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基于 Green 函数分析 Euler-Bernoulli 双曲梁系统的受迫振动*

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摘要： 双曲梁系统通常出现在许多工程领域.与双直梁系统相比,该系统在噪声和振动控制问题上的效率更高.该文采用经典的 Euler-Bernoulli 曲梁模型来模拟双曲梁系统,通过 Green 函数和 Laplace 变换方法得到双曲梁系统稳态受迫振动的闭合形式解,该解可用于任何边界条件.在数值部分,通过与参考文献中的一些结果进行比较来验证本方案的解.讨论了一些重要的几何和物理参数对振动响应的影响以及弹性层刚度与双曲梁系统之间的相互作用.结果表明,梁的半径趋于无穷大时,双曲梁系统退化为双直梁系统,此外,双曲梁系统也可以简化为一个直梁和一个曲梁的组合形式.

关　键　词： Green 函数；　Euler-Bernoulli 梁模型；　双曲梁系统；　Winkler 型弹性层；　Laplace 变换

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Forced Vibration Analysis of Euler-Bernoulli Double-Beam Systems by Means of Green's Functions

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Abstract: Double-curved-beam (DCB) systems are usually seen in many engineering fields. Compared to straight double-beam systems, DCB systems are more efficient in noise and vibration control problems. To obtain closed-form solutions of steady-state forced vibrations of DCB systems, the classical Euler-Bernoulli curved beam (ECB) model was employed to model vibration equations for the DCB systems. Green's functions and the Laplace transform methods were used to get the closed-form solutions to the vibration equations for the DCB systems. These solutions apply to arbitrary boundary conditions. Numerical tests were conducted to verify the present solutions with related results from previous literatures. Effects of some important geometric and physical parameters on vibration responses and the interaction between the elastic layer stiffness and the DCB system, were discussed. The results show that, the DCB system will degenerate to a straight double-beam system when the 2 radii approach infinity, moreover, the DCB system can be simplified as one comprising a straight beam and a curved beam.

Key words: Green's function; Euler-Bernoulli beam model; double-curved-beam system; Winkler elastic layer; Laplace transform

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0 引　言

所谓的双梁系统是由两个独立梁通过弹性层连接而成的结构, 这种结构具有良好的物理性能, 例如比单梁拥有更好的减震效果、更轻的重量以及更高的强度和刚度[1-2], 被广泛应用于机械、航空和土木工程等工程领域.

近年来, 学者们对双梁系统的复杂动力学进行了许多研究, Euler-Bernoulli、Rayleigh 和 Timoshenko 等各种各样的经典模型依次被提出. 针对 Euler-Bernoulli 双梁, Oniszczuk[3-4] 利用 Bernoulli-Fourier 法以及模态展开法, 分析了带有 Winkler 弹性层的简支双直梁系统的自由振动和强迫振动. Zhang 等[5] 讨论了由任意分布的连续载荷引起的双直梁系统的动态响应, 分析了在两种特殊激励载荷下轴向压缩载荷对双直梁系统强迫振动的影响. Wu 等[6] 研究了移动振荡器下 Euler-Bernoulli 双直梁系统的动态响应, 基于 Timoshenko 双梁, 考虑梁的剪切变形和转动惯量的影响. Stojanović等[7] 推导了两个梁的固有频率和相关振幅比的显式表达式, 研究了 Timoshenko-Rayleigh 双直梁系统的经典自由振动、屈曲和强迫振动问题. Zhao 等[8] 利用 Laplace 变换和 Green 函数法, 借助叠加原理得到了轴向压缩荷载作用下耦合 Timoshenko 双直梁系统横向强迫振动的闭式解, 讨论了一些物理参数对闭合解的影响. 在这些研究中, 双梁系统的两个梁均被视为直梁. 随着工程领域的不断发展, 在很多实际的工程问题中[9-10], 受到不同地质、地形条件及环境的影响, 直梁结构不再能满足工程需求, 而弯曲构件凭借其流畅的线条及较高的承载力得以广泛应用[11], 因此出现了很多双曲结构, 例如双曲拱桥、双曲管线和双曲机械构件等[10, 12-13], 这对双梁系统的理论研究朝着弯曲构件的方向发展产生了极大的影响, 其动力学问题也成为研究的焦点. 由于双曲梁系统是存在初始曲率的双梁系统, 其力学特性复杂, 分析起来更加困难, 因此学者们利用不同的方法离散和分析了该系统的控制方程, 这些方法大致可分为数值解法、半解析法和有限元法. Shin 等[14] 将广义微分求积法和微分变换法应用于变截面圆拱的振动分析, 推导了运动控制方程, 得到了各种边界条件下的无量纲固有频率. Sobhani 等[15] 采用 Hamilton 原理和 Green-Gauss 理论, 以广义微分求积算子闻名的鲁棒半解析技术进行求解, 使用标准特征值的解, 获得了双曲梁系统的固有频率. 此外, Stojanović等[16] 还采用 p 型有限元方法研究了一曲一直梁-拱耦合系统的几何非线性响应, 同时采用谐波平衡法和连续法在频域进行非线性分析. 综合笔者的文献调研, 发现以往的研究中大多是通过数值模拟的方法得到系统的数值解, 没有提供双曲梁系统中两个梁挠度的解析解, 在对双曲梁系统的受迫振动进行全面、精确的分析时具有一定的局限性, 使其应用受到较大限制.

因此, 本文研究了由两个 Euler-Bernoulli 曲梁和它们之间的弹性层组成的双曲梁系统的受迫振动, 使用 Green 函数和 Laplace 变换方法, 得到了任意边界条件下双曲梁系统受迫振动的解析解. 该解可以退化为 Euler-Bernoulli 双直梁、Euler-Bernoulli 单曲梁、Euler-Bernoulli 单直梁的广义解析解. 在数值计算中, 通过双曲梁系统的退化解与已发表文献的结果对比, 验证了双曲梁系统解析解的有效性. 最后, 研究了不同条件下中间弹性层刚度、几何特征和外激励频率对双曲梁系统的影响.

