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面内压缩荷载作用下双层微板系统的同步/异步屈曲*

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摘要： 采用修正的偶应力理论和双变量高阶剪切变形理论，发展了层间填充弹性介质的双层微板系统在面内压缩荷载作用下的屈曲模型．基于 Euler-Lagrange 方程推导了系统屈曲的控制微分方程，运用 Navier 法获得了上下层均为四边简支时系统同步/异步屈曲的解析解．通过数值算例讨论了系统各参数对其屈曲特性的影响．结果表明：系统的异步屈曲特性依赖于材料尺度参数、长宽比和弹性介质模量，而同步屈曲特性仅依赖于前两项，并且异步屈曲荷载高于同步屈曲荷载；弹性介质的 Pasternak 模量较之于 Winkler 模量对系统的屈曲特性影响更显著．

关　键　词： 修正的偶应力理论；　双变量高阶剪切变形理论；　双层微板系统；　同步/异步屈曲

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Abstract: A linear buckling model for double-layered microplate systems filled with elastic media between layers was developed under the modified couple stress theory and the 2-variable higher-order shear deformation theory. The governing differential equations for system buckling were derived based on the Euler-Lagrange equation. With the Navier method, the synchronous and asynchronous buckling solutions were analytically obtained in the case of both upper and lower plates being simply supported on 4 edges. The influence of each parameter on the buckling characteristics of the system was discussed by numerical examples. Numerical results show that, the asynchronous buckling characteristics of the system depend on the material length scale parameter, the aspect ratio and the elastic medium modulus, while the synchronous buckling characteristics depend on the 1st 2 only; the asynchronous critical buckling load is noticeably greater than that of the synchronous buckling case; the Pasternak modulus has a more significant effect on the buckling characteristics of the system than the Winkler modulus.

Key words: modified couple stress theory; 2-variable higher-order shear deformation theory; double-layered microplate system; synchronous/asynchronous buckling

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0 引　言

微机电系统 (micro-electro-mechanical systems, MEMS) 是微电路和微机械根据功能需求在芯片上的集成, 特征尺寸通常在微米乃至纳米量级. MEMS 因拥有体积小、重量轻、功耗低、可靠性好、灵敏度高、易于集成等优点, 在空间科学、国防军事、汽车工业、信息技术、消费电子等领域显示出重要的应用价值. MEMS 基本组成单元通常以杆、梁、板、壳和薄膜等结构形式存在. 诸多实验表明, 当构件特征尺寸降低至微纳米尺度时, 其静动态力学特性会呈现出显著的尺度依赖性[1-4]. 例如, Lam 等[1] 在厚度分别为 20 μm, 38 μm, 75 μm 和 115 μm 的环氧树脂悬臂微梁弯曲实验中观察到: 随着微梁厚度的减小, 其无量纲弯曲刚度单调增加, 且厚度为 20 μm 和 115 μm 时无量纲弯曲刚度之比为 2.4. Lei 等[2] 利用激光多普勒测振仪对厚度处于 2.1 ~ 15 μm 范围的镍悬臂梁进行自由振动测试, 发现当微梁厚度从 15 μm 减小到 2.1 μm 时, 其无量纲基频增加至 2.1 倍.

为了定量描述微构件力学性能的尺度效应, 研究者们提出了含有材料内禀尺度参数的高阶连续介质力学理论, 如应变梯度理论[1,5-7] 和偶应力理论[8-10]. 其中, 修正的应变梯度理论[1] 和修正的偶应力理论[9] 在微米量级梁板的力学建模中备受研究者青睐. 例如, 刘松正等[11] 基于修正的偶应力理论与四参数高阶剪切-法向伸缩变形理论, 提出了一种准三维功能梯度微梁的尺度效应模型, 并发展了相应的微分求积有限元法. 徐晓建和邓子辰[12] 建立了考虑应变梯度和速度梯度效应的 Kirchhoff 微板的控制微分方程及其变分自洽边界条件, 修正了已有文献中给出的薄板角点条件, 并采用 Lévy 法给出了均布荷载作用下微板的挠度以及自由振动频率的解析解. 雷剑等[13] 基于修正的偶应力理论, 采用 Ritz 法求解了任意边界条件下变截面二维功能梯度 Timoshenko 微梁的振动频率和临界屈曲载荷的数值解.

随着 MEMS 产品应用场景越来越广泛, 单层微板已无法满足当下日益增长的功能需求. 双层微板系统作为单层微板的重要技术延伸, 在边界条件、层厚比、材料参数以及受载方式选择上具有高度可调节性, 拥有单层微板所不具有的许多优良特性, 如今已广泛应用于高频谐振器、光学调制器、动力吸振器及可延展柔性电路等领域. 当双层微板系统受到面内机械荷载作用时, 一旦其弯曲变形超过了小变形临界点, 系统便会发生失稳屈曲并形成规则的波形, 与之相关的屈曲失效机理研究引起了学术界的关注. Murmu 等[14] 采用 Navier 法获得了非局部双层纳米板系统双轴屈曲的临界荷载, 并对同步屈曲和异步屈曲进行了详细分析. Jamalpoor 等[15] 结合应变梯度理论和 Kirchhoff 板理论建立了外部电磁场作用下双层微板系统的双轴压缩屈曲模型, 利用 Navier 法获得了异步、同步和下层完全固定三种情形下系统的屈曲荷载. Shafiei 等[16] 利用修正的偶应力理论和双变量高阶剪切变形理论建立了 Van der Waals 力作用下多层石墨烯薄片的压缩屈曲模型, 采用有限样条法求解了系统的屈曲载荷. 王少扬[17] 基于修正的偶应力理论建立了黏弹性纳米板双轴屈曲的控制微分方程, 并给出了系统屈曲荷载的 Navier 解. Liu 等[18] 建立了黏弹性地基上双层功能梯度非局部纳米板系统的屈曲模型, 推导了系统屈曲载荷的 Navier 解. 黄明琦[19] 基于 Kirchhoff 板理论建立了双层纳米板非局部屈曲的 Hamilton 求解体系, 采用辛叠加方法获得了各种边界条件下系统的基准解析解.

