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油水渗流动边值问题的迎风差分方法*

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摘要: 可压缩可混溶油、水渗流动边值问题的研究,对重建盆地发育中油气资源运移、聚集的历史和评估油气资源的勘探与开发有重要的价值,其数学模型是一组非线性耦合偏微分方程组的动边值问题.对二维有界域的动边值问题提出一类新的迎风差分格式,应用区域变换、变分形式、能量方法、差分算子乘积交换性理论、高阶差分算子的分解、微分方程先验估计的理论和技巧,得到了最佳误差估计结果.该方法已成功应用到油资评估的数值模拟中.它对这一领域的模型分析,数值方法和软件研制均有重要的价值.

关 键 词: 可压缩渗流; 动边界; 迎风分数步差分; 最佳误差估计; 应用
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引 言

可压缩可混溶油、水渗流动边值问题的研究,对于重建盆地的运移、聚集的历史和评估油气资源的勘探和开发有着重要的价值.文献[1]研究了关于不可压缩油藏盆地发育的动边值问题,而在许多实际情况,需要考虑流体的压缩性,其密度实际上是依赖于压力的[2-5].对于可压缩可混溶二维有界区域的动边值问题,提出一类新的迎风差分格式,应用区域变换、变分形式、能量方法、差分算子乘积交换性理论、高阶差分算子的分解、先验估计理论和技巧,得到了最佳 l^2 误差估计结果.

问题的数学模型是一组非线性抛物型耦合偏微分方程组的动边值问题[2-5]:

$$d(c)\frac{\partial p}{\partial t}+\nabla\cdot\boldsymbol{u}=Q(\boldsymbol{X},t),\quad \boldsymbol{X}=(x_1,x_2)^{\mathrm{T}}\in\Omega(t),\qquad t\in J=(0,T],\tag{1a}$$

$$\boldsymbol{u}=-a(c)\nabla p,\qquad \boldsymbol{X}\in\Omega(t),\ t\in J,\tag{1b}$$

$$\varphi(\boldsymbol{X})\frac{\partial c}{\partial t}+b(c)\frac{\partial p}{\partial t}+\boldsymbol{u}\cdot\nabla c-\nabla\cdot(D\nabla c)=f(\boldsymbol{X},t,c),\qquad \boldsymbol{X}\in\Omega(t),\ t\in J.\tag{2}$$

方程(1a)和(1b)是流动方程, p 为地层压力, $\boldsymbol{u}$ 是 Darcy速度,均为待求函数, $a(c)$ =

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$a(\boldsymbol{X}, c) = k(\boldsymbol{X})\mu(c)^{-1}$, $k(\boldsymbol{X})$ 是地层的渗透率, $\mu(c)$ 为流体速度, $d(c)$, $a(c)$ 均为正定函数, $Q(\boldsymbol{X}, t)$ 是产量项. 方程 (2) 是饱和度方程, c 为饱和度函数, 亦为待求函数. $\varphi(\boldsymbol{X}, t)$ 是地层孔隙度, $D(\boldsymbol{X}, t)$ 是扩散系数, φ, D 亦均为正定函数[2,5]. 这里

$$\Omega(t)=\left\{ \boldsymbol{X} \mid s_1(x_2, t) \leqslant x_1 \leqslant s_2(x_2, t),\ 0 \leqslant x_2 \leqslant L_0(t);\ t\in J\right\},$$

$s_i(x_2, t)\ (i=1, 2)$和$L_0(t)$是已知函数, 对于 $t\in J$具有一阶连续的导函数, 如图 1所示, 记号$\nabla=(\partial/\partial x_1, \partial/\partial x_2)^{\mathrm{T}}$.

定压边界条件:

$$p(\boldsymbol{X}, t) = e(\boldsymbol{X}, t),\ c(\boldsymbol{X}, t) = r(\boldsymbol{X}, t),\quad \boldsymbol{X}\in \partial\Omega(t),\ t\in J, \tag{3}$$

此处 $\partial\Omega(t)$ 是 $\Omega(t)$ 的边界曲线.

初始条件:

$$p(\boldsymbol{X}, 0) = p_0(\boldsymbol{X}),\quad c(\boldsymbol{X}, 0) = c_0(\boldsymbol{X}),\quad \boldsymbol{X}\in \Omega(0). \tag{4}$$

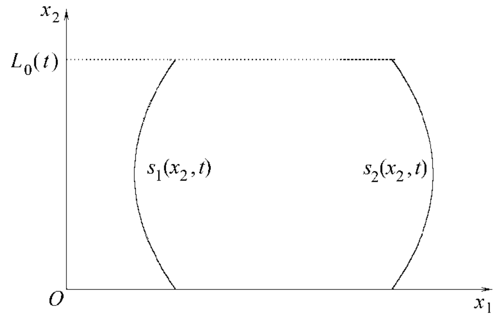

图 1 $\Omega(t)$示意图

对于可压缩、相混溶渗流驱动问题, 其中, 关于压力方程是抛物型方程, 饱和度方程是对流-扩散方程. 由于以对流为主的扩散方程具有很强的双曲特性, 应用中心差分格式, 虽然关于空间步长具有二阶精确度, 但会产生数值弥散和非物理特征的数值振荡, 使数值模拟失真[5-6].

对平面不可压缩两相渗流驱动问题, 在问题的周期性假定下, Douglas, Ewing, Wherler, Russell等提出特征差分方法和特征有限元法, 并给出误差估计[7-11]. 他们将特征线方法和标准的有限差分方法或有限元方法相结合, 真实的反映出对流-扩散方程的一阶双曲特性, 减少截断误差, 克服数值振荡和弥散, 大大提高计算的稳定性和精确度. 对可压缩渗流驱动问题, Douglas等学者同样在周期性假定下, 提出二维可压缩渗流驱动问题的"微小压缩"数学模型、数值方法和分析[2-4], 开创了现代数值模型这一新领域[5]. 作者去掉周期性的假定, 给出新的修正特征差分格式和有限元格式, 并得到 l^2 最佳的模误差估计[1,12-13]. 由于特征线法需要进行插值计算, 并且特征线在求解区域边界附近可能穿出边界, 需要作特殊处理. 特征线与网格边界交点及其相应的函数值需要计算, 这样在算法设计时, 对靠近边界的网格点需要判断其特征线是否越过边界, 从而确定是否需要改变时间步长, 因此, 实际计算还是比较复杂的[12-13].

对抛物型问题, Axelsson, Ewing, Lazarov等提出迎风差分格式[14-17], 来克服数值解的振荡, 同时避免特征差分方法在对靠近边界网点的计算复杂性. 本文从生产实际出发, 对可压缩渗流驱动的一般形式动边值问题, 通过变量替换和分析, 化为基本上是同一类型的问题, 再提出一类迎风差分格式, 该格式既可克服数值振荡和弥散, 同时将高维问题化为连续解几个一维问题, 大大减少计算工作量, 使工程实际计算成为可能. 且将空间的计算精度提高到二阶. 应用区域变换、变分形式、能量方法、差分算子乘积交替性理论、高阶差分算子的分解、微分方程先验估计和特殊的技巧, 得到了最佳 l^2 模误差估计, 成功地解决了 Douglas, Ewing所提的著名问题[18-20]. 该方法已成功地应用到油资源评估和运移聚集的数值模拟中①, ②, 从而在能源数学这一领域起到一定程度的奠基作用.

假定方程 (1) ~ (4) 的精确解是正则的, 即[2,5,7-9]

① 山东大学 数学研究所, 胜利石油管理局计算中心: 三维盆地模拟系统研究, 1995.4

② 山东大学 数学研究所, 胜利油田物探研究院: 石油运移聚集通道数值模拟研究, 2005.10.

(R) $p, c \in L^{\infty}(0, T; W^{4,\infty}(\Omega(t))), \quad \frac{\partial^2 p}{\partial t^2}, \frac{\partial^2 c}{\partial t^2} \in L^{\infty}(0, T; L^{\infty}(\Omega(t)))$.

假定方程(1)和(2)的系数满足:

$$\begin{cases} 0 < a_* \leqslant a(c) \leqslant a^*, \quad 0 < d_* \leqslant d(c) \leqslant d^*, \\ 0 < D_* \leqslant D(\boldsymbol{X}, t) \leqslant D^*, \quad 0 < \varphi_* \leqslant \varphi(\boldsymbol{X}, t) \leqslant \varphi^*, \end{cases} \tag{5a}$$

$$(\mathrm{C}) \quad \left| \frac{\mathrm{d}a}{\mathrm{d}c}(c) \right| \leqslant K^*, \tag{5b}$$

此处 a_*, a^*, d_*, d^*, D_*, D^*, φ_*, φ^*, K^* 均为正常数, $d(c)$, $b(c)$和$f(c)$在解的 ε_0 邻域是Lipschitz连续的, 即存在正常数M, 当$|\varepsilon_i| \leqslant \varepsilon_0$ $(1 \leqslant i \leqslant 6)$时有

$$|d(c(\boldsymbol{X}, t) + \varepsilon_1) - d(c(\boldsymbol{X}, t) + \varepsilon_2)| \leqslant M|\varepsilon_1 - \varepsilon_2|,$$
$$|b(c(\boldsymbol{X}, t) + \varepsilon_3) - b(c(\boldsymbol{X}, t) + \varepsilon_4)| \leqslant M|\varepsilon_3 - \varepsilon_4|,$$
$$|f(c(\boldsymbol{X}, t) + \varepsilon_5) - f(c(\boldsymbol{X}, t) + \varepsilon_6)| \leqslant M|\varepsilon_5 - \varepsilon_6|.$$

本文中记号M和 ε分别表示普通正常数和普通小正数, 在不同处可具有不同的含义.

