{R}^2)=\left\{u=(u_1,u_2):u_i\in X,\ \sup_{x\in\mathbb{R}}\mathrm{e}^{-\eta|x|}|u_i(x)|<\infty,\ x\in\mathbb{R},i=1,2\right\},$$

且其上的范数为$|u|_\eta=\max\left\{\sup_{x\in\mathbb{R}}\mathrm{e}^{-\eta|x|}|u_1(x)|,\ \sup_{x\in\mathbb{R}}\mathrm{e}^{-\eta|x|}|u_2(x)|\right\}$.

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$$\tilde{D}=\left\{(\phi_0(\cdot),\psi_0(\cdot))\in\ \tilde{B}_\eta(\mathbb{R},\mathbb{R}^2)\,\middle|\,\begin{matrix}\phi^-(z,0)\leqslant\phi_0(z)\leqslant\phi^+(z,0)\\ \psi^-(z,0)\leqslant\psi_0(z)\leqslant\psi^+(z,0)\end{matrix}\right\}.$$

下面证明当$(\tilde{\phi}(z,t),\ \tilde{\psi}(z,t))\in D$时, 系统 (18) 有唯一解$(\phi^*(z,t),\psi^*(z,t))\in D$.

首先证明积分系统 (18) 解的不变性, 为此先给出引理 5.

引理 5 函数$\phi^+,\psi^+,\phi^-,\psi^-$满足下面不等式:

$$\phi^+(t)\geqslant T_1(t)\phi^+(0)+\int_0^t T_1(t-s)h_1[\phi^+,\psi^-](s)\mathrm{d}s,\tag{21}$$

$$\psi^+(t)\geqslant T_2(t)\psi^+(0)+\int_0^t T_2(t-s)h_2[\phi^+,\psi^+](s)\mathrm{d}s,\tag{22}$$

$$\phi^-(t)\leqslant T_1(t)\phi^-(0)+\int_0^t T_1(t-s)h_1[\phi^-,\psi^+](s)\mathrm{d}s,\tag{23}$$

$$\psi^-(t)\leqslant T_2(t)\psi^-(0)+\int_0^t T_2(t-s)h_2[\phi^-,\psi^-](s)\mathrm{d}s.\tag{24}$$

证明过程类似于文献 [8] 的引理 7, 故此省略.

基于引理 5, 下面引理给出积分系统 (18) 解的不变性.

引理 6 令$(\phi(z,t;\phi_0,\psi_0),\psi(z,t;\phi_0,\psi_0))$是系统 (18) 关于初值为$(\phi_0,\psi_0)\in\tilde{D}$的解, 则对任意$(z,t)\in\mathbb{R}\times[0,T]$, $(\phi(z,t;\phi_0,\psi_0),\psi(z,t;\phi_0,\psi_0))$满足

$$\phi^-(z,t)\leqslant\phi(z,t;\phi_0,\psi_0)\leqslant\phi^+(z,t),\ \psi^-(z,t)\leqslant\psi(z,t;\phi_0,\psi_0)\leqslant\psi^+(z,t).$$

证明 由$(\tilde{\phi},\tilde{\psi})\in D,(\phi_0,\psi_0)\in\tilde{D}$,则有$\phi^-(z,t)\leqslant\tilde{\phi}(z,t)\leqslant\phi^+(z,t)$, $\psi^-(z,t)\leqslant\tilde{\psi}(z,t)\leqslant\psi^+(z,t)$,$\phi^-(z,0)\leqslant\phi_0(z)\leqslant\phi^+(z,0)$, $\psi^-(z,0)\leqslant\psi_0(z)\leqslant\psi^+(z,0)$.

下证$\phi(z,t;\phi_0,\psi_0)\leqslant\phi^+(z,t)$, 由$\phi(t)=T_1(t)\phi_0+\int_0^t T_1(t-s)h_1[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s$, 结合$\beta_1>\hat{\alpha}\sup\partial_1 f(S,I)+\hat{\mu}$、不等式 (21) 和$\phi_0\leqslant\phi^+(0)$, 可得

$$\int_0^t T_1(t-s)h_1[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s\leqslant\int_0^t T_1(t-s)h_1[\phi^+,\psi^-](s)\mathrm{d}s\leqslant\phi^+(t)-T_1(t)\phi_0.$$

所以$\phi(t)=T_1(t)\phi_0+\int_0^t T_1(t-s)h_1[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s\leqslant\phi^+(t)$, 因为$\phi(z,0)=\phi_0(z)\leqslant\phi^+(z,0)$, 由抛物方程比较原理可知, $\phi(z,t;\phi_0,\psi_0)\leqslant\phi^+(z,t),\forall t\in[0,T],z\in\mathbb{R}$.

接下来证明$\phi^-(z,t)\leqslant\phi(z,t;\phi_0,\psi_0)$. 类似地, 由不等式 (23) 及$\phi_0\geqslant\phi^-(0)$可得$\int_0^t T_1(t-s)h_1[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s\geqslant\int_0^t T_1(t-s)\times h_1[\phi^-,\psi^+](s)\mathrm{d}s\geqslant\phi^-(t)-T_1(t)\phi_0$. 所以$\phi(t)=T_1(t)\phi_0+\int_0^t T_1(t-s)h_1[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s\geqslant\phi^-(t)$. 又$\phi(z,0)=\phi_0(z)\geqslant\phi^-(z,0)$, 由抛物方程比较原理可知, $\phi(z,t;\phi_0,\psi_0)\geqslant\phi^-(z,t),\forall t\in[0,T],z\in\mathbb{R}$.