综上所述, 目前关于双层微板系统的屈曲特性研究主要以非局部理论和三类传统板理论为基础, 而结合修正的偶应力和双变量高阶剪切变形理论的研究工作还很匮乏. 鉴于此, 本文拟在修正的偶应力理论和双变量高阶剪切变形理论下, 建立双层微板系统在面内压缩荷载作用下的屈曲模型, 采用 Navier 法给出了系统同步/异步屈曲的解析解, 探究了不同材料尺度参数、长宽比和弹性介质的模量对系统屈曲特性的影响.

1　双层微板系统的屈曲模型

1.1　几何描述

图 1 所示为由 Winkler-Pasternak 弹性介质连接的各向同性双层微板系统, 描述系统变形的两个直角坐标系分别位于各层板的中面上, 且原点位于中面左上角点. 系统的长度、宽度分别为 L_y, L_x; Winkler 和 Pasternak 模量分别为 k_w, k_p; 第 i 层微板的厚度、弹性模量、Poisson 比、剪切模量以及密度分别为 $h^{(i)}$, $E^{(i)}$, $\nu^{(i)}$, $G^{(i)}$ 和 $\rho^{(i)}$; $P_{xx}^{(i)}$ 和 $P_{yy}^{(i)}$ 分别表示第 i 层微板单位长度上的面内压缩荷载.

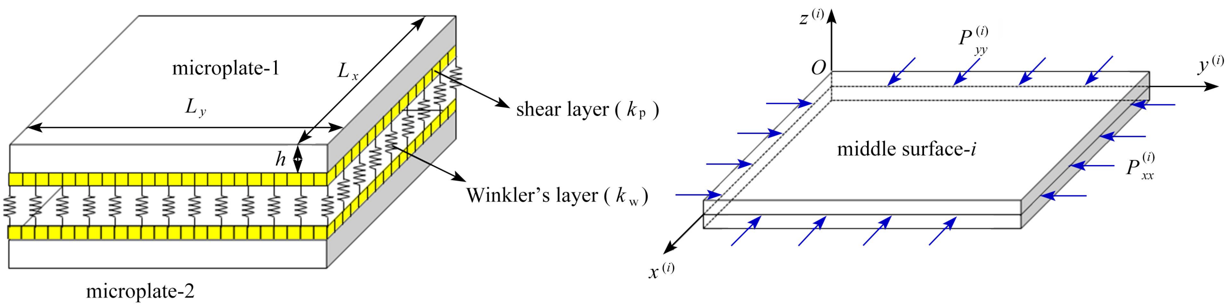


图 1 双层微板系统示意图

Fig. 1 Schematic of the double-layered microplate system

1.2 屈曲控制微分方程

在修正的偶应力理论[9] 下，各向同性线弹性体的应变能U为

$$U=\frac{1}{2}\int_{\Omega}(\sigma_{ij}\varepsilon_{ij}+m_{ij}^{(s)}\chi_{ij}^{(s)})\mathrm{d}\Omega, \tag{1}$$

式中σ_{ij}, ε_{ij}, $m_{ij}^{(s)}$与$\chi_{ij}^{(s)}$分别代表应力张量分量、应变张量分量、偶应力张量分量以及旋转梯度张量对称部分的分量，其具体定义如下：

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\ \chi_{ij}^{(s)}=\frac{1}{2}(\theta_{i,j}+\theta_{j,i}),\ \sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij},\ m_{ij}^{(s)}=2\mu l^2\chi_{ij}^{(s)}, \tag{2}$$

式中u_i是位移矢量分量，θ_i是转动矢量分量，λ和μ是 Lamé常数，δ_{ij}是 Kronecker 符号，l 是材料特征尺度参数. 转动矢量分量θ_i的定义如下：

$$\theta_i=\frac{1}{2}\epsilon_{ijk}u_{k,j}, \tag{3}$$

式中ϵ_{ijk}是置换符号.