1 双曲梁系统模型的建立

双曲梁系统在 $x = x_0$ 处受到简谐集中力载荷下的物理模型如图 1 所示, 该系统由两个梁高 h、梁宽 b、梁长 L、曲率半径 R 的平行 Euler-Bernoulli 曲梁组成, 通过 Winkler 型弹性层连接. 该系统的运动方程可由以下偏微分方程表示[17-18]:

$$\left(E_1A_1+\frac{E_1I_1}{R_1^2}\right)v_1''+\frac{E_1A_1}{R_1}w_1'-\frac{E_1I_1}{R_1}w_1'''=\mu_1\ddot{v}_1, \tag{1}$$

$$-E_1I_1w_1''''+\frac{E_1I_1}{R_1}v_1'''-\frac{E_1A_1}{R_1}v_1'-\frac{E_1A_1}{R_1^2}w_1+K(w_1-w_2)=\mu_1\ddot{w}_1+p_1(x,t), \tag{2}$$

$$\left(E_2A_2+\frac{E_2I_2}{R_2^2}\right)v_2''+\frac{E_2A_2}{R_2}w_2'-\frac{E_2I_2}{R_2}w_2'''=\mu_2\ddot{v}_2, \tag{3}$$

$$-E_2I_2w_2'''' + \frac{E_2I_2}{R_2}v_2''' - \frac{E_2A_2}{R_2}v_2' - \frac{E_2A_2}{R_2^2}w_2 + K(w_2 - w_1) = \mu_2\ddot{w}_2 + p_2(x,t), \tag{4}$$

其中，w_i，v_i分别为曲梁的径向位移、轴向位移；E_iA_i为曲梁的抗拉刚度；E_iI_i为曲梁的抗弯刚度；μ_i为曲梁单位长度的质量；K 为 Winkler 弹性层的刚度模量；$p_i(x,t)$是施加在梁上的简谐集中荷载；R_i表示曲梁的曲率半径；“·”表示对时间t的导数；“′”表示关于空间坐标x的导数；下标i表示梁号，$i=1$时表示上梁，$i=2$时表示下梁.

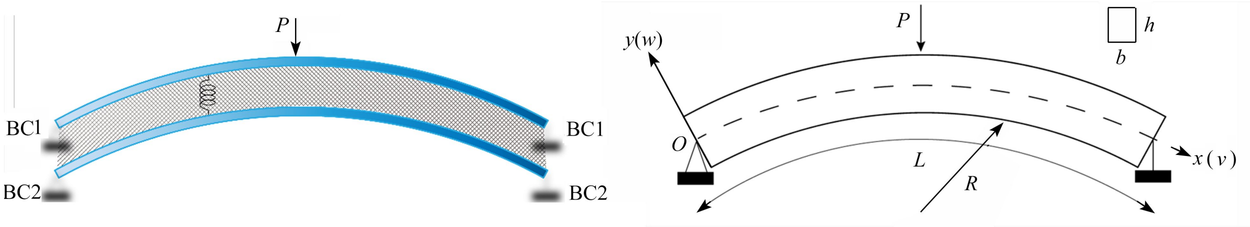

图 1 任意边界条件下的双曲梁系统在$x=x_0$处受到外激励作用

Fig. 1 The DCB system with arbitrary boundary conditions subjected to a load

假设梁上作用的简谐荷载为$p(x,t)=\mathrm{e}^{\mathrm{i}\Omega t}$，其中$\Omega$是荷载的激励频率，则相应的径向位移和轴向位移也可以分别假设为$w(x,t)=W(x)\mathrm{e}^{\mathrm{i}\Omega t}$和$v(x,t)=V(x)\mathrm{e}^{\mathrm{i}\Omega t}$. 为了简化系统，利用$p(x,t)=P(x)\mathrm{e}^{\mathrm{i}\Omega t}$，$w(x,t)=W(x)\mathrm{e}^{\mathrm{i}\Omega t}$和$v(x,t)=V(x)\mathrm{e}^{\mathrm{i}\Omega t}$，将方程 (1)~(4) 分离变量，消除时间变量$t$，则分离变量后的控制方程为

$$-\frac{E_1I_1}{R_1}W_1''' + \frac{E_1A_1}{R_1}W_1' + \left(E_1A_1 + \frac{E_1I_1}{R_1^2}\right)V_1'' + \mu_1\Omega^2V_1 = 0, \tag{5}$$

$$-E_1I_1W_1'''' + \left(\mu_1\Omega^2 - \frac{E_1A_1}{R_1^2} + K\right)W_1 - KW_2 + \frac{E_1I_1}{R_1}V_1''' - \frac{E_1A_1}{R_1}V_1' = P_1, \tag{6}$$

$$-\frac{E_2I_2}{R_2}W_2''' + \frac{E_2A_2}{R_2}W_2' + \left(E_2A_2 + \frac{E_2I_2}{R_2^2}\right)V_2'' + \mu_2\Omega^2V_2 = 0, \tag{7}$$

$$-E_2I_2W_2'''' + \left(\mu_2\Omega^2 - \frac{E_2A_2}{R_2^2} + K\right)W_2 - KW_1 + \frac{E_2I_2}{R_2}V_2''' - \frac{E_2A_2}{R_2}V_2' = P_2. \tag{8}$$

然后分别对式 (5) 和式 (7) 关于x求导得到关于V_i'和V_i'''的表达式，将其分别代入式 (6) 和式 (8)，即可将V_i'表示为

$$\begin{aligned}V_1' = {}& \frac{-E_1A_1 - \dfrac{E_1I_1}{R_1^2}}{\dfrac{E_1A_1E_1A_1}{R_1} + \dfrac{E_1I_1E_1A_1}{R_1^3} + \dfrac{E_1I_1}{R_1}\mu_1\Omega^2}P_1 + \frac{-E_1I_1E_1A_1}{\dfrac{E_1A_1E_1A_1}{R_1} + \dfrac{E_1I_1E_1A_1}{R_1^3} + \dfrac{E_1I_1}{R_1}\mu_1\Omega^2}W_1'''' + \\ & \frac{-\dfrac{E_1I_1E_1A_1}{R_1^2}}{\dfrac{E_1A_1E_1A_1}{R_1} + \dfrac{E_1I_1E_1A_1}{R_1^3} + \dfrac{E_1I_1}{R_1}\mu_1\Omega^2}W_1'' + \\ & \frac{E_1A_1\mu_1\Omega^2 + \dfrac{E_1I_1}{R_1^2}\mu_1\Omega^2 - \dfrac{E_1A_1E_1A_1}{R_1^2} - \dfrac{E_1I_1E_1A_1}{R_1^4} + KE_1A_1 + \dfrac{KE_1I_1}{R_1^2}}{\dfrac{E_1A_1E_1A_1}{R_1} + \dfrac{E_1I_1E_1A_1}{R_1^3} + \dfrac{E_1I_1}{R_1}\mu_1\Omega^2}W_1 + \\ & \frac{-E_1A_1K - \dfrac{E_1I_1K}{R_1^2}}{\dfrac{E_1A_1E_1A_1}{R_1} + \dfrac{E_1I_1E_1A_1}{R_1^3} + \dfrac{E_1I_1}{R_1}\mu_1\Omega^2}W_2,\end{aligned} \tag{9}$$