1 迎风分数步差分格式

对方程(1)~(5)引入下述变量替换

$$\boldsymbol{X} = (x_1, x_2)^{\mathrm{T}} \in \Omega(t) \mapsto$$
$$\boldsymbol{Y} = (y_1, y_2)^{\mathrm{T}} = \left(\frac{x_1 - s_1(x_2, t)}{s_2(x_2, t) - s_1(x_2, t)}, L_0^{-1}(t) x_2 \right)^{\mathrm{T}} \in \hat{\Omega} = \{[0, 1]^2\}, \tag{6a}$$

$$\boldsymbol{Y} = (y_1, y_2)^{\mathrm{T}} \in \hat{\Omega} \mapsto \boldsymbol{X} = (x_1, x_2)^{\mathrm{T}} = ((s_2(L_0(t)y_2, t) - s_1(L_0(t)y_2, t))y_1 +$$
$$s_1(L_0(t)y_2, t), \ L_0(t)y_2)^{\mathrm{T}} \in \Omega(t), \qquad t \in J. \tag{6b}$$

令函数

$$\hat{\varphi}(\boldsymbol{Y}, t) = \varphi((s_2(L_0(t)y_2, t) - s_1(L_0(t)y_2, t))y_1 +$$
$$s_1(L_0(t)y_2, t), L_0(t)y_2)^{\mathrm{T}}, \qquad (\boldsymbol{Y}, t) \in \hat{\Omega} \times J. \tag{7}$$

注意到

$$\frac{\partial \hat{\varphi}}{\partial t} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x_1} \left\{ (s_2'(L_0(t)y_2, t) - s_1'(L_0(t)y_2, t))y_1 + s_1'(L_0(t)y_2, t) \right\} +$$
$$\frac{\partial \varphi}{\partial x_2} \dot{L}_0(t) y_2,$$

$$\frac{\partial \varphi}{\partial x_1} = S^{-1}(y_2, t) \frac{\partial \hat{\varphi}}{\partial y_1}, \quad \frac{\partial \varphi}{\partial x_2} = \left(\frac{\partial \hat{\varphi}}{\partial y_2} - S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial \hat{\varphi}}{\partial y_1} \right) L_0^{-1}(t),$$

此处 $s_i'(L_0(t)y_2, t)$ $(i = 1, 2)$, $\dot{L}_0(t)$表示对 t的导函数,

$$S(y_2, t) = s_2(L_0(t)y_2, t) - s_1(L_0(t)y_2, t),$$
$$\alpha(\boldsymbol{Y}, t) = \frac{\partial S(y_2, t)}{\partial y_2} y_1 + \frac{\partial s_1(L_0(t)y_2, t)}{\partial y_2}.$$

于是可得:

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \hat{\varphi}}{\partial t} - S^{-1}(y_2, t) B(\boldsymbol{Y}, t) \frac{\partial \hat{\varphi}}{\partial y_1} -$$
$$\left(\frac{\partial \hat{\varphi}}{\partial y_2} - S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial \hat{\varphi}}{\partial y_1} \right) L_0^{-1}(t) \dot{L}_0(t) y_2,$$

此处 $B(\boldsymbol{Y}, t) = \dot{S}(y_2, t) y_1 + \dot{s}(L_0(t) y_2, t)$, $\dot{L}_0(t)$, $\dot{S}(y_2, t)$ 和 $\dot{s}(L_0(t) y_2, t)$ 表示对 t 的导数. 假定其满足条件:

$$s_* \leqslant S(y_2, t) \leqslant S^*, \quad l_* \leqslant L_0(t) \leqslant L^*,$$

此处 s_*, S^*, l_* 和 L^* 均为确定的正常数. 由此我们可得

$$\frac{\partial p}{\partial t} = \frac{\partial \hat{p}}{\partial t} - S^{-1}(y_2, t) B(\boldsymbol{Y}, t) \frac{\partial \hat{p}}{\partial y_1} - \left(\frac{\partial \hat{p}}{\partial y_2} - S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial \hat{p}}{\partial y_1} \right) L_0^{-1}(t) \dot{L}_0(t) y_2, \tag{8a}$$

$$\frac{\partial c}{\partial t} = \frac{\partial \hat{c}}{\partial t} - S^{-1}(y_2, t) B(\boldsymbol{Y}, t) \frac{\partial \hat{c}}{\partial y_1} - \left(\frac{\partial \hat{c}}{\partial y_2} - S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial \hat{c}}{\partial y_1} \right) L_0^{-1}(t) \dot{L}_0(t) y_2. \tag{8b}$$

注意到

$$\frac{\partial}{\partial x_1}\left(a(\boldsymbol{X}, t) \frac{\partial \varphi}{\partial x_1} \right) = S^{-1}(y_2, t) \frac{\partial}{\partial y_1}\left(\hat{a}(\boldsymbol{Y}, t) S^{-1}(y_2, t) \frac{\partial \hat{\varphi}}{\partial y_1} \right),$$

$$\begin{aligned} \frac{\partial}{\partial x_2}\left(a(\boldsymbol{X}, t) \frac{\partial \varphi}{\partial x_2} \right) = {} & L_0^{-2}(t) \frac{\partial}{\partial y_2}\left(\hat{a}(\boldsymbol{Y}, t) \frac{\partial \hat{\varphi}}{\partial y_2} \right) + \\ & L_0^{-2}(t) S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial}{\partial y_1}\left(\hat{a}(\boldsymbol{Y}, t) S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial \hat{\varphi}}{\partial y_1} \right) - \\ & L_0^{-2}(t) S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial}{\partial y_1}\left(\hat{a}(\boldsymbol{Y}, t) \frac{\partial \hat{\varphi}}{\partial y_2} \right) - \\ & L_0^{-2}(t) \frac{\partial}{\partial y_2}\left(\hat{a}(\boldsymbol{Y}, t) S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial \hat{\varphi}}{\partial y_1} \right). \end{aligned}$$

由此我们可得

$$\begin{aligned} \nabla \cdot u = {} & -\nabla \cdot (a(c) \nabla p) = -\left\{ \frac{\partial}{\partial x_1}\left(a(c) \frac{\partial p}{\partial x_1} \right) + \frac{\partial}{\partial x_2}\left(a(c) \frac{\partial p}{\partial x_2} \right) \right\} = \\ & -\left\{ S^{-1}(y_2, t) \frac{\partial}{\partial y_1}\left(\hat{a}(\hat{c}) S^{-1}(y_2, t) \frac{\partial \hat{p}}{\partial y_1} \right) + L_0^{-2}(t) S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \times \right. \\ & \frac{\partial}{\partial y_1}\left(S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial \hat{p}}{\partial y_1} \right) - L_0^{-2}(t) S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial}{\partial y_1}\left(\hat{a}(\hat{c}) \frac{\partial \hat{p}}{\partial y_2} \right) - \\ & \left. L_0^{-2}(t) \frac{\partial}{\partial y_2}\left(\hat{a}(\hat{c}) S^{-1}(y_2, t) \alpha(\boldsymbol{Y}, t) \frac{\partial \hat{p}}{\partial y_1} \right) + L_0^{-2}(t) \frac{\partial}{\partial y_2}\left(\hat{a}(\hat{c}) \frac{\partial \hat{p}}{\partial y_2} \right) \right\}. \end{aligned}$$