根据双变量高阶剪切变形理论[20]，微板系统的位移场可以表示为

$$\begin{cases} w^{(i)}(x,y,z,t)=w_{\mathrm{b}}^{(i)}(x,y,t)+w_{\mathrm{s}}^{(i)}(x,y,t), \\ u^{(i)}(x,y,z,t)=\left[\dfrac{z}{4}-\dfrac{5z^3}{3(h^{(i)})^2}\right]\dfrac{\partial w_{\mathrm{s}}^{(i)}}{\partial x}-z\dfrac{\partial w_{\mathrm{b}}^{(i)}}{\partial x}, \\ v^{(i)}(x,y,z,t)=\left[\dfrac{z}{4}-\dfrac{5z^3}{3(h^{(i)})^2}\right]\dfrac{\partial w_{\mathrm{s}}^{(i)}}{\partial y}-z\dfrac{\partial w_{\mathrm{b}}^{(i)}}{\partial y}, \end{cases} \tag{4}$$

式中$u^{(i)}$, $v^{(i)}$和$w^{(i)}$分别表示第i层微板内任一点沿x, y和z方向的位移分量，$w_{\mathrm{b}}^{(i)}$和$w_{\mathrm{s}}^{(i)}$分别表示由纯弯曲和纯剪切变形引起的挠度分量.

利用式 (2) 和 (4) 可得第i层板的应变张量、曲率张量、应力张量以及偶应力张量的非零分量，并将它们代入式 (1) 中，可得双层微板系统的应变能如下：

$$\begin{aligned}\varPi_{\mathrm{S}}=\sum_{i=1}^{2}\int_A\Bigg\{&\varLambda_1^{(i)}\left[\left(\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial y^2}\right)^2\right]+\varLambda_2^{(i)}\left[\left(\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial y^2}\right)^2\right]+\left(\varLambda_3^{(i)}-\varLambda_6^{(i)}\right)\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial x^2}\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial y^2}+\\&\left(\frac{\varLambda_3^{(i)}}{84}-\frac{3\varLambda_6^{(i)}}{8}\right)\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial x^2}\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial y^2}+\varLambda_4^{(i)}\left[\left(\frac{\partial w_{\mathrm{s}}^{(i)}}{\partial x}\right)^2+\left(\frac{\partial w_{\mathrm{s}}^{(i)}}{\partial y}\right)^2\right]+\varLambda_5^{(i)}\left[\left(\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial x\partial y}\right)^2+84\left(\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial x\partial y}\right)^2\right]+\\&\frac{7\varLambda_6^{(i)}}{12}\left(\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial x^2}-\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial y^2}\right)\left(\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial x^2}-\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial y^2}\right)+\frac{7\varLambda_6^{(i)}}{3}\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial x\partial y}\frac{\partial^2 w_b^{(i)}}{\partial x\partial y}+\frac{3\varLambda_6^{(i)}}{4}\left(\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial x\partial y}\right)^2+2\varLambda_6^{(i)}\left(\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial x\partial y}\right)^2\Bigg\}\mathrm{d}A,\end{aligned} \tag{5}$$

式中

$$\varLambda_1^{(i)}=\frac{D^{(i)}}{2}+\frac{G^{(i)}h^{(i)}l^2}{2},\ \varLambda_2^{(i)}=\frac{D^{(i)}}{168}+\frac{3G^{(i)}h^{(i)}l^2}{16},\ \varLambda_3^{(i)}=D^{(i)}v,\ \varLambda_4^{(i)}=\frac{5G^{(i)}h^{(i)}}{12}+\frac{25G^{(i)}l^2}{24h^{(i)}}, \tag{6a}$$

$$\Lambda_5^{(i)}=\frac{G^{(i)}(h^{(i)})^3}{504},\ \Lambda_6^{(i)}=G^{(i)}h^{(i)}l^2,\ D^{(i)}=\frac{E^{(i)}(h^{(i)})^3}{12(1-\nu^2)}. \tag{6b}$$

面内压缩荷载$P_{xx}^{(i)}$和$P_{yy}^{(i)}$作用下系统的屈曲应变能如下:

$$\begin{aligned}\varPi_{\mathrm{B}}=P_{\mathrm{cr}}\sum_{i=1}^{2}\int_A\Bigg\{&\frac{\eta_{xx}^{(i)}(h^{(i)})^2}{2\,016}\left[\left(\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial x\partial y}\right)^2\right]+\frac{\eta_{xx}^{(i)}\big(h^{(i)}\big)^2}{24}\left[\left(\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial x\partial y}\right)^2\right]+\frac{\eta_{xx}^{(i)}}{2}\left[\frac{\partial(w_{\mathrm{b}}^{(i)}+w_{\mathrm{s}}^{(i)})}{\partial x}\right]^2+\\&\frac{\eta_{yy}^{(i)}(h^{(i)})^2}{2\,016}\left[\left(\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w_{\mathrm{s}}^{(i)}}{\partial y^2}\right)^2\right]+\frac{\eta_{yy}^{(i)}(h^{(i)})^2}{24}\left[\left(\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w_{\mathrm{b}}^{(i)}}{\partial y^2}\right)^2\right]+\frac{\eta_{yy}^{(i)}}{2}\left[\frac{\partial(w_{\mathrm{b}}^{(i)}+w_{\mathrm{s}}^{(i)})}{\partial y}\right]^2\Bigg\}\mathrm{d}A,\end{aligned} \tag{7}$$

式中P_{cr}为系统的临界屈曲荷载, $\eta_{xx}^{(i)}=P_{xx}/P_{\mathrm{cr}}$和$\eta_{yy}^{(i)}=P_{yy}/P_{\mathrm{cr}}$表示屈曲荷载因子.