$$V_2' = \frac{-E_2A_2 - \dfrac{E_2I_2}{R_2^2}}{\dfrac{E_2A_2E_2A_2}{R_2} + \dfrac{E_2I_2E_2A_2}{R_2^3} + \dfrac{E_2I_2}{R_2}\mu_2\Omega^2} P_2 + \frac{-E_2I_2E_2A_2}{\dfrac{E_2A_2E_2A_2}{R_2} + \dfrac{E_2I_2E_2A_2}{R_2^3} + \dfrac{E_2I_2}{R_2}\mu_2\Omega^2} W_1'''' +$$
$$\frac{-\dfrac{E_2I_2E_2A_2}{R_2^2}}{\dfrac{E_2A_2E_2A_2}{R_2} + \dfrac{E_2I_2E_2A_2}{R_2^3} + \dfrac{E_2I_2}{R_2}\mu_2\Omega^2} W_2'' +$$
$$\frac{E_2A_2\mu_2\Omega^2 + \dfrac{E_2I_2}{R_2^2}\mu_2\Omega^2 - \dfrac{E_2A_2E_2A_2}{R_2^2} - \dfrac{E_2I_2E_2A_2}{R_2^4} + KE_2A_2 + \dfrac{KE_2I_2}{R_2^2}}{\dfrac{E_2A_2E_2A_2}{R_2} + \dfrac{E_2I_2E_2A_2}{R_2^3} + \dfrac{E_2I_2}{R_2}\mu_2\Omega^2} W_2 +$$
$$\frac{-E_2A_2K - \dfrac{E_2I_2K}{R_2^2}}{\dfrac{E_2A_2E_2A_2}{R_2} + \dfrac{E_2I_2E_2A_2}{R_2^3} + \dfrac{E_2I_2}{R_2}\mu_2\Omega^2} W_1. \tag{10}$$

再将式 (9) 和式 (10) 求导分别代入式 (6) 和式 (8), 即得到双曲梁系统的振动控制方程:

$$W_1^{(6)} + \left(\frac{2}{R_1^2} + \frac{\mu_1\Omega^2}{E_1A_1}\right)W_1^{(4)} + \left(\frac{1}{R_1^4} - \frac{\mu_1\Omega^2}{E_1I_1} - \frac{\mu_1\Omega^2}{E_1A_1R_1^2} - \frac{K}{E_1I_1} - \frac{K}{E_1A_1R_1^2}\right)W_1'' +$$
$$\left(\frac{\mu_1\Omega^2}{E_1I_1R_1^2} - \frac{\mu_1\Omega^2\mu_1\Omega^2}{E_1A_1E_1I_1} - \frac{K\mu_1\Omega^2}{E_1A_1E_1I_1}\right)W_1 + \left(\frac{K}{E_1I_1} + \frac{K}{E_1A_1R_1^2}\right)W_2'' +$$
$$\frac{K\mu_1\Omega^2}{E_1A_1E_1I_1}W_2 = -\left(\frac{1}{E_1I_1} + \frac{1}{E_1A_1R_1^2}\right)P_1'' - \frac{\mu_1\Omega^2}{E_1A_1E_1I_1}P_1, \tag{11}$$

$$W_2^{(6)} + \left(\frac{2}{R_2^2} + \frac{\mu_2\Omega^2}{E_2A_2}\right)W_2^{(4)} + \left(\frac{1}{R_2^4} - \frac{\mu_2\Omega^2}{E_2I_2} - \frac{\mu_2\Omega^2}{E_2A_2R_2^2} - \frac{K}{E_2I_2} - \frac{K}{E_2A_2R_2^2}\right)W_2'' +$$
$$\left(\frac{\mu_2\Omega^2}{E_2I_2R_2^2} - \frac{\mu_2\Omega^2\mu_2\Omega^2}{E_2A_2E_2I_2} - \frac{K\mu_2\Omega^2}{E_2A_2E_2I_2}\right)W_2 + \left(\frac{K}{E_2I_2} + \frac{K}{E_2A_2R_2^2}\right)W_1'' +$$
$$\frac{K\mu_2\Omega^2}{EAEI}W_1 = -\left(\frac{1}{E_2I_2} + \frac{1}{E_2A_2R_2^2}\right)P_2'' - \frac{\mu_2\Omega^2}{E_2A_2E_2I_2}P_2. \tag{12}$$

2　双曲梁系统控制方程的 Green 函数

双曲梁系统控制方程 (11)、(12) 可以简化表示为以下形式:

$$W_1^{(6)} + a_1W_1^{(4)} + a_2W_1'' + a_3W_1 + a_4W_2'' + a_5W_2 = b_1P_1'' + b_2P_1, \tag{13}$$
$$W_2^{(6)} + c_1W_2^{(4)} + c_2W_2'' + c_3W_2 + c_4W_1'' + c_5W_1 = d_1P_2'' + d_2P_2, \tag{14}$$

其中

$$\begin{cases} a_1 = \dfrac{2}{R_1^2} + \dfrac{\mu_1\Omega^2}{E_1A_1},\ a_2 = \dfrac{1}{R_1^4} - \dfrac{\mu_1\Omega^2}{E_1I_1} - \dfrac{\mu_1\Omega^2}{E_1A_1R_1^2} - \dfrac{K}{E_1I_1} - \dfrac{K}{E_1A_1R_1^2},\ a_3 = \dfrac{\mu_1\Omega^2}{E_1I_1R_1^2} - \dfrac{\mu_1\Omega^2\mu_1\Omega^2}{E_1A_1E_1I_1} - \dfrac{K\mu_1\Omega^2}{E_1A_1E_1I_1}, \\ a_4 = \dfrac{K}{E_1I_1} + \dfrac{K}{E_1A_1R_1^2},\ a_5 = \dfrac{K\mu_1\Omega^2}{E_1A_1E_1I_1},\ b_1 = -\left(\dfrac{1}{E_1I_1} + \dfrac{1}{E_1A_1R_1^2}\right),\ b_2 = -\dfrac{\mu_1\Omega^2}{E_1A_1E_1I_1}, \\ c_1 = \dfrac{2}{R_2^2} + \dfrac{\mu_2\Omega^2}{E_2A_2},\ c_2 = \dfrac{1}{R_2^4} - \dfrac{\mu_2\Omega^2}{E_2I_2} - \dfrac{\mu_2\Omega^2}{E_2A_2R_2^2} - \dfrac{K}{E_2I_2} - \dfrac{K}{E_2A_2R_2^2},\ c_3 = \dfrac{\mu_2\Omega^2}{E_2I_2R_2^2} - \dfrac{\mu_2\Omega^2\mu_2\Omega^2}{E_2A_2E_2I_2} - \dfrac{K\mu_2\Omega^2}{E_2A_2E_2I_2}, \\ c_4 = \dfrac{K}{E_2I_2} + \dfrac{K}{E_2A_2R_2^2},\ c_5 = \dfrac{K\mu_2\Omega^2}{EAEI},\ d_1 = -\left(\dfrac{1}{E_2I_2} + \dfrac{1}{E_2A_2R_2^2}\right),\ d_2 = -\dfrac{\mu_2\Omega^2}{E_2A_2E_2I_2}. \end{cases} \tag{15}$$