下面考虑$\psi(z,t;\phi_0,\psi_0)$. 注意到, $\beta_2>\max_{t\in[0,T]}[\mu(t)+\gamma(t)]+2\varepsilon A$, 由不等式 (22) 和$\psi_0\leqslant\psi^+(0)$知$\int_0^t T_2(t-s)h_2\times[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s\leqslant\int_0^t T_2(t-s)h_2[\phi^+,\psi^+](s)\mathrm{d}s\leqslant\psi^+(t)-T_2(t)\psi_0$, 故$\psi(t)=T_2(t)\psi_0+\int_0^t T_2(t-s)h_2[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s\leqslant\psi^+(t)$, 又$\psi(z,0)=\psi_0(z)\leqslant\psi^+(z,0)$, 所以$\psi(z,t;\phi_0,\psi_0)\leqslant\psi^+(z,t),\forall t\in[0,T],z\in\mathbb{R}$. 类似以上讨论, 易证

$$\int_0^t T_2(t-s)h_2[\tilde{\phi},\tilde{\psi}](s)\mathrm{d}s\geqslant\int_0^t T_2(t-s)h_2[\phi^-,\psi^-](s)\mathrm{d}s\geqslant\psi^-(t)-T_2(t)\psi_0,$$

因此

$$\psi(t)=T_2(t)\psi_0+\int_0^t T_2(t-s)h_2[\tilde\phi,\tilde\psi](s)\mathrm{d}s\geqslant\psi^-(t),$$

结合$\psi(z,0)=\psi_0(z)\geqslant\psi^-(z,0)$,故

$$\psi(z,t;\phi_0,\psi_0)\geqslant\psi^-(z,t),\qquad \forall t\in[0,T],\ z\in\mathbb{R}.$$

证毕.

定义系统 (18) 的 T 周期映射为$\Gamma_{(\tilde\phi,\tilde\psi)}(\phi_0(\cdot),\psi_0(\cdot))=(\phi(\cdot,T;\phi_0,\psi_0),\psi(\cdot,T;\phi_0,\psi_0))$. 那么, T 映射$\Gamma_{(\tilde\phi,\tilde\psi)}$存在不动点等价于积分系统 (18) 存在 T 周期解.

定理 1　对任意给定的$(\tilde\phi,\tilde\psi)\in D$, 存在唯一的$(\phi^*,\psi^*)\in D$满足系统 (20).

证明　根据引理 6 以及$\phi^\pm$和$\psi^\pm$的定义知 $\Gamma_{(\tilde\phi,\tilde\psi)}(\tilde D)\subset\tilde D$, 对任意给定的$(\phi_0^*,\psi_0^*),(\phi_0^{**},\psi_0^{**})\in\tilde D$, 由系统 (18) 的第一个等式和式 (19) 可知

$$|\phi(z,T;\phi_0^*,\psi_0^*)-\phi(z,T;\phi_0^{**},\psi_0^{**})|=\left|\mathrm{e}^{-\beta_1T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1T}}\mathrm{e}^{-\frac{(z-cT-x)^2}{4d_1T}}(\phi_0^*(x)-\phi_0^{**}(x))\mathrm{d}x\right|\leqslant$$
$$\mathrm{e}^{-\beta_1T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1T}}\mathrm{e}^{-\frac{(z-cT-x)^2}{4d_1T}}|\phi_0^*(x)-\phi_0^{**}(x)|\mathrm{d}x\leqslant$$
$$\mathrm{e}^{-\beta_1T}\left\|\phi_0^*(\cdot)-\phi_0^{**}(\cdot)\right\|_{L^\infty}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1T}}\mathrm{e}^{-\frac{(z-cT-x)^2}{4d_1T}}\mathrm{d}x=\mathrm{e}^{-\beta_1T}\left\|\phi_0^*(\cdot)-\phi_0^{**}(\cdot)\right\|_{L^\infty}.$$

同理可得

$$|\psi(z,T;\phi_0^*,\psi_0^*)-\psi(z,T;\phi_0^{**},\psi_0^{**})|\leqslant\mathrm{e}^{-\beta_2T}\left\|\psi_0^*(\cdot)-\psi_0^{**}(\cdot)\right\|_{L^\infty}.$$

因为$\mathrm{e}^{-\beta_iT}<1,i=1,2$, 所以$\Gamma_{(\tilde\phi,\tilde\psi)}:\tilde D\to\tilde D$是一个压缩映射, 由 Banach 压缩映像原理可知 Γ 存在唯一不动点$(\phi_0^*,\psi_0^*)\in\tilde D$, 即存在唯一的$(\phi^*,\psi^*)\in D$满足系统 (20). 证毕.

令$(\phi^*(z,t),\psi^*(z,t))=(\phi(z,t;\phi_0^*,\psi_0^*),\psi(z,t;\phi_0^*,\psi_0^*))$, 其中$(\phi_0^*,\psi_0^*)\in\tilde D$是算子 Γ 的唯一不动点, 由定理 1, 定义算子 $F:D\to B_\mu$为$F(\tilde\phi,\tilde\psi)=(\phi^*,\psi^*)$, 从而系统 (6) 周期行波解的存在性转化为算子 F 存在不动点. 为此, 需要以下两个引理成立.

引理 7　算子$F:D\to D$在$B_\mu(\mathbb{R}\times[0,T],\mathbb{R}^2)$上是连续的.

证明　对任意给定的$(\tilde\phi_i,\tilde\psi_i)\in D$, 令$F(\tilde\phi_i,\tilde\psi_i)=(\phi_i^*(z,t;\tilde\phi_i,\tilde\psi_i),\psi_i^*(z,t;\tilde\phi_i,\tilde\psi_i))$, $i=1,2$.