于是对方程 (1a) 和 (1b) 及 (2) 作变量替换 (6), 经整理可得

$$\begin{aligned} & \hat{d} \frac{\partial \hat{p}}{\partial t} - \hat{d}\left\{ S^{-1} B \frac{\partial \hat{p}}{\partial y_1} + \left(\frac{\partial \hat{p}}{\partial y_2} - S^{-1} \alpha \frac{\partial \hat{p}}{\partial y_1} \right) L_0^{-1} \dot{L}_0 y_2 \right\} - \left\{ S^{-1} \frac{\partial}{\partial y_1}\left(\hat{a} S^{-1} \frac{\partial \hat{p}}{\partial y_1} \right) + \right. \\ & L_0^{-2} S^{-1} \alpha \frac{\partial}{\partial y_1}\left(\hat{a} S^{-1} \alpha \frac{\partial \hat{p}}{\partial y_1} \right) - L_0^{-2} S^{-1} \alpha \frac{\partial}{\partial y_1}\left(\hat{a} \frac{\partial \hat{p}}{\partial y_2} \right) - \\ & \left. L_0^{-2} \frac{\partial}{\partial y_2}\left(\hat{a} S^{-1} \alpha \frac{\partial \hat{p}}{\partial y_1} \right) + L_0^{-2} \frac{\partial}{\partial y_2}\left(\hat{a} \frac{\partial \hat{p}}{\partial y_2} \right) \right\} = \\ & \hat{Q}(\boldsymbol{Y}, t), \qquad \boldsymbol{Y} \in \hat{\Omega}, \quad t \in J, \end{aligned} \tag{9a}$$

$$\hat{\boldsymbol{u}} = -\hat{a}(\hat{c})\left(S^{-1}\frac{\partial\hat{p}}{\partial y_1}, \left(\frac{\partial\hat{p}}{\partial y_2} - S^{-1}\alpha\frac{\partial\hat{p}}{\partial y_1}\right)L_0^{-1}(t)\right)^{\mathrm{T}}, \tag{9b}$$

$$\begin{aligned}
&\hat{\varphi}\frac{\partial\hat{c}}{\partial t} - \hat{\varphi}\left\{S^{-1}B\frac{\partial\hat{c}}{\partial y_1} + \left(\frac{\partial\hat{c}}{\partial y_2} - S^{-1}\alpha\frac{\partial\hat{c}}{\partial y_1}\right)L_0^{-1}\dot{L}_0 y_2\right\} + \\
&\quad \hat{\boldsymbol{u}}\cdot\left(S^{-1}\frac{\partial\hat{c}}{\partial y_1}, \left(\frac{\partial\hat{c}}{\partial y_2} - S^{-1}\alpha\frac{\partial\hat{c}}{\partial y_1}\right)L_0^{-1}(t)\right)^{\mathrm{T}} - \\
&\quad \left\{S^{-1}\frac{\partial}{\partial y_1}\left(\hat{D}S^{-1}\frac{\partial\hat{c}}{\partial y_1}\right) + L_0^{-2}S^{-1}\alpha\frac{\partial}{\partial y_1}\left(\hat{D}S^{-1}\alpha\frac{\partial\hat{c}}{\partial y_1}\right) - \right.\\
&\quad \left. L_0^{-2}S^{-1}\alpha\frac{\partial}{\partial y_1}\left(\hat{D}\frac{\partial\hat{c}}{\partial y_2}\right) - L_0^{-2}\frac{\partial}{\partial y_2}\left(\hat{D}S^{-1}\alpha\frac{\partial\hat{c}}{\partial y_1}\right) + L_0^{-2}\frac{\partial}{\partial y_2}\left(\hat{D}\frac{\partial\hat{c}}{\partial y_2}\right)\right\} + \\
&\quad \hat{b}\frac{\partial\hat{p}}{\partial t} - \hat{b}\left\{S^{-1}B\frac{\partial\hat{p}}{\partial y_1} + \left(\frac{\partial\hat{p}}{\partial y_2} - S^{-1}\alpha\frac{\partial\hat{p}}{\partial y_1}\right)L_0^{-1}\dot{L}_0 y_2\right\} = \\
&\quad \hat{f}(\boldsymbol{Y}, t, \hat{c}), \qquad \boldsymbol{Y}\in\hat{\Omega}, \quad t\in J.
\end{aligned} \tag{10}$$

对上述方程(9)和(10), 计算规模是很大的, 当 $\alpha(\boldsymbol{Y}, t)$ 较小时, 工程上通常可舍去混合导数项[5,20], 此时可采用分数步长计算格式, 大大减少计算工作量. 对此简化模型, 注意到

$$S^{-1} = S^{-1}(y_2, t),$$

$$L_0^{-2} = L_0^{-2}(t),$$

$$\alpha\frac{\partial}{\partial y_1}\left(\alpha\hat{a}S^{-2}L_0^{-2}\frac{\partial\hat{p}}{\partial y_1}\right) = \frac{\partial}{\partial y_1}\left(\hat{a}\alpha^2S^{-2}L_0^{-2}\frac{\partial\hat{p}}{\partial y_2}\right) - \frac{\partial}{\partial y_2}S(y_2, t)\cdot\alpha\hat{a}S^{-2}L_0^{-2}\frac{\partial\hat{p}}{\partial y_1}$$

以及

$$S^{-1}\left(B - \frac{\partial}{\partial y_2}S(y_2, t)\cdot\hat{d}^{-1}\alpha\hat{a}S^{-1}L_0^{-2}\right)\frac{\partial\hat{p}}{\partial y_1} + \left(\frac{\partial\hat{p}}{\partial y_2} - S^{-1}\alpha\frac{\partial\hat{p}}{\partial y_1}\right)L_0^{-1}\dot{L}_0 y_2 = -\hat{a}^{-1}\boldsymbol{B}_{p\alpha}\cdot\hat{\boldsymbol{u}},$$

此处

$$\boldsymbol{B}_{p\alpha} = \left(B - \frac{\partial}{\partial y_2}S(y_2, t)\cdot\hat{d}^{-1}\alpha\hat{a}S^{-1}L_0^{-2}, \dot{L}_0 y_2\right)^{\mathrm{T}},$$

则可将流动方程(9)写为下述标准形式:

$$\begin{aligned}
&\hat{d}\frac{\partial\hat{p}}{\partial t} + \hat{d}\hat{a}^{-1}\boldsymbol{B}_{p\alpha}\cdot\hat{\boldsymbol{u}} - \left\{\frac{\partial}{\partial y_1}\left(\hat{a}S^{-2}(1+\alpha^2L_0^{-2})\frac{\partial\hat{p}}{\partial y_1}\right) + \frac{\partial}{\partial y_2}\left(\hat{a}L_0^{-2}\frac{\partial\hat{p}}{\partial y_2}\right)\right\} = \\
&\quad \hat{Q}(\boldsymbol{Y}, t), \qquad \boldsymbol{Y}\in\hat{\Omega}, \quad t\in J.
\end{aligned} \tag{11}$$

类似地, 注意到

$$\begin{aligned}
&-\hat{\varphi}\left\{S^{-1}\left[B - \frac{\partial}{\partial y_2}S(y_2, t)\hat{\varphi}^{-1}\hat{D}\alpha S^{-1}L_0^{-2}\right]\frac{\partial\hat{c}}{\partial y_1} + \left(\frac{\partial\hat{c}}{\partial y_2} - S^{-1}\alpha\frac{\partial\hat{c}}{\partial y_1}\right)L_0^{-1}\dot{L}_0 y_2\right\} + \\
&\quad \hat{\boldsymbol{u}}\cdot\left(S^{-1}\frac{\partial\hat{c}}{\partial y_1}, \left(\frac{\partial\hat{c}}{\partial y_2} - S^{-1}\alpha\frac{\partial\hat{c}}{\partial y_1}\right)L_0^{-1}(t)\right)^{\mathrm{T}} = \\
&\quad -\hat{\varphi}\left\{S^{-1}\left[B - \alpha L_0^{-1}\dot{L}_0 y_2 - \frac{\partial}{\partial y_2}S(y_2, t)\hat{\varphi}^{-1}\hat{D}\alpha S^{-1}L_0^{-2}\right]\frac{\partial\hat{c}}{\partial y_1} + L_0^{-1}\dot{L}_0 y_2\frac{\partial\hat{c}}{\partial y_2}\right\} + \\
&\quad \hat{\boldsymbol{u}}\cdot\left(S^{-1}\frac{\partial\hat{c}}{\partial y_1}, \left(\frac{\partial\hat{c}}{\partial y_2} - S^{-1}\alpha\frac{\partial\hat{c}}{\partial y_1}\right)L_0^{-1}(t)\right)^{\mathrm{T}} = \\
&\quad \left\{\boldsymbol{L}_{c\alpha}\hat{\boldsymbol{u}} - \hat{\varphi}\boldsymbol{B}_{c\alpha}\right\}\cdot\nabla\hat{c},
\end{aligned}$$

此处

$$L_{c\alpha}=\begin{pmatrix} S^{-1} & \alpha S^{-1}L_0^{-1} \\ 0 & L_0^{-1}\end{pmatrix},$$

$$B_{c\alpha}=\left(S^{-1}\left[B-\alpha L_0^{-1}\dot{L}_0 y_2-\frac{\partial}{\partial y_2}S(y_2,t)\hat{\varphi}^{-1}\hat{D}\,\alpha S^{-1}L_0^{-2}\right],\ L_0^{-1}\dot{L}_0 y_2\right)^{\mathrm{T}},$$

则饱和度方程(10)可写成下述标准形式:

$$\hat{\varphi}\frac{\partial\hat{c}}{\partial t}+\left\{L_{c\alpha}\hat{u}-\hat{\varphi}B_{c\alpha}\right\}\cdot\nabla\hat{c}-\left\{\frac{\partial}{\partial y_1}\left(\hat{D}S^{-2}(1+\alpha^2L_0^{-2})\frac{\partial\hat{c}}{\partial y_1}\right)+\frac{\partial}{\partial y_2}\left(\hat{D}L_0^{-2}\frac{\partial\hat{c}}{\partial y_2}\right)\right\}+\hat{b}\frac{\partial\hat{p}}{\partial t}+\hat{b}\,\hat{a}^{-1}B_{po}\cdot\hat{u}=\hat{f}(Y,t,\hat{c}),\qquad Y\in\hat{\Omega},\ t\in J,\tag{12}$$