双曲梁系统控制方程 (13)、(14) 实际上是一种广义的系统形式, 可以通过改变方程中的系数a_i, b_i, c_i, d_i获

得一些退化模型, 如 Euler-Bernoulli 双直梁、Euler-Bernoulli 单曲梁、Euler-Bernoulli 单直梁等. 根据线性系统的叠加原理, 施加荷载$P_1(x)$和$P_2(x)$的双曲梁系统的动态响应是分别受到$P_1(x)$和$P_2(x)$两个系统响应的总和. 本文考虑简谐作用力作用于上梁情况的解, 则双曲梁系统在荷载$P_1(x)$作用下的受迫振动控制方程为

$$W_1^{(6)}+a_1W_1^{(4)}+a_2W_1''+a_3W_1+a_4W_2''+a_5W_2=b_1P_1''+b_2P_1, \tag{16}$$

$$W_2^{(6)}+c_1W_2^{(4)}+c_2W_2''+c_3W_2+c_4W_1''+c_5W_1=0. \tag{17}$$

从物理上讲, 双曲梁系统稳态强迫振动的 Green 函数是受单位谐波载荷作用系统的动态响应, 即$P(x)=\delta(x-x_0)$, 则从数学上来说, 方程 (16)、(17) 的 Green 函数是以下方程的解:

$$W_1^{(6)}+a_1W_1^{(4)}+a_2W_1''+a_3W_1+a_4W_2''+a_5W_2=b_1\delta''(x-x_0)+b_2\delta(x-x_0), \tag{18}$$

$$W_2^{(6)}+c_1W_2^{(4)}+c_2W_2''+c_3W_2+c_4W_1''+c_5W_1=0, \tag{19}$$

其中, $\delta(\cdot)$是 Dirac 函数, x_0表示施加简谐荷载的位置. 对式 (18) 和 (19) 进行关于变量x的 Laplace 变换得

$$\begin{aligned}&(s^6+a_1s^4+a_2s^2+a_3)\hat{W}_1(s,x_0)+(a_4s^2+a_5)\hat{W}_2t(s,x_0)=\\&\quad(b_1s^2+b_2)\mathrm{e}^{-sx_0}+(s^5+a_1s^3+a_2s)W_1(0)+(s^4+a_1s^2+a_2)W_1'(0)+(s^3+a_1s)W_1''(0)+\\&\quad(s^2+a_1)W_1'''(0)+sW_1^{(4)}(0)+W_1^{(5)}(0)+(a_4s)W_2(0)+a_4W_2'(0),\end{aligned} \tag{20}$$

$$\begin{aligned}&(c_4s^2+c_5)\hat{W}_1(s,x_0)+(s^6+c_1s^4+c_2s^2+c_3)\hat{W}_2(s,x_0)=\\&\quad(c_4s)W_1(0)+c_4W_1'(0)+(s^5+c_1s^3+c_2s)W_2(0)+(s^4+c_1s^2+c_2)W_2'(0)+(s^3+c_1s)W_2''(0)+\\&\quad(s^2+c_1)W_2'''(0)+sW_2^{(4)}(0)+W_2^{(5)}(0),\end{aligned} \tag{21}$$

其中, $\hat{W}_1(s,x_0)$和$\hat{W}_2(s,x_0)$分别是 Laplace 变换$W_1(x)$和$W_2(x)$的图像函数, s代表复变量. 再将式 (20) 和 (21) 进行逆变换, 即可以得到双曲梁系统的 Green 函数:

$$\begin{aligned}G_{11}(x,x_0)=&\,\mathrm{H}(x-x_0)\phi_{11}(x-x_0)+\phi_{21}(x)W_1(0)+\phi_{31}(x)W_1'(0)+\phi_{41}(x)W_1''(0)+\phi_{51}(x)W_1'''(0)+\\&\phi_{61}(x)W_1^{(4)}(0)+\phi_{71}(x)W_1^{(5)}(0)+\phi_{81}(x)W_2(0)+\phi_{91}(x)W_2'(0)+\phi_{10,1}(x)W_2''(0)+\phi_{11,1}(x)W_2'''(0)+\\&\phi_{12,1}(x)W_2^{(4)}(0)+\phi_{13,1}(x)W_2^{(5)}(0),\end{aligned} \tag{22}$$

$$\begin{aligned}G_{12}(x,x_0)=&\,\mathrm{H}(x-x_0)\phi_{12}(x-x_0)+\phi_{22}(x)W_1(0)+\phi_{32}(x)W_1'(0)+\phi_{42}(x)W_1''(0)+\phi_{52}(x)W_1'''(0)+\\&\phi_{62}(x)W_1^{(4)}(0)+\phi_{72}(x)W_1^{(5)}(0)+\phi_{82}(x)W_2(0)+\phi_{92}(x)W_2'(0)+\phi_{10,2}(x)W_2''(0)+\phi_{11,2}(x)W_2'''(0)+\\&\phi_{12,2}(x)W_2^{(4)}(0)+\phi_{13,2}(x)W_2^{(5)}(0),\end{aligned} \tag{23}$$