由式 (19) 和系统 (20) 的第一个方程可知

$$\phi_i^*(z,T;\tilde\phi_i,\tilde\psi_i)=\mathrm{e}^{-\beta_1T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1T}}\mathrm{e}^{-\frac{(z-cT-y)^2}{4d_1T}}\phi_i^*(y,0)\mathrm{d}y+$$
$$\int_0^T\mathrm{e}^{-\beta_1(T-s)}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1(T-s)}}\mathrm{e}^{-\frac{(z-c(T-s)-y)^2}{4d_1(T-s)}}h_1[\tilde\phi_i,\tilde\psi_i](y,s)\mathrm{d}y\mathrm{d}s=$$
$$\mathrm{e}^{-\beta_1T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1T}}\mathrm{e}^{-\frac{(z-cT-y)^2}{4d_1T}}\phi_i^*(y,0)\mathrm{d}y+\int_0^T\mathrm{e}^{-\beta_1s}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1s}}\mathrm{e}^{-\frac{(z-cs-y)^2}{4d_1s}}h_1[\tilde\phi_i,\tilde\psi_i](y,T-s)\mathrm{d}y\mathrm{d}s.$$

令$m=\sup\partial_2f(S,0),S\in[0,\max_{t\in[0,T]}S_0(t)]$, 选取充分小的$\mu$使得$\mathrm{e}^{(d_1\mu^2+\mu c-\beta_1)T}\leqslant1/4$, 因此,

$$|\phi_1^*(z,T;\tilde\phi_1,\tilde\psi_1)-\phi_2^*(z,T;\tilde\phi_2,\tilde\psi_2)|\mathrm{e}^{-\mu|z|}=\left|\mathrm{e}^{-\beta_1T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1T}}\mathrm{e}^{-\frac{(z-cT-y)^2}{4d_1T}}[\phi_1^*(y,0)-\phi_2^*(y,0)]\mathrm{e}^{-\mu|z|}\mathrm{d}y+\right.$$
$$\left.\int_0^T\mathrm{e}^{-\beta_1s}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1s}}\mathrm{e}^{-\frac{(z-cs-y)^2}{4d_1s}}[h_1[\tilde\phi_1,\tilde\psi_1]-h_1[\tilde\phi_2,\tilde\psi_2]](y,T-s)\mathrm{e}^{-\mu|z|}\mathrm{d}y\mathrm{d}s\right|\leqslant$$
$$\mathrm{e}^{-\beta_1T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1T}}\mathrm{e}^{-\frac{(z-cT-y)^2}{4d_1T}}|\phi_1^*(y,0)-\phi_2^*(y,0)|\mathrm{e}^{-\mu|z|}\mathrm{d}y+$$
$$\int_0^T\mathrm{e}^{-\beta_1s}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1s}}\mathrm{e}^{-\frac{(z-cs-y)^2}{4d_1s}}|h_1[\tilde\phi_1,\tilde\psi_1]-h_1[\tilde\phi_2,\tilde\psi_2]|(y,T-s)\mathrm{e}^{-\mu|z|}\mathrm{d}y\mathrm{d}s\leqslant$$

$$
\begin{aligned}
&\mathrm{e}^{-\beta_1 T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 T}}\mathrm{e}^{-\frac{(z-cT-y)^2}{4d_1T}}|\phi_1^*(y,0)-\phi_2^*(y,0)|\mathrm{e}^{-\mu|z|}\mathrm{d}y+\\
&\int_0^T\mathrm{e}^{-\beta_1 s}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 s}}\mathrm{e}^{-\frac{(z-cs-y)^2}{4d_1s}}(2\beta_1+\hat{\alpha}m)(|\tilde{\phi}_1(y,T-s)-\tilde{\phi}_2(y,T-s)|+\\
&|\tilde{\psi}_1(y,T-s)-\tilde{\psi}_2(y,T-s)|)\mathrm{e}^{-\mu|z|}\mathrm{d}y\mathrm{d}s\leqslant\\
&\mathrm{e}^{-\beta_1 T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 T}}\mathrm{e}^{-\frac{(z-cT-y)^2}{4d_1T}}|\phi_1^*(y,0)-\phi_2^*(y,0)|\mathrm{e}^{-\mu|y|}\mathrm{e}^{\mu|y-z|}\mathrm{d}y+\\
&\int_0^T\mathrm{e}^{-\beta_1 s}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 s}}\mathrm{e}^{-\frac{(z-cs-y)^2}{4d_1s}}(2\beta_1+\hat{\alpha}m)(|\tilde{\phi}_1(y,T-s)-\tilde{\phi}_2(y,T-s)|\mathrm{e}^{-\mu|y|}+\\
&|\tilde{\psi}_1(y,T-s)-\tilde{\psi}_2(y,T-s)|\mathrm{e}^{-\mu|y|})\mathrm{e}^{\mu|y-z|}\mathrm{d}y\mathrm{d}s\leqslant\\
&\mathrm{e}^{-\beta_1 T+\mu cT}\|\phi_1^*(0)-\phi_2^*(0)\|_\mu\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 T}}\mathrm{e}^{-\frac{(z-cT-y)^2}{4d_1T}}\mathrm{e}^{\mu|z-y-cT|}\mathrm{d}y+\\
&(2\beta_1+\hat{\alpha}m)(\|\tilde{\phi}_1-\tilde{\phi}_2\|_\mu+\|\tilde{\psi}_1-\tilde{\psi}_2\|_\mu)\int_0^T\mathrm{e}^{-\beta_1 s}\mathrm{e}^{\mu cs}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 s}}\mathrm{e}^{-\frac{(z-cs-y)^2}{4d_1s}}\mathrm{e}^{\mu|z-y-cs|}\mathrm{d}y\mathrm{d}s\leqslant\\
&2\mathrm{e}^{(d_1\mu^2+\mu c-\beta_1)T}\|\phi_1^*(0)-\phi_2^*(0)\|_\mu+(2\beta_1+\hat{\alpha}m)(\|\tilde{\phi}_1-\tilde{\phi}_2\|_\mu+\|\tilde{\psi}_1-\tilde{\psi}_2\|_\mu)\int_0^T2\mathrm{e}^{(d_1\mu^2+\mu c-\beta_1)s}\mathrm{d}s\leqslant\\
&2\mathrm{e}^{(d_1\mu^2+\mu c-\beta_1)T}\|\phi_1^*(0)-\phi_2^*(0)\|_\mu+\frac{2(2\beta_1+\hat{\alpha}m)(\mathrm{e}^{(d_1\mu^2+\mu c-\beta_1)T}-1)}{d_1\mu^2+\mu c-\beta_1}(\|\tilde{\phi}_1-\tilde{\phi}_2\|_\mu+\|\tilde{\psi}_1-\tilde{\psi}_2\|_\mu)\leqslant\\
&\frac{1}{2}\|\phi_1^*(0)-\phi_2^*(0)\|_\mu+\frac{2(2\beta_1+\hat{\alpha}m)(\mathrm{e}^{(d_1\mu^2+\mu c-\beta_1)T}-1)}{d_1\mu^2+\mu c-\beta_1}(\|\tilde{\phi}_1-\tilde{\phi}_2\|_\mu+\|\tilde{\psi}_1-\tilde{\psi}_2\|_\mu).
\end{aligned}
$$