Winkler-Pasternak 弹性介质的势能为

$$\begin{aligned}\varPi_{\mathrm{F}}=\int_A\Bigg\{&\frac{k_w}{2}(w_{\mathrm{b}}^{(2)}+w_{\mathrm{s}}^{(2)}-w_{\mathrm{b}}^{(1)}-w_{\mathrm{s}}^{(1)})^2+\frac{k_{\mathrm{p}}}{2}\left[\frac{\partial(w_{\mathrm{b}}^{(2)}+w_{\mathrm{s}}^{(2)}-w_{\mathrm{b}}^{(1)}-w_{\mathrm{s}}^{(1)})}{\partial x}\right]^2+\\&\frac{k_{\mathrm{p}}}{2}\left[\frac{\partial(w_{\mathrm{b}}^{(2)}+w_{\mathrm{s}}^{(2)}-w_{\mathrm{b}}^{(1)}-w_{\mathrm{s}}^{(1)})}{\partial y}\right]^2\Bigg\}\mathrm{d}A.\end{aligned} \tag{8}$$

依据文献 [21], 双层微板系统的 Euler-Lagrange 方程表示为

$$\frac{\partial\Re}{\partial\varDelta}-\left[\frac{\partial}{\partial x}\left(\frac{\partial\Re}{\partial\varDelta_{,x}}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\Re}{\partial\varDelta_{,y}}\right)\right]+\frac{\partial^2}{\partial x^2}\left(\frac{\partial\Re}{\partial\varDelta_{,xx}}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial\Re}{\partial\varDelta_{,xy}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial\Re}{\partial\varDelta_{,yy}}\right)=0, \tag{9}$$

式中$\Re=\varPi_{\mathrm{B}}-\varPi_{\mathrm{S}}+\varPi_{\mathrm{F}}$, $\varDelta$表示$w_{\mathrm{b}}^{(1)}$, $w_{\mathrm{s}}^{(1)}$, $w_{\mathrm{b}}^{(2)}$或$w_{\mathrm{s}}^{(2)}$.

将式 (5)、(7) 和 (8) 代入式 (9), 可得到系统屈曲的控制微分方程:

$$\begin{aligned}\delta w_{\mathrm{b}}^{(i)}:\ &\frac{P_{\mathrm{cr}}(h^{(i)})^2}{12}\left[\left(\eta_{xx}^{(i)}\frac{\partial^4 w_{\mathrm{b}}^{(i)}}{\partial x^4}+\eta_{yy}^{(i)}\frac{\partial^4 w_{\mathrm{b}}^{(i)}}{\partial y^4}\right)+(\eta_{xx}^{(i)}+\eta_{yy}^{(i)})\frac{\partial^4 w_{\mathrm{b}}^{(i)}}{\partial x^2\partial y^2}\right]-P_{\mathrm{cr}}\left(\eta_{xx}^{(i)}\frac{\partial^2}{\partial x^2}+\eta_{yy}^{(i)}\frac{\partial^2}{\partial y^2}\right)(w_{\mathrm{b}}^{(i)}+w_{\mathrm{s}}^{(i)})-\\&\frac{7\varSigma_6^{(i)}}{12}\nabla^4 w_{\mathrm{s}}^{(i)}-2\varSigma_1^{(i)}\nabla^4 w_{\mathrm{b}}^{(i)}+(-1)^i(k_{\mathrm{p}}\nabla^2-k_{\mathrm{w}})(w_{\mathrm{b}}^{(2)}+w_{\mathrm{s}}^{(2)}-w_{\mathrm{b}}^{(1)}-w_{\mathrm{s}}^{(1)})=0,\end{aligned} \tag{10}$$

$$\begin{aligned}\delta w_{\mathrm{s}}^{(i)}:\ &\frac{P_{\mathrm{cr}}(h^{(i)})^2}{1\,008}\left[\eta_{xx}^{(i)}\frac{\partial^4 w_{\mathrm{s}}^{(i)}}{\partial x^4}+\eta_{yy}^{(i)}\frac{\partial^4 w_{\mathrm{s}}^{(i)}}{\partial y^4}+(\eta_{xx}^{(i)}+\eta_{yy}^{(i)})\frac{\partial^4 w_{\mathrm{s}}^{(i)}}{\partial x^2\partial y^2}\right]-P_{\mathrm{cr}}\left(\eta_{xx}^{(i)}\frac{\partial^2}{\partial x^2}+\eta_{yy}^{(i)}\frac{\partial^2}{\partial y^2}\right)(w_{\mathrm{b}}^{(i)}+w_{\mathrm{s}}^{(i)})+\\&2\varSigma_4^{(i)}\nabla^2 w_{\mathrm{s}}^{(i)}-2\varSigma_2^{(i)}\nabla^4 w_{\mathrm{s}}^{(i)}+(-1)^i(k_{\mathrm{p}}\nabla^2-k_{\mathrm{w}})(w_{\mathrm{b}}^{(2)}+w_{\mathrm{s}}^{(2)}-w_{\mathrm{b}}^{(1)}-w_{\mathrm{s}}^{(1)})-2\varSigma_5^{(i)}\frac{\partial^4 w_{\mathrm{s}}^{(i)}}{\partial x^2\partial y^2}-\frac{7\varSigma_6^{(i)}}{12}\nabla^4 w_{\mathrm{b}}^{(i)}=0,\end{aligned} \tag{11}$$