此处 $B_{po}=(B,\dot{L}_0y_2)^{\mathrm{T}},\ \nabla=\left(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2}\right)^{\mathrm{T}}.$

为了书写简便,记

$$\hat{a}_1=\hat{a}_1(\hat{c})=\hat{a}(\hat{c})S^{-2}(y_2,t)(1+L_0^{-2}(t)\alpha^2(Y,t)),\quad \hat{a}_2=\hat{a}_2(\hat{c})=\hat{a}(\hat{c})L_0^{-2}(t);$$

$$\hat{D}_1=\hat{D}_1(Y,t)=\hat{D}(Y,t)S^{-2}(y_2,t)(1+L_0^{-2}(t)\alpha^2(Y,t)),$$

$$\hat{D}_2=\hat{D}_2(Y,t)=\hat{D}(Y,t)L_0^{-2}(t).$$

于是方程(11)和(12)写为下述形式:

$$\hat{d}\frac{\partial\hat{p}}{\partial t}+\hat{d}\,\hat{a}^{-1}B_{p\alpha}\cdot\hat{u}-\left\{\frac{\partial}{\partial y_1}\left(\hat{a}_1(\hat{c})\frac{\partial\hat{p}}{\partial y_1}\right)+\frac{\partial}{\partial y_2}\left(\hat{a}_2(\hat{c})\frac{\partial\hat{p}}{\partial y_2}\right)\right\}=\hat{Q}(Y,t),\qquad Y\in\hat{\Omega},\ t\in J,\tag{13a}$$

$$\hat{u}=-\hat{a}(\hat{c})\left(S^{-1}\frac{\partial\hat{p}}{\partial y_1},\left(\frac{\partial\hat{p}}{\partial y_2}-S^{-1}\alpha\frac{\partial\hat{p}}{\partial y_1}\right)L_0^{-1}(t)\right)^{\mathrm{T}},\tag{13b}$$

$$\hat{\varphi}\frac{\partial\hat{c}}{\partial t}+\left\{L_{c\alpha}\hat{u}-\hat{\varphi}B_{c\alpha}\right\}\cdot\nabla\hat{c}-\left\{\frac{\partial}{\partial y_1}\left(\hat{D}_1\frac{\partial\hat{c}}{\partial y_1}\right)+\frac{\partial}{\partial y_2}\left(\hat{D}_2\frac{\partial\hat{c}}{\partial y_2}\right)\right\}+\hat{b}\frac{\partial\hat{p}}{\partial t}+\hat{b}\,\hat{a}^{-1}B_{po}\cdot\hat{u}=\hat{f}(Y,t,\hat{c}),\qquad Y\in\hat{\Omega},\ t\in J.\tag{14}$$

这样将原方程(1a)和(1b)及(2)简化为标准域 $\hat{\Omega}=\left\{[0,1]^2\right\}$ 上求解方程(13)和(14).

为了用差分方法求解,我们用网格区域 $\hat{\Omega}_h$ 代替 $\hat{\Omega}$,用 $\partial\hat{\Omega}_h$ 表示 $\hat{\Omega}_h$ 的边界,取定 $\Delta t=T/L$,对 $\hat{\Omega}_h$ 采用等距剖分, $0=y_{10}<y_{11}<y_{12}<\cdots<y_{1N}=1$; $0=y_{20}<y_{21}<\cdots<y_{2N}=1$; $h=1/N$. 记

$$Y_{ij}=(ih_1,jh_2)^{\mathrm{T}},\quad t^n=n\Delta t,\quad W(Y_{ij},t^n)=W_{ij}^n,$$

$$A_{k,i+1/2,j}^n=\frac{1}{2}\left[\hat{a}_k(Y_{ij},\hat{C}_{h,ij}^n)+\hat{a}_k(Y_{i+1,j},\hat{C}_{h,i+1,j}^n)\right],$$

$$A_{k,i,j+1/2}^n=\frac{1}{2}\left[\hat{a}_k(Y_{ij},\hat{C}_{h,ij}^n)+\hat{a}_k(Y_{i,j+1},\hat{C}_{h,i,j+1}^n)\right],$$

$$a_{k,i+1/2,j}^n=\frac{1}{2}\left[\hat{a}_k(Y_{ij},\hat{c}_{ij}^n)+\hat{a}_k(Y_{i+1,j},\hat{c}_{i+1,j}^n)\right],$$

$$a_{k,i,j+1/2}^n=\frac{1}{2}\left[\hat{a}_k(Y_{ij},\hat{c}_{ij}^n)+\hat{a}_k(Y_{i,j+1},\hat{c}_{i,j+1}^n)\right],$$

$$\delta_{\bar{y}_1}(A_1^n\delta_{y_1}\hat{P}_h^{n+1})_{ij}=h^{-2}\left\{A_{1,i+1/2,j}^n(\hat{P}_{h,i+1,j}^{n+1}-\hat{P}_{h,ij}^{n+1})-A_{1,i-1/2,j}^n(\hat{P}_{h,ij}^{n+1}-\hat{P}_{h,i-1,j}^{n+1})\right\},$$

$$\delta_{y_2}(A_2^n\delta_{y_2}\hat{P}_h^{n+1})_{ij}=h^{-2}\left\{A_{2,i,j+1/2}^n(\hat{P}_{h,i,j+1}^{n+1}-\hat{P}_{h,ij}^{n+1})-A_{2,i,j-1/2}^n(\hat{P}_{h,ij}^{n+1}-\hat{P}_{h,i,j-1}^{n+1})\right\},$$

$$\nabla_h(\boldsymbol{A}^n\nabla_h\hat{P}_h^{n+1})_{ij}=\delta_{y_1}(A_1^n\delta_{y_1}\hat{P}_h^{n+1})_{ij}+\delta_{y_2}(A_2^n\delta_{y_2}\hat{P}_h^{n+1})_{ij},$$

此处 $$\boldsymbol{A}^n(t)=\begin{pmatrix}A_1^n & 0\\ 0 & A_2^n\end{pmatrix}.$$

则方程(13a)($t=t^{n+1}$)的分数步差分格式:

$$\hat{d}(\hat{C}_{h,ij}^n)\frac{\hat{P}_{h,ij}^{n+1/2}-\hat{P}_{h,ij}^n}{\Delta t}=\delta_{y_1}(A_1^n\delta_{y_1}\hat{P}_h^{n+1/2})_{ij}+\delta_{y_2}(A_2^n\delta_{y_2}\hat{P}_h^n)_{ij}-$$
$$\hat{d}(\hat{C}_{h,ij}^n)\hat{a}^{-1}(\hat{C}_{h,ij}^n)\boldsymbol{B}_{p\alpha}(\boldsymbol{Y}_{ij},t^{n+1})\cdot\hat{\boldsymbol{U}}_{h,ij}^n+\hat{Q}(\boldsymbol{Y}_{ij},t^{n+1}),\qquad 1\leqslant i\leqslant N-1,\tag{15a}$$

$$\hat{P}_{h,ij}^{n+1/2}=\hat{e}_{ij}^{n+1},\qquad \boldsymbol{Y}_{ij}\in\partial\hat{\Omega}_h,\tag{15b}$$

$$\hat{d}(\hat{C}_{h,ij}^n)\frac{\hat{P}_{h,ij}^{n+1}-\hat{P}_{h,ij}^{n+1/2}}{\Delta t}=\delta_{y_2}(A_2^n\delta_{y_2}(\hat{P}_h^{n+1}-\hat{P}_h^n))_{ij},\qquad 1\leqslant j\leqslant N-1,\tag{15c}$$

$$\hat{P}_{h,ij}^{n+1}=\hat{e}_{ij}^{n+1},\qquad \boldsymbol{Y}_{ij}\in\partial\hat{\Omega}_h.\tag{15d}$$

近似 Darcy速度$\hat{\boldsymbol{U}}_h^n=(\hat{\boldsymbol{V}}^n,\hat{\boldsymbol{W}}^n)^{\mathrm{T}}$,按下述公式计算:

$$\hat{V}_{ij}^n=-(2S(y_2,t^n)h)^{-1}(A_{i+1/2,j}^n(\hat{P}_{h,i+1,j}^n-\hat{P}_{h,ij}^n)+A_{i-1/2,j}^n(\hat{P}_{h,ij}^n-\hat{P}_{h,i-1,j}^n)),\tag{16a}$$

$$\hat{W}_{ij}^n=-(2L_0(t^n)h)^{-1}\Big\{(A_{i,j+1/2}^n(\hat{P}_{h,i,j+1}^n-\hat{P}_{h,ij}^n)+A_{i,j-1/2}^n(\hat{P}_{h,ij}^n-\hat{P}_{h,i,j-1}^n))-$$
$$S^{-1}(y_2,t^n)(A_{i+1/2,j}^n\alpha_{i+1/2,j}^n(\hat{P}_{h,i+1,j}^n-\hat{P}_{h,ij}^n)+$$
$$A_{i-1/2,j}^n\alpha_{i-1/2,j}^n(\hat{P}_{h,ij}^n-\hat{P}_{h,i-1,j}^n))\Big\},\tag{16b}$$