记$L=\dfrac{4(2\beta_1+\hat{\alpha}m)(\mathrm{e}^{(d_1\mu^2+\mu c-\beta_1)T}-1)}{d_1\mu^2+\mu c-\beta_1}$, 定义$\phi_i^*(z,T;\tilde{\phi}_i,\tilde{\psi}_i)=\phi_i^*(z,0), i=1,2$, 则$\|\phi_1^*(0)-\phi_2^*(0)\|_\mu\leqslant L(\|\tilde{\phi}_1-\tilde{\phi}_2\|_\mu+\|\tilde{\psi}_1-\tilde{\psi}_2\|_\mu)$, 又因为$\phi_i^*(z,t;\tilde{\phi}_i,\tilde{\psi}_i)$满足

$$
\begin{aligned}
\phi_i^*(z,t;\tilde{\phi}_i,\tilde{\psi}_i)=&\,\mathrm{e}^{-\beta_1 t}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 t}}\mathrm{e}^{-\frac{(z-ct-y)^2}{4d_1t}}\phi_i^*(y,0)\mathrm{d}y+\\
&\int_0^t\mathrm{e}^{-\beta_1 s}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 s}}\mathrm{e}^{-\frac{(z-cs-y)^2}{4d_1s}}h_1[\tilde{\phi}_i,\tilde{\psi}_i](y,t-s)\mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

因此, 不难验证$\phi^*(z,t;\tilde{\phi},\tilde{\psi})$关于$(\tilde{\phi},\tilde{\psi})$是连续的. 类似可证$\psi^*(z,t;\tilde{\phi},\tilde{\psi})$关于$(\tilde{\phi},\tilde{\psi})$也是连续的. 证毕.

引理 8 算子$F:D\to D$在$B_\mu(\mathbb{R}\times[0,T],\mathbb{R}^2)$上是紧的.

证明 对任意的$(\tilde{\phi},\tilde{\psi})\in D$, 有$h_i[\tilde{\phi},\tilde{\psi}](\cdot,t)\in C(\mathbb{R}\times[0,T],C(\mathbb{R})), i=1,2$. 此外, $h_i[\tilde{\phi},\tilde{\psi}]$关于$(\tilde{\phi},\tilde{\psi})\in D$是一致有界的, 由文献 [21] 的定理 5.1.2 可知, 系统 (19) 定义的$(\phi,\psi)\in C(\mathbb{R}\times[0,T],\mathbb{R})\cap C^{2\theta,\theta}(\mathbb{R}\times[\varsigma,T],\mathbb{R})$, 其中$\varsigma\in(0,T)$, $\theta\in(0,1)$, 且存在$C_1(\varsigma,\theta)>0, C_2(\varsigma,\theta)>0$使得

$$\|\phi(\cdot,T)\|_{C^{2\theta}(\mathbb{R})}\leqslant C_1(\varsigma,\theta)(\varsigma^{-\theta}\|\phi_0\|_\infty+\left\|h_1[\tilde{\phi},\tilde{\psi}]\right\|_\infty), \tag{25}$$

$$\|\psi(\cdot,T)\|_{C^{2\theta}(\mathbb{R})}\leqslant C_2(\varsigma,\theta)(\varsigma^{-\theta}\|\psi_0\|_\infty+\left\|h_2[\tilde{\phi},\tilde{\psi}]\right\|_\infty). \tag{26}$$

令$F(\tilde{\phi},\tilde{\psi})=(\phi^*,\psi^*)$, 其中$(\phi^*,\psi^*)$是系统 (18) 的解. 由式 (25) 和 (26) 可知, 存在正数K使得$\|\phi^*(0)\|_{C^{2\theta}(\mathbb{R})}=\|\phi^*(T)\|_{C^{2\theta}(\mathbb{R})}\leqslant K$且$\|\psi^*(0)\|_{C^{2\theta}(\mathbb{R})}=\|\psi^*(T)\|_{C^{2\theta}(\mathbb{R})}\leqslant K$, 故由文献 [21] 的定理 5.1.2 可知, 对任意的$\theta\in(0,1)$, $\phi^*,\psi^*\in C^{2\theta,\theta}(\mathbb{R}\times[0,T],\mathbb{R})$且存在$C_1,C_2>0$和$\hat{K}>0$使得

$$\|\phi^*\|_{C^{2\theta,\theta}(\mathbb{R}\times[0,T])}\leqslant C_1(\|\phi^*(0)\|_{C^{2\theta}(\mathbb{R})}+\|h_1[\tilde{\phi},\tilde{\psi}]\|_\infty)\leqslant\hat{K}, \tag{27}$$

$$\|\psi^*\|_{C^{2\theta,\theta}(\mathbb{R}\times[0,T])}\leqslant C_2(\|\psi^*(0)\|_{C^{2\theta}(\mathbb{R})}+\|h_2[\tilde{\phi},\tilde{\psi}]\|_\infty)\leqslant\hat{K}. \tag{28}$$