式中

$$\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2,\ \nabla^4=(\partial^2/\partial x^2+\partial^2/\partial y^2)^2.$$

鉴于篇幅的限制, 本文仅考虑系统上下层均为四边简支且材料、厚度、面内载荷完全相同的情形. 此情形下, $\Lambda_1^{(1)}=\Lambda_1^{(2)}=\Lambda_1^{(0)}$, $\Lambda_2^{(1)}=\Lambda_2^{(2)}=\Lambda_2^{(0)}$, $\Lambda_3^{(1)}=\Lambda_3^{(2)}=\Lambda_3^{(0)}$, $\Lambda_4^{(1)}=\Lambda_4^{(2)}=\Lambda_4^{(0)}$, $\Lambda_5^{(1)}=\Lambda_5^{(2)}=\Lambda_5^{(0)}$, $\Lambda_6^{(1)}=\Lambda_6^{(2)}=\Lambda_6^{(0)}$, $h^{(1)}=h^{(2)}=h^{(0)}$, $\eta_{xx}^{(1)}=\eta_{xx}^{(2)}=\eta_{xx}^{(0)}$, $\eta_{yy}^{(1)}=\eta_{yy}^{(2)}=\eta_{yy}^{(0)}$.

当系统发生异步屈曲时, 上下层之间存在相对位移, 即$w_{\mathrm{b}}^{(0)}+w_{\mathrm{s}}^{(0)}=w_{\mathrm{b}}^{(1)}+w_{\mathrm{s}}^{(1)}-w_{\mathrm{b}}^{(2)}-w_{\mathrm{s}}^{(2)}\neq 0$. 从式 (10) 和 (11) 出发, 可得系统异步屈曲的控制微分方程如下:

$$\begin{aligned}\delta w_{\mathrm{b}}^{(0)}:\ &\frac{P_{\mathrm{cr}}(h^{(0)})^2}{12}\left[\eta_{xx}^{(0)}\frac{\partial^4 w_{\mathrm{b}}^{(0)}}{\partial x^4}+\eta_{yy}^{(0)}\frac{\partial^4 w_{\mathrm{b}}^{(0)}}{\partial y^4}+(\eta_{xx}^{(0)}+\eta_{yy}^{(0)})\frac{\partial^4 w_{\mathrm{b}}^{(0)}}{\partial x^2\partial y^2}\right]+2k_{\mathrm{p}}\nabla^2(w_{\mathrm{b}}^{(0)}+w_{\mathrm{s}}^{(0)})-2k_{\mathrm{w}}(w_{\mathrm{b}}^{(0)}+w_{\mathrm{s}}^{(0)})-\\&\frac{7\Lambda_6^{(0)}}{12}\nabla^4 w_{\mathrm{s}}^{(0)}-2\Lambda_1^{(0)}\nabla^4 w_{\mathrm{b}}^{(0)}-P_{\mathrm{cr}}\left[\eta_{xx}^{(0)}\frac{\partial^2(w_{\mathrm{b}}^{(0)}+w_{\mathrm{s}}^{(0)})}{\partial x^2}+\eta_{yy}^{(0)}\frac{\partial^2(w_{\mathrm{b}}^{(0)}+w_{\mathrm{s}}^{(0)})}{\partial y^2}\right]=0,\end{aligned} \tag{12}$$

$$\begin{aligned}\delta w_{\mathrm{s}}^{(0)}:\ &\frac{P_{\mathrm{cr}}(h^{(0)})^2}{1\,008}\left[\eta_{xx}^{(0)}\frac{\partial^4 w_{\mathrm{s}}^{(0)}}{\partial x^4}+\eta_{yy}^{(0)}\frac{\partial^4 w_{\mathrm{s}}^{(0)}}{\partial y^4}+(\eta_{xx}^{(0)}+\eta_{yy}^{(0)})\frac{\partial^4 w_{\mathrm{s}}^{(0)}}{\partial x^2\partial y^2}\right]-P_{\mathrm{cr}}\left(\eta_{xx}^{(0)}\frac{\partial^2}{\partial x^2}+\eta_{yy}^{(0)}\frac{\partial^2}{\partial y^2}\right)(w_{\mathrm{b}}^{(0)}+w_{\mathrm{s}}^{(0)})+\\&2\Lambda_4^{(0)}\nabla^2 w_{\mathrm{s}}^{(0)}-2\Lambda_2^{(0)}\nabla^4 w_{\mathrm{s}}^{(0)}-2\Lambda_5^{(0)}\frac{\partial^4 w_{\mathrm{s}}^{(0)}}{\partial x^2\partial y^2}-\frac{7\Lambda_6^{(0)}}{12}\nabla^4 w_{\mathrm{b}}^{(0)}+2(k_{\mathrm{p}}\nabla^2-k_{\mathrm{w}})(w_{\mathrm{b}}^{(0)}+w_{\mathrm{s}}^{(0)})=0.\end{aligned} \tag{13}$$

当系统发生同步屈曲时,上下层之间相对位移为 0,弹性介质不起作用,此时系统的屈曲等同于单层微板的屈曲.令式 (10) 和 (11) 中弹性介质模量为 0,即可得到系统同步屈曲的控制微分方程.