此处

$$A_{i+1/2,j}^n=\frac{1}{2}[\hat{a}(\boldsymbol{Y}_{ij},\hat{C}_{h,ij}^n)+\hat{a}(\boldsymbol{Y}_{i+1,j},\hat{C}_{h,i+1,j}^n)],$$

$$A_{i,j+1/2}^n=\frac{1}{2}[\hat{a}(\boldsymbol{Y}_{ij},\hat{C}_{h,ij}^n)+\hat{a}(\boldsymbol{Y}_{ij},\hat{C}_{h,i,j+1}^n)].$$

下面考虑饱和度方程(14)的二阶迎风差分格式,记其对流系数$\boldsymbol{E}(\boldsymbol{Y},t,\hat{\boldsymbol{u}})=\boldsymbol{L}_{c\alpha}\hat{\boldsymbol{u}}-\hat{\varphi}\boldsymbol{B}_{c\alpha}=(E_1,E_2)^{\mathrm{T}}$及其近似对流系数$\boldsymbol{E}_h^n=\boldsymbol{E}(\boldsymbol{Y},t^n,\hat{\boldsymbol{U}}_h^n)=\boldsymbol{L}_{c\alpha}^n\hat{\boldsymbol{U}}_h^n-\hat{\varphi}^n\boldsymbol{B}_{c\alpha}^n=(E_{1h}^n,E_{2h}^n)^{\mathrm{T}}$.针对新的对流-扩散方程(14),我们提出一类修正迎风分数步差分格式[15-16].

饱和度方程(14)的分数步迎风差分格式:

$$\hat{\varphi}_{ij}^{n+1}\frac{\hat{C}_{h,ij}^{n+1/2}-\hat{C}_{h,ij}^n}{\Delta t}=\left(1+\frac{h}{2}|E_{1h}^n|(\hat{D}_1^{n+1})^{-1}\right)_{ij}^{-1}\delta_{y_1}(\hat{D}_1^{n+1}\delta_{y_1}\hat{C}_h^{n+1/2})_{ij}+$$
$$\left(1+\frac{h}{2}|E_{2h}^n|(\hat{D}_2^{n+1})^{-1}\right)_{ij}^{-1}\delta_{y_2}(\hat{D}_2^{n+1}\delta_{y_2}\hat{C}_h^n)_{ij}-\delta_{E_{1h}^n}\hat{C}_{h,ij}^n-\delta_{E_{2h}^n}\hat{C}_{h,ij}^n-$$
$$\hat{b}(\hat{C}_{h,ij}^n)\frac{\hat{P}_{h,ij}^{n+1}-\hat{P}_{h,ij}^n}{\Delta t}-\hat{b}(\hat{C}_{h,ij}^n)\hat{a}^{-1}(\hat{C}_{h,ij}^n)\boldsymbol{B}_{po}^n(\boldsymbol{Y}_{ij},t^{n+1})\cdot\hat{\boldsymbol{U}}_{h,ij}^n+\hat{f}(\boldsymbol{Y}_{ij},t^n,\hat{C}_{h,ij}^n),$$
$$1\leqslant i\leqslant N-1,\tag{17a}$$

$$\hat{C}_{h,ij}^{n+1/2}=\hat{r}_{ij}^{n+1},\qquad \boldsymbol{Y}_{ij}\in\partial\hat{\Omega}_h,\tag{17b}$$

$$\hat{\varphi}_{ij}^{n+1}\frac{\hat{C}_{h,ij}^{n+1}-\hat{C}_{h,ij}^{n+1/2}}{\Delta t}=\left(1+\frac{h}{2}|E_{2h}^n|(\hat{D}_2^{n+1})^{-1}\right)^{-1}\delta_{y_2}(\hat{D}_2^{n+1}\delta_{y_2}(\hat{C}_h^{n+1}-\hat{C}_h^n))_{ij},$$
$$1\leqslant j\leqslant N-1,\tag{17c}$$

$$\hat{C}_{h,ij}^{n+1}=\hat{r}_{ij}^{n+1},\qquad Y_{ij}\in\partial\hat{\Omega}_h,\tag{17d}$$

此处

$$\delta_{E_{1h}^n}\hat{C}_{h,ij}^n=E_{1h,ij}^n\Big\{H(E_{1h,ij}^n)(\hat{D}_1^n)_{ij}^{-1}\hat{D}_{1,i-1/2,j}^n\delta_{\bar{y}_1}\hat{C}_{h,ij}^n+(1-H(E_{1h,ij}^n))(\hat{D}_1^n)_{ij}^{-1}\hat{D}_{1,i+1/2,j}^n\delta_{y_1}\hat{C}_{h,ij}^n\Big\},$$

$$\delta_{E_{2h}^n}\hat{C}_{h,ij}^n=E_{2h,ij}^n\Big\{H(E_{2h,ij}^n)(\hat{D}_2^n)_{ij}^{-1}\hat{D}_{2,i,j-1/2}^n\delta_{\bar{y}_2}\hat{C}_{h,ij}^n+(1-H(E_{2h,ij}^n))(\hat{D}_2^n)_{ij}^{-1}\hat{D}_{2,i,j+1/2}^n\delta_{y_2}\hat{C}_{h,ij}^n\Big\},$$

$$H(z)=\begin{cases}1, & z\geqslant 0,\\ 0, & z<0.\end{cases}$$

此处需要指明的是迎风分数步格式(17)是二阶的,它可按文献[15-16]的方法证明其是对空间达到二阶精度.

初始逼近:

$$\hat{P}_{h,ij}^0=\hat{p}_0(Y_{ij}),\ \ \hat{C}_{h,ij}^0=\hat{C}_0(Y_{ij}),\qquad 0\leqslant i,\ j\leqslant N.\tag{18}$$

分数步长特征差分格式 I的计算程序是:当$\{\hat{P}_{h,ij}^n,\hat{C}_{h,ij}^n\}$已知时,首先由方程(15a)和(15b)沿 y_1方向用追赶法求出过渡层的解$\{\hat{P}_{h,ij}^{n+1/2}\}$,再由方程(15c)和(15d)沿 y_2方向用追赶法求出$\{\hat{P}_{h,ij}^{n+1}\}$,与此同时,并行的由方程(17a)和(17b)沿 y_1方向用追赶法求出过渡层的解$\{\hat{C}_{h,ij}^{n+1/2}\}$,再由方程(17c)和(17d)沿 y_2方向用追赶法求出$\{\hat{C}_{h,ij}^{n+1}\}$. 由正定性条件(C),方程(5)和(17)的解存在且唯一.

2 收敛性分析

为了记号简便,将标号"^"省略,此时 $\Omega=\{[0,1]^2\}$,迎风差分格式为

$$d(C_{h,ij}^n)\frac{P_{h,ij}^{n+1/2}-P_{h,ij}^n}{\Delta t}=\delta_{\bar{y}_1}(A_1^n\delta_{y_1}P_h^{n+1/2})_{ij}+\delta_{\bar{y}_2}(A_2^n\delta_{y_2}P_h^n)_{ij}-d(C_{h,ij}^n)\,a^{-1}(C_{h,ij}^n)B_{p\alpha}(Y_{ij},t^{n+1})\cdot U_{h,ij}^n+Q(Y_{ij},t^{n+1}),\qquad 1\leqslant i\leqslant N-1,\tag{19a}$$

$$P_{h,ij}^{n+1/2}=e_{ij}^{n+1},\qquad Y_{ij}\in\partial\Omega_h,\tag{19b}$$

$$d(C_{h,ij}^n)\frac{P_{h,ij}^{n+1}-P_{h,ij}^{n+1/2}}{\Delta t}=\delta_{\bar{y}_2}(A_2^n\delta_{y_2}(P_h^{n+1}-P_h^n))_{ij},\qquad 1\leqslant j\leqslant N-1,\tag{19c}$$

$$P_{h,ij}^{n+1}=e_{ij}^{n+1},\qquad Y_{ij}\in\partial\Omega_h,\tag{19d}$$

$$\varphi_{ij}^{n+1}\frac{C_{h,ij}^{n+1/2}-C_{h,ij}^n}{\Delta t}=\left(1+\frac{h}{2}|E_{1h}^n|(D_1^{n+1})^{-1}\right)_{ij}^{-1}\delta_{\bar{y}_2}(D_1^{n+1}\delta_{y_1}C_h^{n+1/2})_{ij}+\left(1+\frac{h}{2}|E_{2h}^n|(D_2^{n+1})^{-1}\right)_{ij}^{-1}\delta_{\bar{y}_2}(D_2^{n+1}\delta_{y_2}C_h^n)_{ij}-\delta_{E_{1h}^n}C_{h,ij}^n-\delta_{E_{2h}^n}C_{h,ij}^n-b(C_{h,ij}^n)\frac{P_{h,ij}^{n+1}-P_{h,ij}^n}{\Delta t}-b(C_{h,ij}^n)a^{-1}(C_{h,ij}^n)B_{po}(Y_{ij},t^{n+1})\cdot U_{h,ij}^n+f(Y_{ij},t^n,C_{h,ij}^n),\qquad 1\leqslant i\leqslant N-1,\tag{20a}$$

$$C_{h,ij}^{n+1/2}=r_{ij}^{n+1},\qquad Y_{ij}\in\partial\Omega_h,\tag{20b}$$

$$\varphi_{ij}^{n+1}\frac{C_{h,ij}^{n+1}-C_{h,ij}^{n+1/2}}{\Delta t}=\left(1+\frac{h}{2}|E_{2h}^{n}|(D_2^{n+1})^{-1}\right)_{ij}^{-1}\delta_{\bar{y}_2}(D_2^{n+1}\delta_{y_2}(C_h^{n+1}-C_h^{n}))_{ij},$$
$$1\leqslant j\leqslant N-1, \tag{20c}$$

$$C_{h,ij}^{n+1}=r_{ij}^{n+1},\qquad Y_{ij}\in\partial\Omega_h. \tag{20d}$$

设 $\pi=p-P_h$, $\xi=c-C_h$, 这里 p, c 为问题的精确解, P_h, C_h 为差分解. 为了进行误差分析, 定义网格函数空间 H_h 的内积 [11-13].