因此, 在$C_{\mathrm{loc}}(\mathbb{R}\times[0,T],\mathbb{R}^2)$中可以找到函数$(\phi^{**},\psi^{**})$, 使得$\{(\phi_n^*,\psi_n^*)\}$存在收敛于$(\phi^{**},\psi^{**})$的子列, 不妨将该子列仍记为$\{(\phi_n^*,\psi_n^*)\}$, 即对任意的$N\in\mathbb{R}_+$, 有

$$\lim_{n\to\infty}\left\|(\phi_n^*(z,t),\psi_n^*(z,t))-(\phi^{**}(z,t),\psi^{**}(z,t))\right\|_{C([-N,N]\times[0,T],\mathbb{R}^2)}=0. \tag{29}$$

显然$(\phi^{**},\psi^{**})\in D$.

下证$\lim_{n\to\infty}\left\|(\phi_n^*(z,t),\psi_n^*(z,t))-(\phi^{**}(z,t),\psi^{**}(z,t))\right\|_\mu=0$.

由于D关于范数$\|\cdot\|_\mu$一致有界, 因此, 范数$\left\|(\phi_n^*,\psi_n^*)-(\phi^{**},\psi^{**})\right\|_\mu$对所有的$n\in\mathbb{N}$一致有界. 任意给定的$\rho>0$,可以找到一个$Z^*>0$, 使得对任意$t\in[0,T]$, $|z|>Z^*$和$n\in\mathbb{N}$,

$$\mathrm{e}^{-\mu|z|}|(\phi_n^*(z,t),\psi_n^*(z,t))-(\phi^{**}(z,t),\psi^{**}(z,t))|<\rho.$$

又由式 (29) 知, 存在$H\in\mathbb{N}$, 使得对任意$t\in[0,T]$, $z\in[-Z^*,Z^*]$和$n>H$,

$$\mathrm{e}^{-\mu|z|}|(\phi_n^*(z,t),\psi_n^*(z,t))-(\phi^{**}(z,t),\psi^{**}(z,t))|<\rho.$$

所以当$n\to\infty$时, $(\phi_n^*(z,t),\psi_n^*(z,t))\to(\phi^{**}(z,t),\psi^{**}(z,t))$. 因此, 算子$F:D\to D$在$B_\mu(\mathbb{R}\times[0,T],\mathbb{R}^2)$上是紧的. 证毕.

2.3 周期行波解

本小节建立当$\mathfrak{R}_0>1,c>c^*$时系统 (5) 周期行波解的存在性.

引理 9　假设$\mathfrak{R}_0>1,c>c^*$, 则系统 (9) 存在周期解$(\phi^*(z,t),\psi^*(z,t))$.

证明　由引理 7 和引理 8, 结合 Schauder 不动点定理可知, 算子 F 存在不动点$(\phi^*(z,t),\psi^*(z,t))\in D$, $(\phi^*(\cdot,0),\psi^*(\cdot,0))=(\phi^*(\cdot,T),\psi^*(\cdot,T))$, 且满足

$$\begin{cases}\phi^*(t)=T_1(t)\phi^*(0)+\int_0^t T_1(t-s)h_1[\phi^*,\psi^*](s)\mathrm{d}s, & t\in[0,T],\\ \psi^*(t)=T_2(t)\psi^*(0)+\int_0^t T_2(t-s)h_2[\phi^*,\psi^*](s)\mathrm{d}s, & t\in[0,T].\end{cases}$$

定义$(\bar\phi^*(z,t),\bar\psi^*(z,t))=(\phi^*(z,t-nT),\psi^*(z,t-nT))$, $n\in\mathbb{Z}$且满足$nT\leqslant t\leqslant(n+1)T$. 显然对任意的$(z,t)\in\mathbb{R}\times\mathbb{R}$, $\bar\phi^*(z,t)$, $\bar\psi^*(z,t)$是 T 周期的, 故$(\bar\phi^*(z,t),\bar\psi^*(z,t))$满足

$$\begin{cases}\bar\phi^*(t)=T_1(t)\bar\phi^*(0)+\int_0^t T_1(t-s)h_1[\bar\phi^*,\bar\psi^*](s)\mathrm{d}s, & t\in\mathbb{R},\\ \bar\psi^*(t)=T_2(t)\bar\psi^*(0)+\int_0^t T_2(t-s)h_2[\bar\phi^*,\bar\psi^*](s)\mathrm{d}s, & t\in\mathbb{R}.\end{cases}\tag{30}$$

又$(\phi^*,\psi^*)\in C^{2\theta,\theta}(\mathbb{R}\times[0,T],\mathbb{R}^2)$, 则$(\bar\phi^*(z,t),\bar\psi^*(z,t))\in C^{2\theta,\theta}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$. 记$(\bar\phi^*,\bar\psi^*)$仍为(ϕ^*,ψ^*). 由文献 [21] 的定理 5.1.2 ~ 5.1.4 可知, $(\phi^*,\psi^*)\in C^{2+2\theta,1}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$满足

$$\begin{cases}\phi_t^*(z,t)=d_1\phi_{zz}^*(z,t)-c\phi_z^*(z,t)+\Lambda(t)-\alpha(t)f(\phi^*(z,t),\psi^*(z,t))-\mu(t)\phi^*(z,t),\\ \psi_t^*(z,t)=d_2\psi_{zz}^*(z,t)-c\psi_z^*(z,t)+\alpha(t)f(\phi^*(z,t),\psi^*(z,t))-(\mu(t)+\gamma(t))\psi^*(z,t)-\varepsilon[\psi^*(z,t)]^2,\end{cases}\tag{31}$$

其中$(z,t)\in\mathbb{R}\times\mathbb{R}$. 此外, 对任意$\theta\in(0,1)$,

$$\|\phi^*\|_{C^{2+2\theta,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})}+\|\psi^*\|_{C^{2+2\theta,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})}<\infty.\tag{32}$$

定理 2　假设$\mathfrak{R}_0>1,c>c^*$, 则系统 (7) 存在周期解$(\phi^*(z,t),\psi^*(z,t))$.