1.3 同步/异步屈曲的解析解

系统发生异步屈曲时,相对位移的 Navier 级数形式为

$$w_{\mathrm{b}}^{(0)}=w_{\mathrm{b}}^{(1)}-w_{\mathrm{b}}^{(2)}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin(\alpha_m x)\sin(\beta_n y),\ w_{\mathrm{s}}^{(0)}=w_{\mathrm{s}}^{(1)}-w_{\mathrm{s}}^{(2)}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin(\alpha_m x)\sin(\beta_n y), \tag{14}$$

式中m,n为半波数,A_{mn},B_{mn}为相对位移幅值,$\alpha_m=m\pi/L_x,\beta_n=n\pi/L_y$.

将式 (14) 代入式 (12) 和 (13), 可得

$$\left\{\begin{bmatrix}K_{11}&K_{12}\\K_{21}&K_{22}\end{bmatrix}+P_{\mathrm{cr}}\begin{bmatrix}G_{11}&G_{12}\\G_{21}&G_{22}\end{bmatrix}\right\}\begin{Bmatrix}A\\B\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}, \tag{15}$$

式中

$$\begin{cases}K_{11}=4\Lambda_1^{(0)}\alpha_m^2\beta_n^2+2\Lambda_1^{(0)}(\alpha_m^4+\beta_n^4)+2k_w+2k_p(\alpha_m^2+\beta_n^2),\\K_{12}=K_{21}=\dfrac{7\Lambda_6^{(0)}}{12}(\alpha_m^2+\beta_n^2)^2+2k_w+2k_p(\alpha_m^2+\beta_n^2),\\K_{22}=4\Lambda_2^{(0)}\alpha_m^2\beta_n^2+2\Lambda_2^{(0)}(\alpha_m^4+\beta_n^4)+2(k_p+\Lambda_4^{(0)})(\alpha_m^2+\beta_n^2)+2k_w,\end{cases} \tag{16}$$

$$\begin{cases}G_{11}=\eta_{xx}^{(0)}\alpha_m^2+\eta_{yy}^{(0)}\beta_n^2+\dfrac{[\eta_{xx}^{(0)}\alpha_m^4+(\eta_{xx}^{(0)}+\eta_{yy}^{(0)})\alpha_m^2\beta_n^2+\eta_{yy}^{(0)}\beta_n^4](h^{(0)})^2}{12},\\G_{12}=G_{21}=\eta_{xx}^{(0)}\alpha_m^2+\eta_{yy}^{(0)}\beta_n^2,\\G_{22}=\eta_{xx}^{(0)}\alpha_m^2+\eta_{yy}^{(0)}\beta_n^2+\dfrac{[\eta_{xx}^{(0)}\alpha_m^4+(\eta_{xx}^{(0)}+\eta_{yy}^{(0)})\alpha_m^2\beta_n^2+\eta_{yy}^{(0)}\beta_n^4](h^{(0)})^2}{1\,008}.\end{cases} \tag{17}$$

通过求解式 (15) 的特征值,即可得到系统异步屈曲荷载的解析解.$\eta_{xx}^{(0)}=1$(或$\eta_{yy}^{(0)}=1$) 和$\eta_{xx}^{(0)}=\eta_{yy}^{(0)}=1$分别代表单轴屈曲和双轴屈曲.当弹性介质模量设为 0 时,式 (15) 退化为同步屈曲的特征方程.

2 数值算例及分析

基于前一节建立的理论模型,本节将探究长宽比、弹性介质模量和无量纲材料尺度参数对双层微板系统单轴和双轴屈曲特性的影响.为了便于今后参照,引入以下无量纲参数:

$$\overline{N}_{\mathrm{cr}}=\frac{P_{\mathrm{cr}}L_y^2}{D\pi^2},\ \bar{k}_{\mathrm{w}}=\frac{k_{\mathrm{w}}L_y^4}{D},\ \bar{k}_{\mathrm{p}}=\frac{k_{\mathrm{p}}L_y^2}{D}. \tag{18}$$

表 1 给出了四边简支 (SSSS) 情形下宏观单层方板无量纲临界屈曲荷载.当同步屈曲模型中$\eta_{xx}^{(0)}=1$, $\eta_{yy}^{(0)}=0$, $\overline{N}_{\mathrm{cr}}=P_{\mathrm{cr}}L_y^2/(D\pi^2)$, $l=0$时,本文模型可退化为宏观的情形.对比本文 Navier 法、文献 [22] 中 Lévy 法、文献 [23] 中解析法、文献 [24] 中样条法所预测的屈曲荷载,可以看出,本文结果与已有文献中结果在宏观情形下吻合较好,且介于文献 [23] 与文献 [24] 中结果之间.以上分析表明,本文模型和计算方法是准确可靠的.