$$\langle v,w\rangle=\sum_{i,j=1}^{N}v_{ij}w_{ij}h^2,\quad [v,w)_1=\sum_{i=0}^{N-1}\sum_{j=1}^{N}v_{ij}w_{ij}h^2,\quad [v,w)_2=\sum_{i=1}^{N}\sum_{j=0}^{N-1}v_{ij}w_{ij}h^2,$$
$$\forall\, v,w\in H_h.$$

首先研究压力方程, 由方程(19a)~(19d)消去 $P_h^{n+1/2}$ 可得下述等价差分方程:

$$d(C_{h,ij}^{n})\frac{P_{h,ij}^{n+1}-P_{h,ij}^{n}}{\Delta t}=-\left\{\delta_{\bar{y}_1}(A_1^n\delta_{y_1}P_h^{n+1})_{ij}+\delta_{\bar{y}_2}(A_2^n\delta_{y_2}P_h^{n+1})_{ij}\right\}=$$
$$-d(C_{h,ij}^{n})\,a^{-1}(C_{h,ij}^{n})\boldsymbol{B}_{p\alpha}(\boldsymbol{Y}_{ij},t^{n+1})\cdot\boldsymbol{U}_{h,ij}^{n}+Q(\boldsymbol{Y}_{ij},t^{n+1})-$$
$$(\Delta t)^2\delta_{\bar{y}_1}(A_1^n\delta_{y_1}(d^{-1}(C_h^n)(\delta_{\bar{y}_2}(A_2^n\delta_{y_2}d_tP_h^n))))_{ij},\qquad 1\leqslant i,\ j\leqslant N-1, \tag{21a}$$

$$P_{h,ij}^{n+1}=e_{ij}^{n+1},\qquad \boldsymbol{Y}_{ij}\in\partial\Omega_h, \tag{21b}$$

此处 $\quad d_tP_{h,ij}^{n}=(P_{h,ij}^{n+1}-P_{h,ij}^{n})/\Delta t.$

由方程(13a)($t=t^{n+1}$)和(21)可得压力方程的误差方程:

$$d(C_{h,ij}^{n})d_t\pi_{ij}^{n}-\left\{\delta_{\bar{y}_1}(A_1^n\delta_{y_1}\pi^{n+1})_{ij}+\delta_{\bar{y}_2}(A_2^n\delta_{y_2}\pi^{n+1})_{ij}\right\}=$$
$$-(\Delta t)^2\delta_{\bar{y}_1}(A_1^n\delta_{y_1}(d^{-1}(C_h^n)(\delta_{\bar{y}_2}(A_2^n\delta_{y_2}d_t\pi^n))))_{ij}+$$
$$(\Delta t)^2\delta_{\bar{y}_1}(A_1^n\delta_{y_1}(d^{-1}(C_h^n)(\delta_{\bar{y}_2}(A_2^n\delta_{y_2}d_tp^n))))_{ij}+$$
$$d(C_{h,ij}^{n})a^{-1}(C_{h,ij}^{n})\boldsymbol{B}_{p\alpha}(\boldsymbol{Y}_{ij},t^{n+1})\cdot(\boldsymbol{U}_{h,ij}^{n}-\boldsymbol{u}_{h,ij}^{n})+$$
$$\left\{d(C_{h,ij}^{n})a^{-1}(C_{h,ij}^{n})-d(c_{ij}^{n+1})a^{-1}(c_{ij}^{n+1})\right\}\boldsymbol{B}_{p\alpha}(\boldsymbol{Y}_{ij},t^{n+1})\cdot\boldsymbol{u}_{h,ij}^{n}-$$
$$[d(c_{ij}^{n+1})-d(C_{h,ij}^{n})]\,d_tp_{ij}^{n}+\left\{\delta_{\bar{y}_1}((a_1^{n+1}-A_1^n)\delta_{y_1}p^{n+1})_{ij}+\right.$$
$$\left.\delta_{\bar{y}_2}((a_2^{n+1}-A_2^n)\delta_{y_2}p^{n+1})_{ij}\right\}+\sigma_{ij}^{n+1},\qquad 1\leqslant i,\ j\leqslant N-1, \tag{22a}$$

$$\pi_{ij}^{n+1}=0,\qquad \boldsymbol{Y}_{ij}\in\partial\Omega_h, \tag{22b}$$

此处

$$d_t\pi^n=(\pi^{n+1}-\pi^n)/\Delta t,\quad |\sigma_{ij}^{n+1}|\leqslant M\left\{h^2+\Delta t\right\}.$$

对方程(22)乘以 $\delta_t\pi_{ij}^n=d_t\pi_{ij}^n\Delta t$ 作内积并分部求和:

$$\langle d(C_h^n)d_t\pi^n,d_t\pi^n\rangle\Delta t+$$
$$\langle A_1^n\delta_{y_1}\pi^{n+1},\delta_{y_1}(\pi^{n+1}-\pi^n)\rangle+\langle A_2^n\delta_{y_2}\pi^{n+1},\delta_{y_2}(\pi^{n+1}-\pi^n)\rangle=$$
$$-(\Delta t)^2\left\{\langle\delta_{\bar{y}_1}(A_1^n\delta_{y_1}(d^{-1}(C_h^n)(\delta_{\bar{y}_2}(A_2^n\delta_{y_2}d_t\pi^n)))),d_t\pi^n\rangle-\right.$$
$$\left.\langle\delta_{\bar{y}_1}(A_1^n\delta_{y_1}(d^{-1}(C_h^n)(\delta_{\bar{y}_2}(A_2^n\delta_{y_2}d_tp^n)))),d_t\pi^n\rangle\right\}\Delta t+$$
$$\langle d(C_h^n)a^{-1}(C_h^n)\boldsymbol{B}_{p\alpha}(\boldsymbol{Y},t^{n+1})\cdot(\boldsymbol{U}_h^n-\boldsymbol{u}^{n+1}),d_t\pi^n\rangle\Delta t+$$
$$\langle[d(C_h^n)a^{-1}(C_{h,ij}^{n})-d(c^{n+1})a^{-1}(c^{n+1})]\boldsymbol{B}_{p\alpha}(\boldsymbol{Y},t^{n+1})\cdot\boldsymbol{u}^{n+1},d_t\pi^n\rangle\Delta t+$$

$$\langle\delta_{y_1}((a_1^{n+1}-A_1^n)\delta_{y_1}p^{n+1})+\delta_{y_2}((a_2^{n+1}-A_2^n)\delta_{y_2}p^{n+1}),d_t\pi^n\rangle\Delta t-$$
$$\langle[d(c^{n+1})-d(C_h^n)]d_tp^n,d_t\pi^n\rangle\Delta t+\langle\sigma^{n+1},d_t\pi^n\rangle\Delta t.\tag{23}$$

对方程(23)的左、右两端依次进行估计,并求和 $0\leqslant n\leqslant L$,注意到 $\pi^0=\xi^0=0$,有

$$\sum_{n=1}^{L}\|d_t\pi^n\|_0^2\Delta t+\left\{\langle A_1^L\delta_{y_1}\pi^{L+1},\delta_{y_1}\pi^{L+1}\rangle+\langle A_2^L\delta_{y_2}\pi^{L+1},\delta_{y_2}\pi^{L+1}\rangle\right\}-$$
$$\left\{\langle A_1^0\delta_{y_1}\pi^0,\delta_{y_1}\pi^0\rangle+\langle A_2^0\delta_{y_2}\pi^0,\delta_{y_2}\pi^0\rangle\right\}\leqslant$$
$$\sum_{n=1}^{L}\left\{\langle(A_1^n-A_1^{n-1})\delta_{y_1}\pi^n,\delta_{y_1}\pi^n\rangle+\langle(A_2^n-A_2^{n-1})\delta_{y_2}\pi^n,\delta_{y_2}\pi^n\rangle\right\}+$$
$$M\left\{h^4+(\Delta t)^2+\sum_{n=1}^{L}[\|\nabla_h\pi^n\|_0^2+\|\xi^n\|_0^2+\|\nabla_h\xi^n\|_0^2]\Delta t\right\}.\tag{24}$$