证明　令$\{\varepsilon_n\}\subset(0,1)$是一个递减序列, 且当$n\to\infty$时$\varepsilon_n\to0$. 由引理 9 可知, 系统 (9) 的一个解序列$\{(\phi_n^*(z,t),\psi_n^*(z,t))\}$在$\varepsilon=\varepsilon_n(n=1,2,\cdots)$时满足

$$\begin{cases}(\phi_n^*)_t=d_1(\phi_n^*)_{zz}-c(\phi_n^*)_z+\Lambda(t)-\alpha(t)f(\phi_n^*,\psi_n^*)-\mu(t)\phi_n^*,\\ (\psi_n^*)_t=d_2(\psi_n^*)_{zz}-c(\psi_n^*)_z+\alpha(t)f(\phi_n^*,\psi_n^*)-(\mu(t)+\gamma(t))\psi_n^*-\varepsilon_n{\psi_n^*}^2.\end{cases}\tag{33}$$

此外, $\phi^-(z,t)\leqslant\phi_n^*(z,t)\leqslant\phi^+(z,t)$, $\psi^-(z,t)\leqslant\psi_n^*(z,t)\leqslant\psi^+(z,t)$, $n\in\mathbb{N}_+$, 因此, $(\phi_n^*(z,t),\psi_n^*(z,t))$是一致有界且等度连续的, 由 Arzela-Ascoli 定理可知, 当$k\to\infty$时, 存在一个一致收敛的子序列$(\phi_{n_k}^*(z,t),\psi_{n_k}^*(z,t))$收敛到$(\phi^*(z,t),\psi^*(z,t))$, 所以当$k\to\infty$时

$$\begin{cases}\phi_t^*(z,t)=d_1\phi_{zz}^*(z,t)-c\phi_z^*(z,t)+\Lambda(t)-\alpha(t)f(\phi^*(z,t),\psi^*(z,t))-\mu(t)\phi^*(z,t),\\ \psi_t^*(z,t)=d_2\psi_{zz}^*(z,t)-c\psi_z^*(z,t)+\alpha(t)f(\phi^*(z,t),\psi^*(z,t))-(\mu(t)+\gamma(t))\psi^*(z,t).\end{cases}\tag{34}$$

即系统 (7) 存在一个周期解$(\phi^*(z,t),\psi^*(z,t))$. 证毕.

为证明系统 (6) 的周期行波解满足渐近边界条件, 还需要下面的条件成立:

(A3) 当$\xi\in(0,\max_{t\in[0,T]}S_0(t))$, $\eta\in[0,A]$时, 有$\min_{t\in[0,T]}(\alpha(t)\partial_2f(\xi,\eta))\geqslant\max_{t\in[0,T]}(\mu(t)+\gamma(t))$.

下面证明行波系统 (7) 的周期解满足渐近边界条件.

定理 3　假设$\mathfrak{R}_0>1,c>c^*$, 则系统 (7) 的周期解满足

$$\lim_{z\to-\infty}\phi(z,t)=S_0(t),\ \lim_{z\to-\infty}\psi(z,t)=0,\ \lim_{z\to+\infty}\phi(z,t)=S^*(t),\ \lim_{z\to+\infty}\psi(z,t)=I^*(t).$$

证明 由$\phi^{\pm},\psi^{\pm}$的定义,容易验证$\lim_{z\to-\infty}\phi(z,t)=S_0(t),\lim_{z\to-\infty}\psi(z,t)=0$. 由估计式 (32) 和文献 [22] 中的 Landau 型不等式可得

$$|\phi_z^*|_{L^\infty(\mathbb{R}\times[0,T])}\leqslant 2|\phi^*-S_0(t)|^{1/2}_{L^\infty(\mathbb{R}\times[0,T])}|\phi_{zz}^*|^{1/2}_{L^\infty(\mathbb{R}\times[0,T])},$$

$$|\psi_z^*|_{L^\infty(\mathbb{R}\times[0,T])}\leqslant 2|\psi^*|^{1/2}_{L^\infty(\mathbb{R}\times[0,T])}|\psi_{zz}^*|^{1/2}_{L^\infty(\mathbb{R}\times[0,T])}.$$

因此, $\lim_{z\to-\infty}(\phi_z^*,\psi_z^*)=(0,0)$关于$t\in\mathbb{R}$一致成立. 另外, 由强最大值原理可知, $\phi^*(t,z)>0,\psi^*(t,z)>0,\forall t>0,z\in\mathbb{R}$, 对系统 (34) 两边关于 z 微分可得

$$\begin{cases}(\phi_z^*)_t=d_1(\phi_z^*)_{zz}-c(\phi_z^*)_z-\alpha(t)[\partial_1 f(\phi^*,\psi^*)\phi_z^*+\partial_2 f(\phi^*,\psi^*)\psi_z^*]-\mu(t)\phi_z^*,\\(\psi_z^*)_t=d_2(\psi_z^*)_{zz}-c(\psi_z^*)_z+\alpha(t)[\partial_1 f(\phi^*,\psi^*)\phi_z^*+\partial_2 f(\phi^*,\psi^*)\psi_z^*]-[\mu(t)+\gamma(t)]\psi_z^*.\end{cases}\tag{35}$$