表 1 宏观单层 SSSS 方板的无量纲临界屈曲荷载

Table 1 Dimensionless critical buckling loads on the SSSS single-layered square macroplate

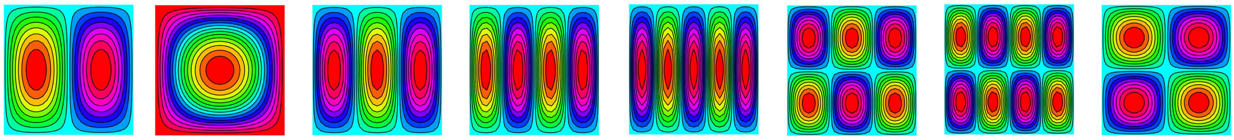
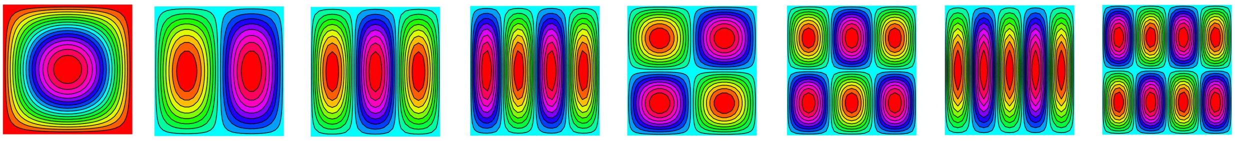
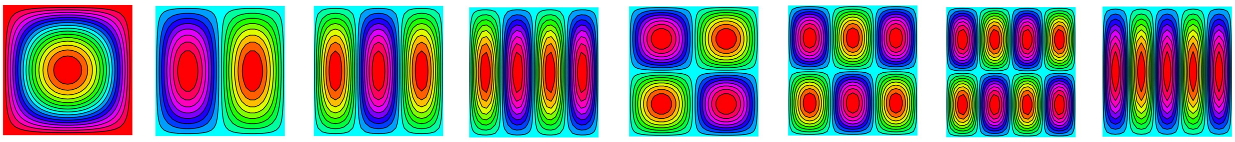
model	L_x/h			
	5	10	20	1 000
ref. [22]	3.265 3	3.786 5	3.944 3	–
ref. [23]	3.255 8	3.783 8	3.943 7	4.000
ref. [24]	3.119 0	3.729 0	3.928 0	4.000
present	3.162 0	3.744 5	3.932 3	4.000

表 2 给出了上下层均为四边简支时双层微板系统在单轴压缩荷载作用下异步屈曲荷载和模态随无量纲材料尺度参数l/h的变化,此处$\eta_{xx}^{(0)}=1$, $\bar{k}_{\mathrm{w}}=100$, $\bar{k}_{\mathrm{p}}=10$, $L_y/h=8$, $L_x/L_y=1$.从表 2 可以看到:随着l/h的增大,异步屈曲荷载显著增大.$l/h=0.5$时,第 1, 2, 5, 7, 8 阶模态半波数发生显著改变,而$l/h=1.0$时,第 1, 2, 5, 8 阶模

态半波数发生显著改变, 这意味着微板的异步屈曲荷载和模态均具有尺度依赖性.

表 2　尺度效应对 SSSS-SSSS 双层微板系统异步屈曲荷载与模态的影响

Table 2　Asynchronous buckling loads and modes of the SSSS-SSSS double-layered microplate system under uniaxial compression

l/h	asynchronous buckling loads and modes							
0								
	$\bar{N}_{cr}^{(2,1)}=7.8790$	$\bar{N}_{cr}^{(1,1)}=9.5908$	$\bar{N}_{cr}^{(3,1)}=9.7135$	$\bar{N}_{cr}^{(4,1)}=11.9743$	$\bar{N}_{cr}^{(5,1)}=14.1063$	$\bar{N}_{cr}^{(3,2)}=14.3461$	$\bar{N}_{cr}^{(4,2)}=15.1594$	$\bar{N}_{cr}^{(2,2)}=15.6784$
0.5								
	$\bar{N}_{cr}^{(1,1)}=13.8852$	$\bar{N}_{cr}^{(2,1)}=14.2003$	$\bar{N}_{cr}^{(3,1)}=20.0869$	$\bar{N}_{cr}^{(4,1)}=27.4524$	$\bar{N}_{cr}^{(2,2)}=31.0655$	$\bar{N}_{cr}^{(3,2)}=31.1938$	$\bar{N}_{cr}^{(5,1)}=35.4414$	$\bar{N}_{cr}^{(4,2)}=35.9428$
1.0								
	$\bar{N}_{cr}^{(1,1)}=26.7590$	$\bar{N}_{cr}^{(2,1)}=33.1370$	$\bar{N}_{cr}^{(3,1)}=51.1440$	$\bar{N}_{cr}^{(4,1)}=73.7790$	$\bar{N}_{cr}^{(2,2)}=77.1413$	$\bar{N}_{cr}^{(3,2)}=81.6258$	$\bar{N}_{cr}^{(4,2)}=98.1460$	$\bar{N}_{cr}^{(5,1)}=99.2938$