其次研究饱和度方程的误差估计,由方程(20a)~(20d)消去 $C_h^{n+1/2}$ 可得下述等价差分方程:

$$\varphi_{ij}^{n+1}\frac{C_{h,ij}^{n+1}-C_{h,ij}^n}{\Delta t}-\left\{\left(1+\frac{h}{2}|E_{1h}^n|(D_1^{n+1})^{-1}\right)_{ij}^{-1}\delta_{y_1}(D_1^{n+1}\delta_{y_1}C_h^{n+1})_{ij}+\right.$$
$$\left.\left(1+\frac{h}{2}|E_{2h}^n|(D_2^{n+1})^{-1}\right)_{ij}^{-1}\delta_{y_2}(D_2^{n+1}\delta_{y_2}C_h^{n+1})_{ij}\right\}=$$
$$-\delta_{E_{1h}^n}C_{h,ij}^n-\delta_{E_{2h}^n}C_{h,ij}^n-b(C_{h,ij}^n)\frac{P_{h,ij}^{n+1}-P_{h,ij}^n}{\Delta t}+$$
$$b(C_{h,ij}^n)a^{-1}(C_{h,ij}^n)\boldsymbol{B}_{po}(\boldsymbol{Y}_{ij},t^{n+1})\cdot\boldsymbol{U}_{h,ij}^n+$$
$$f(\boldsymbol{Y}_{ij},t^n,\hat{C}_{h,ij}^n)-(\Delta t)^2\left(1+\frac{h}{2}|E_{1h}^n|(D_1^{n+1})^{-1}\right)_{ij}^{-1}\delta_{y_1}\left(D_1^{n+1}\delta_{y_1}\left((\varphi^{n+1})^{-1}\times\right.\right.$$
$$\left.\left.\left(1+\frac{h}{2}|E_{2h}^n|(D_2^{n+1})^{-1}\right)^{-1}\delta_{y_2}(D_2^{n+1}\delta_{y_2}d_tC_h^n)\right)\right)_{ij},\qquad 1\leqslant i,\ j\leqslant N-1,\tag{25a}$$
$$C_{h,ij}^{n+1}=r_{ij}^{n+1},\qquad \boldsymbol{Y}_{ij}\in\partial\Omega_h.\tag{25b}$$

为以后书写简便,记

$$B_1^{n+1}=\left(1+\frac{h}{2}|E_{1h}^n|(D_1^{n+1})^{-1}\right)^{-1},\quad B_2^{n+1}=\left(1+\frac{h}{2}|E_{2h}^n|(D_2^{n+1})^{-1}\right)^{-1},$$
$$B_1^{n+1}=\left(1+\frac{h}{2}|E_1^{n+1}|(D_1^{n+1})^{-1}\right)^{-1},\quad B_2^{n+1}=\left(1+\frac{h}{2}|E_2^{n+1}|(D_2^{n+1})^{-1}\right)^{-1}.$$

由方程(14)($t=t^{n+1}$)和(25)可得饱和度的误差方程:

$$\varphi_{ij}^{n+1}\frac{\xi_{ij}^{n+1}-\xi_{ij}^n}{\Delta t}-\left\{B_{1,ij}^{n+1}\delta_{y_1}(D_1^{n+1}\delta_{y_1}\xi^{n+1})_{ij}+B_{2,ij}^{n+1}\delta_{y_2}(D_2^{n+1}\delta_{y_2}\xi^{n+1})_{ij}\right\}=$$
$$[\delta_{E_{1h}^n}C_{h,ij}^n-\delta_{E_1^{n+1}}c_{ij}^{n+1}]+[\delta_{E_{2h}^n}C_{h,ij}^n-\delta_{E_2^{n+1}}c_{ij}^{n+1}]+\left\{[B_{1,ij}^{n+1}-B_{1,ij}^{n+1}]\delta_{y_1}(D_1^{n+1}\delta_{y_1}C_h^{n+1})_{ij}+\right.$$
$$\left.[B_{2,ij}^{n+1}-B_{2,ij}^{n+1}]\delta_{y_2}(D_2^{n+1}\delta_{y_2}C_h^{n+1})_{ij}\right\}-b(C_{h,ij}^n)d_t\pi_{ij}^n-[b(c_{ij}^{n+1})-b(C_{h,ij}^n)]d_tp_{ij}^n-$$
$$b(C_{h,ij}^n)a^{-1}(C_{h,ij}^n)\boldsymbol{B}_{po}(\boldsymbol{Y}_{ij},t^{n+1})\cdot(\boldsymbol{u}_{ij}^{n+1}-\boldsymbol{U}_{h,ij}^n)+$$
$$[b(C_{h,ij}^n)a^{-1}(C_{h,ij}^n)-b(c_{ij}^{n+1})a^{-1}(c_{ij}^{n+1})]\boldsymbol{B}_{po}(\boldsymbol{Y}_{ij},t^{n+1})\cdot\boldsymbol{u}_{ij}^{n+1}+f(\boldsymbol{Y}_{ij},t^{n+1},c_{ij}^{n+1})-f(\boldsymbol{Y}_{ij},t^n,\hat{C}_{h,ij}^n)-$$
$$(\Delta t)^2\left\{B_{1,ij}^{n+1}\delta_{y_1}(D_1^{n+1}\delta_{y_1}((\varphi^{n+1})^{-1}B_{2,ij}^{n+1}\delta_{y_2}(D_2^{n+1}\delta_{y_2}d_tc^n)))_{ij}-\right.$$

$$B_{1,ij}^{n+1}\delta_{y_1}(D_1^{n+1}\delta_{y_1}((\varphi^{n+1})^{-1}B_{2,ij}^{n+1}\delta_{y_2}(D_2^{n+1}\delta_{y_2}d_tC_h^n)))_{ij}\Big\} + \varepsilon_{ij}^{n+1},$$

$$1\leqslant i,\ j\leqslant N-1. \tag{26a}$$

$$\xi_{ij}^{n+1}=0,\qquad Y_{ij}\in\partial\Omega_h, \tag{26b}$$

此处 $\quad |\varepsilon_{ij}^{n+1}|\leqslant M(h^2+\Delta t)$.

对方程(26)乘以 $\delta_t\xi_{ij}^n=\xi_{ij}^{n+1}-\xi_{ij}^n=d_t\xi_{ij}^n\Delta t$ 作内积,并分部求和可得:

$$\begin{aligned}
&\langle\varphi^{n+1}d_t\xi^n,\ d_t\xi^n\rangle\Delta t+\langle D_1^{n+1}\delta_{y_1}\xi^{n+1},\ \delta_{y_1}[B_1^{n+1}(\xi^{n+1}-\xi^n)]\rangle+\\
&\quad\langle D_2^{n+1}\delta_{y_2}\xi^{n+1},\ \delta_{y_2}[B_2^{n+1}(\xi^{n+1}-\xi^n)]\rangle=\\
&\quad\Big\{\langle\delta_{E_{1h}^n}C_h^n-\delta_{E_1^{n+1}}c^{n+1},\ d_t\xi^n\rangle+\langle\delta_{E_{2h}^n}C_h^n-\delta_{E_2^{n+1}}c^{n+1},\ d_t\xi^n\rangle\Big\}\Delta t+\\
&\quad\Big\{\langle[B_1^{n+1}-B_1^{n+1}]\delta_{y_1}(D_1^{n+1}\delta_{y_1}C_h^{n+1}),\ d_t\xi^n\rangle+\\
&\quad\langle[B_2^{n+1}-B_2^{n+1}]\delta_{y_2}(D_2^{n+1}\delta_{y_2}C_h^{n+1}),d_t\xi^n\rangle\Big\}\Delta t+\Big\{-\langle b(C_h^n)d_t\pi^n,\ d_t\xi^n\rangle\Delta t+\\
&\quad\langle[b(C_h^n)-b(c^{n+1})]d_tp^n,d_t\xi^n\rangle+\langle f(\boldsymbol{Y},t^{n+1},c^{n+1})-f(\boldsymbol{Y},t^n,\hat{C}_h^n),d_t\xi^n\rangle\Big\}\Delta t-\\
&\quad\Big\{\langle b(C_h^n)a^{-1}(C_h^n)\boldsymbol{B}_{po}(\boldsymbol{Y},t^{n+1})\cdot(\boldsymbol{u}^{n+1}-\boldsymbol{U}_h^n),d_t\xi^n\rangle-\\
&\quad\langle[b(C_h^n)a^{-1}(C_h^n)-b(c^{n+1})a^{-1}(c^{n+1})]\boldsymbol{B}_{po}(\boldsymbol{Y},t^{n+1})\cdot\boldsymbol{u}^{n+1},d_t\xi^n\rangle\Big\}\Delta t-\\
&\quad(\Delta t)^3\Big\{\langle B_1^{n+1}\delta_{y_1}(D_1^{n+1}\delta_{y_1}((\varphi^{n+1})^{-1}B_2^{n+1}\delta_{y_2}(D_2^{n+1}\delta_{y_2}d_t\xi^n))),\ d_t\xi^n\rangle+\\
&\quad\langle B_1^{n+1}\delta_{y_1}(D_1^{n+1}\delta_{y_1}((\varphi^{n+1})^{-1}B_2^{n+1}\delta_{y_2}(D_2^{n+1}\delta_{y_2}d_tc^n)))-\\
&\quad B_1^{n+1}\delta_{y_1}(D_1^{n+1}\delta_{y_1}((\varphi^{n+1})^{-1}B_2^{n+1}\delta_{y_2}(D_2^{n+1}\delta_{y_2}d_tc^n))),d_t\xi^n\rangle\Big\}+\langle\varepsilon^{n+1},\ d_t\xi^n\rangle\Delta t.
\end{aligned}\tag{27}$$