类似地, 由ϕ^*,ψ^*的周期性及文献 [21] 的定理 5.1.3、5.1.4 可知, 对任意的$\theta\in(0,1)$, $\|\phi_z^*\|_{C^{2+2\theta,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})}<+\infty$, $\|\psi_z^*\|_{C^{2+2\theta,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})}<+\infty$, 对$\phi^*,\psi^*$运用类似的方法, 由 Landau 型不等式可知$\lim_{z\to-\infty}(\phi_{zz}^*,\psi_{zz}^*)=(0,0)$关于$t\in\mathbb{R}$一致成立. 令$\Psi(z)=\dfrac{1}{T}\int_0^T\psi^*(z,t)\mathrm{d}t$, 显然$\lim\limits_{z\to-\infty}\Psi_z(z)=0$, 由系统 (34) 的第二个方程可知

$$d_2\Psi_{zz}=c\Psi_z-\frac{1}{T}\int_0^T[\alpha(t)f(\phi^*,\psi^*)-(\mu(t)+\gamma(t))\psi^*]\mathrm{d}t.\tag{36}$$

对式 (36) 从$-\infty$到 z 积分, 可得

$$d_2\Psi_z(z)=c\Psi(z)-\frac{1}{T}\int_{-\infty}^{z}\int_0^T[\alpha(t)f(\phi^*(y,t),\psi^*(y,t))-(\mu(t)+\gamma(t))\psi^*(y,t)]\mathrm{d}t\mathrm{d}y.\tag{37}$$

由于$\psi^*(z,t),\psi_z^*(z,t)$一致有界, 则$\Psi(z)=\dfrac{1}{T}\int_0^T\psi^*(z,t)\mathrm{d}t$, $\Psi_z(z)=\dfrac{1}{T}\int_0^T\psi_z^*(z,t)\mathrm{d}t$一致有界, 从而$\dfrac{1}{T}\int_0^T[\alpha(t)f(\phi^*,\psi^*)-(\mu(t)+\gamma(t))\psi^*]\mathrm{d}t$在$\mathbb{R}$上可积, 由式 (36) 可得

$$(\mathrm{e}^{-\frac{cz}{d_2}}\Psi_z)_z=\mathrm{e}^{-\frac{cz}{d_2}}\left[\Psi_{zz}-\frac{c}{d_2}\Psi_z\right]=-\frac{\mathrm{e}^{-\frac{cz}{d_2}}}{d_2T}\int_0^T[\alpha(t)f(\phi^*,\psi^*)-(\mu(t)+\gamma(t))\psi^*]\mathrm{d}t.$$

对上式从 z 到$+\infty$积分并利用微分中值定理可得

$$\begin{aligned}\mathrm{e}^{-\frac{cz}{d_2}}\Psi_z(z)&=\frac{1}{d_2T}\int_z^{+\infty}\mathrm{e}^{-\frac{cy}{d_2}}\int_0^T[\alpha(t)f(\phi^*(y,t),\psi^*(y,t))-(\mu(t)+\gamma(t))\psi^*(y,t)]\mathrm{d}t\mathrm{d}y=\\&\frac{1}{d_2T}\int_z^{+\infty}\mathrm{e}^{-\frac{cy}{d_2}}\int_0^T[\alpha(t)\partial_2 f(\xi,\eta)-(\mu(t)+\gamma(t))]\psi^*(y,t)\mathrm{d}t\mathrm{d}y,\end{aligned}$$

这里$\xi\in(0,\max_{t\in[0,T]}S_0(t)),\eta\in[0,A]$. 结合条件 (A3) 可知$\Psi_z(z)\geqslant 0$, 又$\Psi(z)$一致有界且$\Psi(-\infty)=\dfrac{1}{T}\int_0^T\psi^*(-\infty,t)\mathrm{d}t=0$, 则存在$\Psi^*$使得$\Psi(+\infty)=\Psi^*$. 因此, $\Psi(+\infty)=\dfrac{1}{T}\int_0^T\psi^*(+\infty,t)\mathrm{d}t=\Psi^*$, 即存在$I^*(t)$使得$\lim_{z\to+\infty}\psi^*(z,t)=I^*(t)$.

下证$\lim_{z\to+\infty}\phi^*(z,t)=S^*(t)$. 假设$\lim_{z\to+\infty}\phi^*(z,t)=0$, 根据系统 (34) 的第一个方程可得$\Lambda(t)=0$, 这与已知矛盾, 故$\lim_{z\to+\infty}\phi^*(z,t)=S^*(t)$. 证毕.

至此, 由定理 2 和定理 3 得到了系统 (5) 周期行波解的存在性.

3 周期行波解的不存在性

本节借助分析技术证明当$\Re_0\leqslant 1,c\geqslant 0$时系统 (5) 周期行波解的不存在性.

定理 4 假设$\Re_0\leqslant 1,c\geqslant 0$, 则系统 (6) 不存在满足

$$\lim_{z\to-\infty}\phi(z,t)=S_0(t),\ \lim_{z\to-\infty}\psi(z,t)=0,\ \lim_{z\to+\infty}\phi(z,t)=S^*(t),\ \lim_{z\to+\infty}\psi(z,t)=I^*(t)\tag{38}$$

的周期行波解.