其中, $\mathrm{H}(x-x_0)$是 Heaviside 函数, $W_i(0)$, $W_i'(0)$, $W_i''(0)$, $W_i'''(0)$, $W_i^{(4)}(0)$和$W_i^{(5)}(0)$是可由边界条件确定的 12 个未知常数. ϕ_{11}和ϕ_{12}表示双曲梁系统的受迫振动项, 双曲梁系统的自由振动项为$\phi_{ij}(x)\,(i=2,3,\cdots,13,\ j=1,2)$, 它们的表达式如下:

$$\phi_{11}(x)=\sum_{i=1}^{12}\mathrm{H}(x-x_0)A_i(x-x_0)(b_1s_i^2+b_2)(s_i^6+c_1s_i^4+c_2s_i^2+c_3),$$

$$\phi_{21}(x)=\sum_{i=1}^{12}A_i(x)[(s_i^5+a_1s_i^3+a_2s_i)(s_i^6+c_1s_i^4+c_2s_i^2+c_3)-(c_4s_i)(a_4s_i^2+a_5)],$$

$$\phi_{31}(x)=\sum_{i=1}^{12}A_i(x)[(s_i^4+a_1s_i^2+a_2)(s_i^6+c_1s_i^4+c_2s_i^2+c_3)-c_4(a_4s_i^2+a_5)],$$

$$\phi_{41}(x)=\sum_{i=1}^{12}A_i(x)[(s_i^3+a_1s_i)(s_i^6+c_1s_i^4+c_2s_i^2+c_3)],\quad \phi_{51}(x)=\sum_{i=1}^{12}A_i(x)[(s_i^2+a_1)(s_i^6+c_1s_i^4+c_2s_i^2+c_3)],$$

$$\phi_{61}(x)=\sum_{i=1}^{12}A_i(x)s_i(s_i^6+c_1s_i^4+c_2s_i^2+c_3),\quad \phi_{71}(x)=\sum_{i=1}^{12}A_i(x)(s_i^6+c_1s_i^4+c_2s_i^2+c_3),$$

$$\phi_{81}(x)=\sum_{i=1}^{12}A_i(x)[(a_4s_i)(s_i^6+c_1s_i^4+c_2s_i^2+c_3)-(s_i^5+c_1s_i^3+c_2s_i)(a_4s_i^2+a_5)],$$

$$\phi_{91}(x)=\sum_{i=1}^{12}A_i(x)[a_4(s_i^6+c_1s_i^4+c_2s_i^2+c_3)-(s_i^4+c_1s_i^2+c_2)(a_4s_i^2+a_5)],$$

$$\phi_{10,1}(x)=\sum_{i=1}^{12}A_i(x)[-(s_i^3+c_1s_i)(a_4s_i^2+a_5)],$$

$$\phi_{11,1}(x)=\sum_{i=1}^{12}A_i(x)[-(s_i^2+c_1)(a_4s_i^2+a_5)],\ \phi_{12,1}(x)=\sum_{i=1}^{12}-A_i(x)s_i(a_4s_i^2+a_5),\ \phi_{13,1}(x)=\sum_{i=1}^{12}-A_i(x)(a_4s_i^2+a_5),$$

$$\phi_{12}(x)=\sum_{i=1}^{12}-\mathrm{H}(x-x_0)A_it(x-x_0)(b_1s_i^2+b_2)(c_4s_i^2+c_5),$$

$$\phi_{22}(x)=\sum_{i=1}^{12}A_i(x)[(c_4s_i)(s_i^6+a_1s_i^4+a_2s_i^2+a_3)-(s_i^5+a_1s_i^3+a_2s_i)(c_4s_i^2+c_5)],$$

$$\phi_{32}(x)=\sum_{i=1}^{12}A_i(x)[c_4(s_i^6+a_1s_i^4+a_2s_i^2+a_3)-(s_i^4+a_1s_i^2+a_2)(c_4s_i^2+c_5)],$$

$$\phi_{42}(x)=\sum_{i=1}^{12}A_i(x)[-(s_i^3+a_1s_i)(c_4s_i^2+c_5)],\ \phi_{52}(x)=\sum_{i=1}^{12}A_i(x)[-(s_i^2+a_1)(c_4s_i^2+c_5)],$$

$$\phi_{62}(x)=\sum_{i=1}^{12}-A_i(x)s_i(c_4s_i^2+c_5),\ \phi_{72}(x)=\sum_{i=1}^{12}-A_i(x)(c_4s_i^2+c_5),$$

$$\phi_{82}(x)=\sum_{i=1}^{12}A_i(x)[(s_i^5+c_1s_i^3+c_2s_i)(s_i^6+a_1s_i^4+a_2s_i^2+a_3)-(a_4s_i)(c_4s_i^2+c_5)],$$

$$\phi_{92}(x)=\sum_{i=1}^{12}A_i(x)[(s_i^4+c_1s_i^2+c_2)(s_i^6+a_1s_i^4+a_2s_i^2+a_3)-a_4(c_4s_i^2+c_5)],$$

$$\phi_{10,2}(x)=\sum_{i=1}^{12}A_i(x)[(s_i^3+c_1s_i)(s_i^6+a_1s_i^4+a_2s_i^2+a_3)],\ \phi_{11,2}(x)=\sum_{i=1}^{12}A_i(x)[(s_i^2+c_1)(s_i^6+a_1s_i^4+a_2s_i^2+a_3)],$$

$$\phi_{12,2}(x)=\sum_{i=1}^{12}A_i(x)s_i(s_i^6+a_1s_i^4+a_2s_i^2+a_3),\ \phi_{13,2}(x)=\sum_{i=1}^{12}A_i(x)(s_i^6+a_1s_i^4+a_2s_i^2+a_3).$$

如果令ϕ_{11}和ϕ_{12}及其各阶导数都为 0, 则可以从受迫振动系统导出系统自由振动的解析表达式, 并在确定了两个曲梁的边界条件下, 常数 $W_i(0)$, $W_i'(0)$, $W_i''(0)$, $W_i'''(0)$, $W_i^{(4)}(0)$和$W_i^{(5)}(0)$可以被求解.

3　确定 Green 函数的待定系数

在本节中, Green 函数中涉及的未知常数$W_i(0)$, $W_i'(0)$, $W_i''(0)$, $W_i'''(0)$, $W_i^{(4)}(0)$和$W_i^{(5)}(0)$(i=1,2) 由$x=0,L$处的边界条件确定.