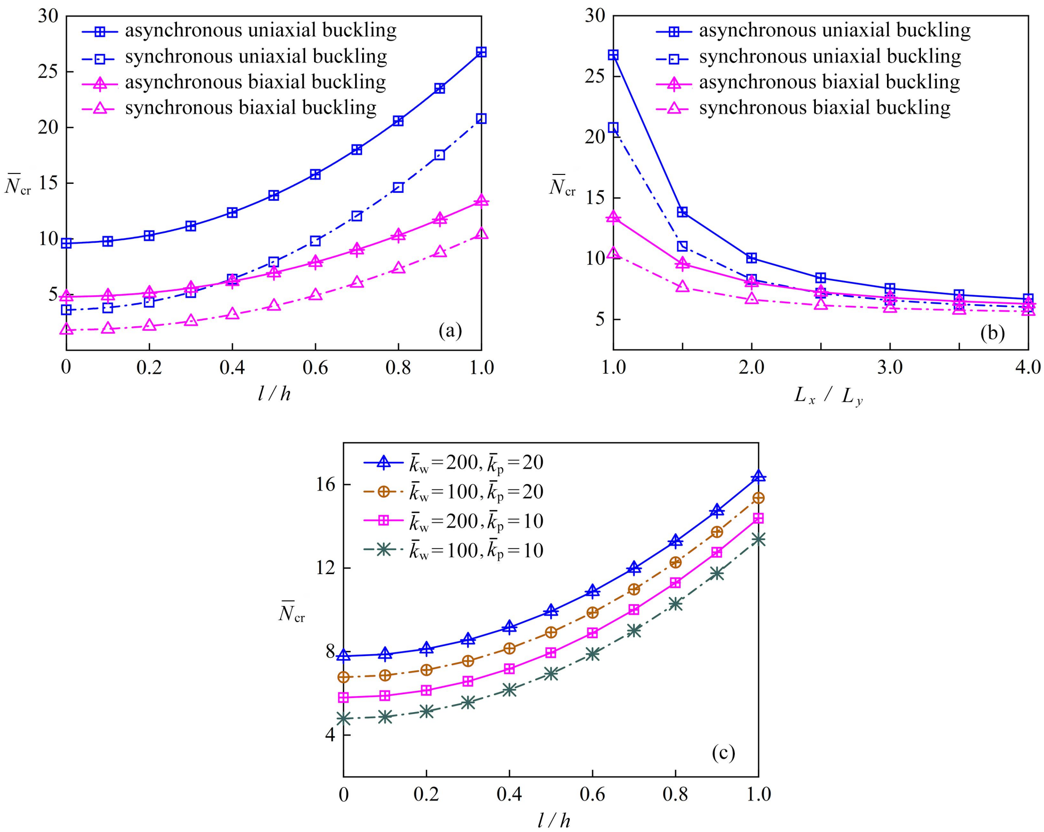


图 2　各参数对双层微板系统临界屈曲荷载影响: (a) 无量纲材料尺度参数; (b) 长宽比; (c) 弹性介质模量

Fig. 2　Effects of different parameters on the critical buckling loads on the double-layered microplate system: (a) the dimensionless material length scale parameter; (b) the aspect ratio; (c) the elastic medium modulus

图 2(a) 展示了双层微板系统的单轴/双轴压缩屈曲荷载随无量纲材料尺度参数l/h的变化, 此处分别取$\eta_{xx}^{(0)}=1$和$\eta_{xx}^{(0)}=\eta_{yy}^{(0)}=1$. 从图 2(a) 可以看出: 随着$l/h$的增加, 系统的屈曲荷载随之增大, 这说明引入尺度效应后

系统的刚度增强,或者说系统的屈曲稳定性提高;尺度效应对单轴压缩屈曲的影响大于双轴压缩屈曲,原因在于尺度效应对系统刚度起到增强作用,而压缩荷载正好相反,两者呈现出反向竞争关系;考虑弹性介质作用下,异步屈曲荷载均大于同步屈曲的情形.

图 2(b) 呈现了双层微板系统的单轴/双轴压缩屈曲荷载随长宽比L_x/L_y的变化,此处分别取$\eta_{xx}^{(0)}=1$和$\eta_{xx}^{(0)}=\eta_{yy}^{(0)}=1$.图中结果显示,随着$L_x/L_y$的增加,单轴压缩和双轴压缩屈曲荷载的差异逐渐减小,这说明系统长宽比的改变会显著影响其承载能力.

图 2(c) 给出了不同 Winkler 和 Pasternak 模量下,双层微板系统的双轴压缩屈曲荷载随无量纲材料尺度参数l/h的变化,此处$\eta_{xx}^{(0)}=\eta_{yy}^{(0)}=1$.由于弹性介质对同步屈曲不起作用,图中仅考虑了异步屈曲.从图 2 中可以看出:随着材料尺度参数或弹性介质模量的增大,系统异步屈曲稳定性逐渐增强;Pasternak 模量相较于 Winkler 模量对系统异步屈曲影响更为显著.

3 结 论

本文基于修正的偶应力理论和双变量高阶剪切变形理论,建立了以 Winkler-Pasternak 弹性介质连接的双层微板系统的屈曲模型;推导了系统屈曲的控制微分方程,给出了上下层均为四边简支时系统屈曲的临界荷载的解析解;讨论了尺度效应、长宽比和弹性介质对系统屈曲特性的影响.结果表明:尺度效应不仅会显著增强系统的临界屈曲荷载,而且会导致系统某些阶次的屈曲模态发生改变;长宽比和弹性介质模量的改变会使得系统屈曲特性的改变,弹性介质模量越大,临界屈曲荷载越大,且较之于 Winkler 模量,Pasternak 模量对系统屈曲特性的影响更显著.

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