证明 利用反证法. 假设系统 (6) 存在满足 (38) 的周期行波解$(\phi^*(z,t),\psi^*(z,t))$. 则对任意的$t>0$, $z\in\mathbb{R}$, 有

$$\psi_t^*=d_2\psi_{zz}^*-c\psi_z^*+\alpha(t)f(\phi^*,\psi^*)+\alpha(t)\partial_2 f(S_0(t),0)\psi^*-(\mu(t)+\gamma(t))\psi^*-\alpha(t)\partial_2 f(S_0(t),0)\psi^*.\tag{39}$$

令$\tilde{\Psi}(t)=\int_{\mathbb{R}}\psi^*(z,t)\mathrm{d}z$. 将式 (39) 两边关于变量$z$在$\mathbb{R}$上积分并结合渐近边界条件 (38), 可得$\tilde{\Psi}_t(t)=[\alpha(t)\partial_2 f(S_0(t),0)-$

$\mu(t)-\gamma(t)]\tilde{\Psi}(t)+q(t)$, 其中$q(t)=\alpha(t)\int_{\mathbb{R}}[f(\phi^*,\psi^*)-\partial_2 f(S_0(t),0)\psi^*]\mathrm{d}z-cI^*(t)$. 由条件 (A1) 和 (A2) 可知$q(t)<0$. 又$\tilde{\Psi}(t)$和$q(t)$都是$T$周期函数, 从而有

$$\frac{\tilde{\Psi}_t(t)}{\tilde{\Psi}(t)}=\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)+\frac{q(t)}{\tilde{\Psi}(t)}. \tag{40}$$

对式 (40) 从0到 T 积分, 结合函数$\tilde{\Psi}(t)$的周期性, 可得

$$0=\int_0^T[\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]\mathrm{d}t+\int_0^T\frac{q(t)}{\tilde{\Psi}(t)}\mathrm{d}t. \tag{41}$$

另外, 当$\Re_0=\dfrac{\int_0^T\alpha(t)\partial_2 f(S_0(t),0)\mathrm{d}t}{\int_0^T[\mu(t)+\gamma(t)]\,\mathrm{d}t}\leqslant 1$时, 有$\int_0^T[\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]\mathrm{d}t\leqslant 0$. 则由式 (41) 和$q(t)<0$可知

$$0=\int_0^T[\alpha(t)\partial_2 f(S_0(t),0)-\mu(t)-\gamma(t)]\mathrm{d}t+\int_0^T\frac{q(t)}{\tilde{\Psi}(t)}\mathrm{d}t<0.$$

这出现了矛盾, 故假设不成立, 证毕.

4 应用举例

本节给出系统 (5) 周期行波解存在性和不存在性的具体应用.

当$f(S(x,t),I(x,t))=\dfrac{S(x,t)I(x,t)}{S(x,t)+I(x,t)}$时, 系统 (5) 化为具有标准发生率的 SIR 传染病系统:

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=d_1\Delta S(x,t)+\Lambda(t)-\dfrac{\alpha(t)S(x,t)I(x,t)}{S(x,t)+I(x,t)}-\mu(t)S(x,t),\\ \dfrac{\partial I(x,t)}{\partial t}=d_2\Delta I(x,t)+\dfrac{\alpha(t)S(x,t)I(x,t)}{S(x,t)+I(x,t)}-(\mu(t)+\gamma(t))I(x,t),\\ \dfrac{\partial R(x,t)}{\partial t}=d_3\Delta R(x,t)+\gamma(t)I(x,t)-\mu(t)R(x,t).\end{cases} \tag{42}$$

不难验证, 当$S,I>0$时, $f(0,I)=f(S,0)=0$, 且当$S,I\geqslant 0$时, $f(S,I)=\dfrac{SI}{S+I}$是二阶连续可微的, 即条件 (A1) 成立. 另外, 当$S>0,I\geqslant 0$时, $\partial_2 f(S,I)=\left(\dfrac{S}{S+I}\right)^2>0, \partial_{22}f(S,I)=-\dfrac{2S^2}{(S+I)^3}<0$; 当$S\geqslant 0,I>0$时, $0<\partial_1 f(S,I)=\left(\dfrac{I}{S+I}\right)^2\leqslant 1$, 所以条件 (A2) 成立. 此外, 若系统 (42) 还满足当$\xi\in(0,\max_{t\in[0,T]}S_0(t)),\eta\in[0,A]$时, 有

$$\min_{t\in[0,T]}\left[\alpha(t)\left(\frac{\xi}{\xi+\eta}\right)^2\right]\geqslant\max_{t\in[0,T]}[\mu(t)+\gamma(t)],$$

则条件 (A3) 成立.

因此, 由定理 2 ~ 4 可得系统 (42) 周期行波解的存在性和不存在性, 即推论 1 和推论 2.

推论 1　若$\Re_0>1,c>c^*$, 则系统 (42) 存在周期行波解$(\phi^*(z,t),\psi^*(z,t))$, 满足渐近边界条件

$$\lim_{z\to-\infty}\phi(z,t)=S_0(t), \lim_{z\to-\infty}\psi(z,t)=0, \lim_{z\to+\infty}\phi(z,t)=S^*(t), \lim_{z\to+\infty}\psi(z,t)=I^*(t).$$

推论 2　若$\Re_0\leqslant 1,c\geqslant 0$, 则系统 (42) 不存在满足渐近边界条件

$$\lim_{z\to-\infty}\phi(z,t)=S_0(t), \lim_{z\to-\infty}\psi(z,t)=0, \lim_{z\to+\infty}\phi(z,t)=S^*(t), \lim_{z\to+\infty}\psi(z,t)=I^*(t)$$

的周期行波解.

5 结　　论

本文研究了一类带有外部输入项的时间周期 SIR 传染病模型周期行波解的存在性和不存在性. 首先, 通过构造适当的上下解定义闭凸锥, 结合 Schauder 不动点定理建立了$\Re_0>1,c>c^*$时系统 (5) 周期行波解的存在性. 接下来, 借助分析技术证明了$\Re_0\leqslant 1,c\geqslant 0$时系统 (5) 周期行波解的不存在性. 注意到, 由于系统 (5) 中的扩散项是 Laplace 扩散, 它主要适用于小范围的传染病扩散. 因此, 系统 (5) 对应的非局部扩散系统周期行波解的

存在性和不存在性是今后可以继续研究的问题.

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