以简支双曲梁为例, 将简支边界条件代入方程 (22) 和 (23), 获得以下矩阵形式的方程:

$$\boldsymbol{TW}_0=\begin{bmatrix}-\phi_{11}(L_1-x_0)\\ -\lambda_{15}\phi_{11}^{(4)}(L_1-x_0)-\lambda_{16}\phi_{11}''(L_1-x_0)\\ 0\\ -\lambda_{11}\phi_{11}^{(5)}(L_1-x_0)-\lambda_{12}\phi_{11}'''(L_1-x_0)-\lambda_{13}\phi_{11}'(L_1-x_0)-\lambda_{14}\phi_{12}'(L_2-x_0)\\ 0\\ -\phi_{12}(L_2-x_0)\\ -\lambda_{25}\phi_{12}^{(4)}(L_2-x_0)-\lambda_{26}\phi_{12}''(L_2-x_0)\\ 0\\ -\lambda_{21}\phi_{12}^{(5)}(L_2-x_0)-\lambda_{22}\phi_{12}'''(L_2-x_0)-\lambda_{23}\phi_{12}'(L_2-x_0)-\lambda_{24}\phi_{11}'(L_2-x_0)\\ 0\end{bmatrix},\tag{24}$$

其中

$$\boldsymbol{T}=\begin{bmatrix}\boldsymbol{T}_{11} & \boldsymbol{T}_{12}\\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22}\end{bmatrix}, \tag{25}$$

$$\boldsymbol{W}_0=[\ W_1'(0)\quad W_1''(0)\quad W_1'''(0)\quad W_1^{(4)}(0)\quad W_1^{(5)}(0)\quad W_2'(0)\quad W_2''(0)\quad W_2'''(0)\quad W_2^{(4)}(0)\quad W_2^{(5)}(0)\ ]^{\mathrm{T}}, \tag{26}$$

$\boldsymbol{T}$ 为一个10×10的方阵,为简化此方阵将其写为分块矩阵形式:

$$\begin{cases}\boldsymbol{T}_{11}=\begin{bmatrix}\phi_{31} & \phi_{41} & \phi_{51} & \phi_{61} & \phi_{71}\\ \bar{\phi}_{31} & \bar{\phi}_{41} & \bar{\phi}_{51} & \bar{\phi}_{61} & \bar{\phi}_{71}\\ 0 & \lambda_{15} & 0 & \lambda_{14} & 0\\ \tilde{\phi}_{31} & \tilde{\phi}_{41} & \tilde{\phi}_{51} & \tilde{\phi}_{61} & \tilde{\phi}_{71}\\ \lambda_{13} & 0 & \lambda_{12} & 0 & \lambda_{11}\end{bmatrix}, \boldsymbol{T}_{12}=\begin{bmatrix}\phi_{91} & \phi_{10,1} & \phi_{11,1} & \phi_{12,1} & \phi_{13,1}\\ \bar{\phi}_{91} & \bar{\phi}_{10,1} & \bar{\phi}_{11,1} & \bar{\phi}_{12,1} & \bar{\phi}_{13,1}\\ 0 & 0 & 0 & 0 & 0\\ \tilde{\phi}_{91} & \tilde{\phi}_{10,1} & \tilde{\phi}_{11,1} & \tilde{\phi}_{12,1} & \tilde{\phi}_{13,1}\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},\\ \boldsymbol{T}_{21}=\begin{bmatrix}\phi_{32} & \phi_{42} & \phi_{52} & \phi_{62} & \phi_{72}\\ \bar{\phi}_{32} & \bar{\phi}_{42} & \bar{\phi}_{52} & \bar{\phi}_{62} & \bar{\phi}_{72}\\ 0 & 0 & 0 & 0 & 0\\ \tilde{\phi}_{32} & \tilde{\phi}_{42} & \tilde{\phi}_{52} & \tilde{\phi}_{62} & \tilde{\phi}_{72}\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}, \boldsymbol{T}_{22}=\begin{bmatrix}\phi_{92} & \phi_{10,2} & \phi_{11,2} & \phi_{12,2} & \phi_{13,2}\\ \bar{\phi}_{92} & \bar{\phi}_{10,2} & \bar{\phi}_{11,2} & \bar{\phi}_{12,2} & \bar{\phi}_{13,2}\\ 0 & \lambda_{25} & 0 & \lambda_{24} & 0\\ \tilde{\phi}_{92} & \tilde{\phi}_{10,2} & \tilde{\phi}_{11,2} & \tilde{\phi}_{12,2} & \tilde{\phi}_{13,2}\\ \lambda_{23} & 0 & \lambda_{22} & 0 & \lambda_{21}\end{bmatrix},\end{cases} \tag{27}$$

$$\begin{cases}\bar{\phi}_{i1}=\lambda_{15}\phi_{i1}^{(4)}+\lambda_{16}\phi_{i1}'',\ \tilde{\phi}_{i1}=\lambda_{11}\phi_{i1}^{(5)}+\lambda_{12}\phi_{i1}'''+\lambda_{13}\phi_{i1}'+\lambda_{14}\phi_{i2}',\\ \bar{\phi}_{i2}=\lambda_{25}\phi_{i2}^{(4)}+\lambda_{26}\phi_{i2}'',\ \tilde{\phi}_{i2}=\lambda_{21}\phi_{i2}^{(5)}+\lambda_{22}\phi_{i2}'''+\lambda_{23}\phi_{i2}'+\lambda_{24}\phi_{i1}',\end{cases}\quad i=3,4,\cdots,13. \tag{28}$$

通过以上 10 个方程的求解,可以确定 10 个未知常数$W_i'(0)$, $W_i''(0)$, $W_i'''(0)$, $W_i^{(4)}(0)$和$W_i^{(5)}(0)$(i=1, 2). 因此,两端简支双曲梁系统的 Green 函数形式为

$$\begin{aligned}G_{11}(x,x_0)=&\mathrm{H}(x-x_0)\phi_{11}(x-x_0)+\phi_{31}(x)W_1'(0)+\phi_{41}(x)W_1''(0)+\phi_{51}(x)W_1'''(0)+\\ &\phi_{61}(x)W_1^{(4)}(0)+\phi_{71}(x)W_1^{(5)}(0)+\phi_{91}(x)W_2'(0)+\phi_{10,1}(x)W_2''(0)+\\ &\phi_{11,1}(x)W_2'''(0)+\phi_{12,1}(x)W_2^{(4)}(0)+\phi_{13,1}(x)W_2^{(5)}(0),\end{aligned} \tag{29}$$

$$\begin{aligned}G_{12}(x,x_0)=&\mathrm{H}(x-x_0)\phi_{12}(x-x_0)+\phi_{32}(x)W_1'(0)+\phi_{42}(x)W_1''(0)+\phi_{52}(x)W_1'''(0)+\\ &\phi_{62}(x)W_1^{(4)}(0)+\phi_{72}(x)W_1^{(5)}(0)+\phi_{92}(x)W_2'(0)+\phi_{10,2}(x)W_2''(0)+\\ &\phi_{11,2}(x)W_2'''(0)+\phi_{12,2}(x)W_2^{(4)}(0)+\phi_{13,2}(x)W_2^{(5)}(0).\end{aligned} \tag{30}$$

如表 1 所示,对于其他的边界条件,如两端固支、两端自由、固支-简支等边界条件,与求解两端简支曲梁的 Green 函数一样,按照同一过程也可以得到相应的 Green 函数.

表 1 双曲梁的不同边界条件

Table 1 Boundary conditions (BCs) of the DCB

BC	beam	ECB
pinned	upper beam	$W_1\vert_{x=0,L}=0,\ \lambda_{15}W_1^{(4)}+\lambda_{16}W_1''\vert_{x=0,L}=0,\ \lambda_{11}W_1^{(5)}+\lambda_{12}W_1'''+\lambda_{13}W_1'+\lambda_{14}W_2'\vert_{x=0,L}=0$
	bottom beam	$W_2\vert_{x=0,L}=0,\ \lambda_{25}W_2^{(4)}+\lambda_{26}W_2''\vert_{x=0,L}=0,\ \lambda_{21}W_2^{(5)}+\lambda_{22}W_2'''+\lambda_{23}W_2'+\lambda_{24}W_1'\vert_{x=0,L}=0$
fixed	upper beam	$W_1\vert_{x=0,L}=0,\ W_1'\vert_{x=0,L}=0,\ \lambda_{11}W_1^{(5)}+\lambda_{12}W_1'''\vert_{x=0,L}=0$
	bottom beam	$W_2\vert_{x=0,L}=0,\ W_2'\vert_{x=0,L}=0,\ \lambda_{21}W_1^{(5)}+\lambda_{22}W_1'''\vert_{x=0,L}=0$
free	upper beam	$W_1''\vert_{x=0,L}=0,\ W_1'''\vert_{x=0,L}=0,\ \lambda_{11}W_1^{(5)}+\lambda_{13}W_1'+\lambda_{14}W_2'\vert_{x=0,L}=0$
	bottom beam	$W_2''\vert_{x=0,L}=0,\ W_2'''\vert_{x=0,L}=0,\ \lambda_{21}W_2^{(5)}+\lambda_{23}W_2'+\lambda_{24}W_1'\vert_{x=0,L}=0$

4 数值分析及讨论

在本节中,考虑上梁为两端简支的曲梁,在$x=L/2$处受到单位简谐集中力作用. 为了方便起见,引入无量纲化参数:

$$\beta=\frac{h}{L},\ \xi=\frac{x}{L},\ \xi_0=\frac{x_0}{L},\ g(x;x_0)=\frac{G(x;x_0)}{w_{\max}^{s}},\ \varOmega'=\frac{\varOmega}{\varOmega_0},\ K_0=\frac{K}{E}, \tag{31}$$

其中, $w_{\max}^{s}=L^3/(48EI)$是简支梁中受到单位集中力产生的最大静挠度, $\varOmega_0=\pi^2\sqrt{EI/(\rho A)}/L^2$是 EB 的一阶固有频率.

4.1 解的有效性验证

本小节通过双曲梁系统的退化解与已发表文献的结果对比, 验证了该系统解析解的有效性. 在上文中已经提到双曲梁系统的 Green 函数解可以退化为单曲梁和双直梁系统的 Green 函数解. 利用这一点可以对简支双曲梁系统的 Green 函数解进行退化验证. 根据曲梁的公式推导判断出半径趋于无穷大时, 曲梁的 Green 函数解可以退化为直梁的 Green 函数解. 同理, 双曲梁系统也可在半径趋于无穷大时退化为双直梁的 Green 函数解, 再将弹性层系数设为零, 可以得到单直梁的退化解. 因此, 可将双曲梁系统的退化解与文献 [8] 进行对照. 如图 2 所示, 设定与文献 [8] 中相同的材料参数, 由于不存在半径为无穷大的结构, 本文设置半径 $R = 10\,000$ m得到的结果与文献 [17] 及有限元算例基本吻合.

为了进一步验证本文解的有效性, 本文设置半径 $R = 10\,000$ m, 曲梁的弹性模量 $E = 2.0\times10^{11}$ Pa, 弹性层刚度 $K_0 = 0.000\,25E$, 将双曲梁系统退化为双直梁系统得到的结果也与文献 [8] 基本吻合, 如图 3 所示. 因此, 本文解的有效性得以验证.

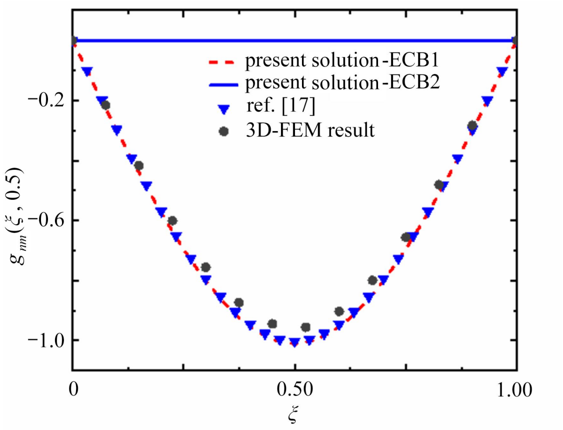

图 2　退化简支曲梁在外部动力作用下的无量纲挠度

Fig. 2　Dimensionless displacement $g(\xi, 0.5)(n=1, m=1, 2)$ of the degenerated ECB as a function of dimensionless coordinate ξ

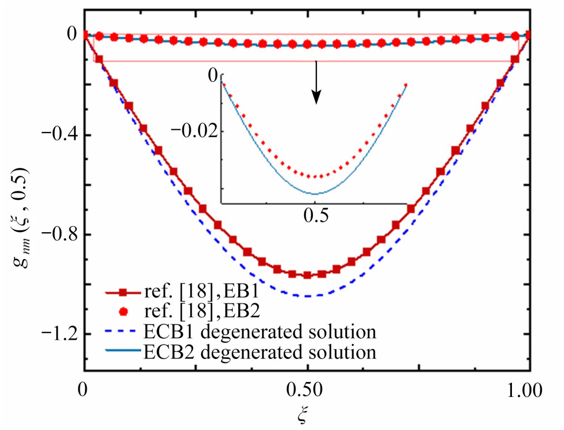

图 3　退化简支双曲梁在外部动力作用下的无量纲挠度

Fig. 3　Dimensionless displacement $g(\xi, 0.5)(n=1, m=1, 2)$ of the degenerated DCB as a function of dimensionless coordinate ξ

4.2 几何物理参数对解的影响

本小节分别分析了 Winkler 弹性层弹性模量、曲梁半径、不同的外激励频率对振动响应的影响以及双曲梁之间的相互影响.

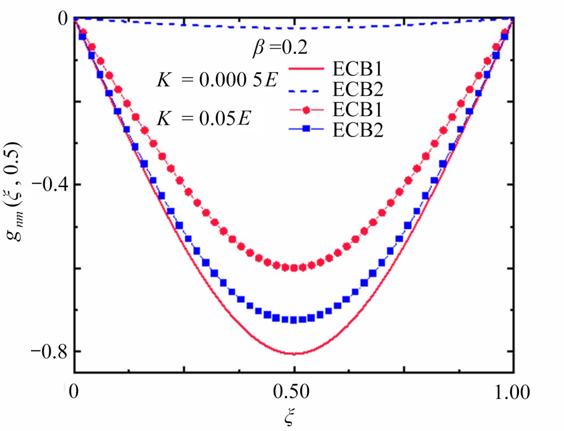

图 4　相同半径不同弹性层弹性模量下双曲梁的无量纲化位移 $g_{nm}(\xi, 0.5)(n=1, m=1, 2)$ $(R = 100, \Omega' = 0.5)$

Fig. 4　Dimensionless Green's function $g_{nm}(\xi, 0.5)(n=1, m=1, 2)$ as a function of dimensionless coordinate x for different values of stiffness modulus $K(R = 100, \Omega' = 0.5)$

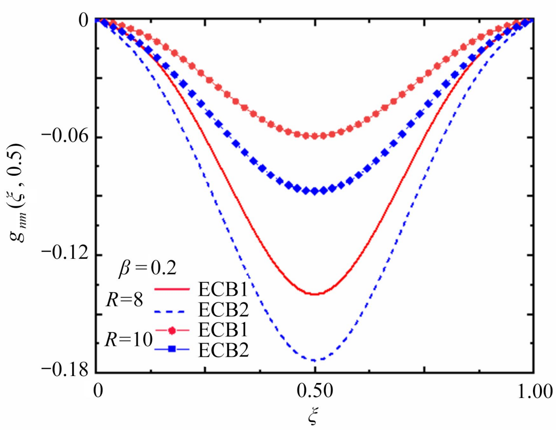

图 5　不同半径相同弹性层弹性模量和相同外激励下双曲梁的无量纲化位移 $g_{nm}(\xi, 0.5)(n=1, m=1, 2)$ $(K_0 = 0.1, \Omega' = 0.5)$

Fig. 5　Dimensionless Green's function $g_{nm}(\xi, 0.5)(n=1, m=1, 2)$ as a function of dimensionless coordinate x for different radius values $(K_0 = 0.1, \Omega' = 0.5)$

图 4 所示是相同半径不同弹性层弹性模量和相同外激励下双曲梁的无量纲化位移,其横坐标是无量纲化的曲梁跨度ξ,从图中可以看出随着弹性层弹性模量的增大,上梁的无量纲位移减小,下梁的无量纲位移增大.从物理角度来看,这表明上下梁的相互约束得到加强.当 Winkler 弹性层弹性模量大到一定程度时,双梁系统可视为一个整体,其上梁和下梁的无量纲挠度将趋于一致,即$g_{11}(0.5,0.5) = g_{12}(0.5,0.5)$.

图 5 所示是不同半径相同弹性层弹性模量和相同外激励下双曲梁的无量纲化位移,从图中可以看出半径越大双曲梁系统的挠度越大.从物理意义上看,当曲率半径越大时,曲梁越平缓,当曲率半径趋于无穷时,曲梁的几何特征将会无限趋近于直梁,与直梁相比曲梁的稳定性较强,因此,当半径越大时趋于平滑的曲梁挠度也会越大.

图 6 所示是相同半径相同弹性层弹性模量和不同外激励下双曲梁的无量纲化位移,从图中可以看出外激励频率越大挠度越小,且两个梁的振动方向相反.当外激励频率与固有振动频率相等时系统发生共振,振幅达到最大,而系统受迫振动稳定后的频率等于外激励的频率,表明外激励频率与固有振动频率相差越大时,系统的振幅就越小.

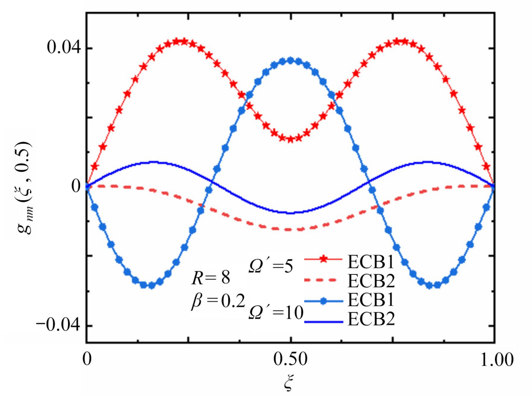

图 6 相同半径相同弹性层弹性模量不同外激励下双曲梁的无量纲化位移($K_0 = 0.1, \Omega' = 5, 10$)

Fig. 6 Dimensionless Green's function $g_{nm}(\xi, 0.5)$(n=1, m=1, 2) as a function of dimensionless coordinate x for different external frequency values($K_0 = 0.1, \Omega' = 5, 10$)

5 结 论

本文通过 Laplace 变换,系统地研究了两端简支、固支、自由边界条件下双曲梁强迫振动的 Green 函数.并以两端简支边界条件为例,证实了所得到的双曲梁强迫振动 Green 函数可以退化到双直梁、单曲梁和单直梁的强迫振动 Green 函数.通过探究 Winkler 弹性层弹性模量、曲梁半径、不同的外激励频率以及双曲梁之间的相互作用对双曲梁挠度的影响,得出以下结论:

1) 随着弹性层弹性模量的增大,双曲梁中上梁的无量纲挠度逐渐变大,下梁的无量纲挠度逐渐变小,最终上梁和下梁的无量纲挠度趋于一致;

2) 随着半径 R 的增大,双曲梁的无量纲挠度逐渐变大,并最终收敛于双直梁的无量纲挠度;

3) 外激励频率越大双曲梁系统振动频率越大,且外激励频率与固有频率相差越大时系统的振幅越小.

文中所研究的理论公式可为相关领域的分析和计算提供理论参考.

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