对误差估计方程(27)的左右两端依次进行估计可得:

$$\begin{aligned}
&\|d_t\xi^n\|^2\Delta t+\frac{1}{2}\Big\{[\langle D_1^{n+1}\delta_{y_1}\xi^{n+1},\ B_1^{n+1}\delta_{y_1}\xi^{n+1}\rangle+\langle D_2^{n+1}\delta_{y_2}\xi^{n+1},\ B_2^{n+1}\delta_{y_2}\xi^{n+1}\rangle]-\\
&\quad[\langle D_1^{n+1}\delta_{y_1}\xi^n,B_1^{n+1}\delta_{y_1}\xi^n\rangle+\langle D_2^{n+1}\delta_{y_2}\xi^n,B_2^{n+1}\delta_{y_2}\xi^n\rangle]\Big\}\leqslant\\
&\quad\varepsilon\|d_t\xi^n\|^2\Delta t+M\Big\{\|\nabla_h\pi^n\|^2+\|d_t\pi^n\|^2+\|\nabla_h\xi^{n+1}\|^2+\|\nabla_h\xi^n\|^2+\\
&\quad\|\xi^{n+1}\|^2+\|\xi^n\|^2+(\Delta t)^2+h^4\Big\}\Delta t.
\end{aligned}\tag{28}$$

对上式关于 t 求和 $0\leqslant n\leqslant L$,注意到 $\xi^0=0$,可得

$$\begin{aligned}
&\sum_{n=0}^{L}\|d_t\xi^n\|^2\Delta t+\frac{1}{2}\Big\{[\langle D_1^{L+1}\delta_{y_1}\xi^{L+1},\ B_1^{L+1}\delta_{y_1}\xi^{L+1}\rangle+\langle D_2^{L+1}\delta_{y_2}\xi^{L+1},\ B_2^{L+1}\delta_{y_2}\xi^{L+1}\rangle]-\\
&\quad[\langle D_1^0\delta_{y_1}\xi^0,B_1^0\delta_{y_1}\xi^0\rangle+\langle D_2^0\delta_{y_2}\xi^0,B_2^0\delta_{y_2}\xi^0\rangle]\Big\}\leqslant\\
&\quad\sum_{n=0}^{L}\Big\{\langle D_1^{n+1}\delta_{y_1}\xi^n,[B_1^{n+1}-B_1^n]\delta_{y_1}\xi^n\rangle+\langle D_2^{n+1}\delta_{y_2}\xi^n,[B_2^{n+1}-B_2^n]\delta_{y_2}\xi^n\rangle\Big\}+\\
&\quad\varepsilon\sum_{n=0}^{L}\|d_t\xi^n\|^2\Delta t+M\sum_{n=0}^{L}\Big\{\|\xi^{n+1}\|_1^2+\|\nabla_h\pi^n\|^2+\|d_t\pi^n\|^2+(\Delta t)^2+h^4\Big\}\Delta t,
\end{aligned}\tag{29}$$

注意到 $|B_{k,ij}^{n+1}-B_{k,ij}^n|\leqslant Mh\Delta t$, $k=1,2$,则误差方程(30)可写为

$$\begin{aligned}
&\sum_{n=0}^{L}\|d_t\xi^n\|^2\Delta t+\frac{1}{2}[\langle D_1^{L+1}\delta_{y_1}\xi^{L+1},B_1^{L+1}\delta_{y_1}\xi^{L+1}\rangle+\langle D_2^{L+1}\delta_{y_2}\xi^{L+1},B_2^{L+1}\delta_{y_2}\xi^{L+1}\rangle]\leqslant\\
&\quad M\sum_{n=0}^{L}\|d_t\pi^n\|^2\Delta t+M\sum_{n=0}^{L}\Big\{\|\xi^{n+1}\|_1^2+\|\nabla_h\pi^n\|^2+(\Delta t)^2+h^4\Big\}\Delta t.
\end{aligned}\tag{30}$$

对压力函数误差方程(24),注意到

$$\sum_{n=0}^{L}\left\{\langle (A_1^n - A_1^{n-1})\delta_{y_1}\pi^n,\ \delta_{y_1}\pi^n\rangle + \langle (A_2^n - A_2^{n-1})\delta_{y_2}\pi^n,\ \delta_{y_2}\pi^n\rangle\right\} \leqslant$$

$$\varepsilon\sum_{n=1}^{L}\|d_t\xi^{n-1}\|^2\Delta t + M\sum_{n=1}^{L}\|\nabla_h\pi^n\|^2\Delta t, \tag{31a}$$

$$\begin{cases}\|\pi^{L+1}\|^2 \leqslant \varepsilon\sum_{n=0}^{L}\|d_t\pi^n\|_0^2\Delta t + M\sum_{n=0}^{L}\|\pi^n\|^2\Delta t,\\ \|\xi^{L+1}\|^2 \leqslant \varepsilon\sum_{n=0}^{L}\|d_t\xi^n\|^2\Delta t + M\sum_{n=0}^{L}\|\xi^n\|^2\Delta t,\end{cases} \tag{31b}$$

组合方程(24)和(30)可得

$$\sum_{n=0}^{L}\left\{\|d_t\pi^n\|_0^2 + \|d_t\xi^n\|_0^2\right\}\Delta t + \|\pi^{L+1}\|_1^2 + \|\xi^{L+1}\|_1^2 \leqslant$$

$$M\sum_{n=0}^{L}\left\{\|\pi^{n+1}\|_1^2 + \|\xi^{n+1}\|_1^2 + h^4 + (\Delta t)^2\right\}\Delta t. \tag{32}$$

应用 Gronwall引理可得:

$$\sum_{n=0}^{L}\left\{\|d_t\pi^n\|_0^2 + \|d_t\xi^n\|_0^2\right\}\Delta t + \|\pi^{L+1}\|_1^2 + \|\xi^{L+1}\|_1^2 \leqslant M\left\{h^4 + (\Delta t)^2\right\}. \tag{33}$$

定理 假定方程(13)和(14)的精确解满足光滑性条件:

$$p,\ c\in L^\infty(W^{4,\infty}),\quad \frac{\partial^2 p}{\partial t^2},\ \frac{\partial^2 c}{\partial t^2}\in L^\infty(L^\infty).$$

采用分数步长特征差分方程(15)和(17)逐层计算,则下述误差估计方程成立:

$$\|p - P_h\|_{L^\infty([0,T],h^1)} + \|c - C_h\|_{L^\infty([0,T],h^1)} +$$

$$\|d_t(p - P_h)\|_{L^2([0,T],l^2)} + \|d_t(c - C_h)\|_{L^2([0,T],l^2)} \leqslant$$

$$M^*\left\{\Delta t + h^2\right\}, \tag{34}$$

此处常数 M^* 依赖于 $p,\ c$ 及其导函数.

3 应 用

3.1 本文所提出的数值方法已成功应用到油气资源评估的数值模拟系统②[21],其数学模型是

$$\nabla\cdot\left(\frac{\kappa}{\mu}\nabla p\right) = \varphi\frac{\partial p}{\partial t} - f\frac{\partial s}{\partial t} + \varphi\frac{\partial P_n}{\partial t},\qquad X = (x_1, x_2, x_3)^{\mathrm{T}}\in\Omega_1(t),$$

$$t\in J = (0, T],\tag{35a}$$

$$p = 0,\qquad X\in\Omega_2(t),\ t\in J\ (\text{流动方程}), \tag{35b}$$

$$\nabla\cdot(\kappa_s\nabla T) - c_w\rho_w\nabla\cdot(\boldsymbol{v}T) + Q = c_s\rho_s\frac{\partial T}{\partial t},$$

$$X\in\Omega(t),\ t\in J\ (\text{古温度方程}), \tag{36}$$

$$\frac{\partial\Phi}{\partial t} = -f\left(\frac{\partial s}{\partial t} - \frac{\partial p}{\partial t} - \frac{\partial P_n}{\partial t}\right),\qquad X\in\Omega(t),\ t\in J\ (\text{孔隙度方程}), \tag{37}$$

此处 $\Omega(t) = \Omega_1(t)\cup\Omega_2(t)$ 是盆地的三维有界区域,超压函数 $p = p(X, t)$ 在超压区 Ω_1 满足方程(35a),在非超压区 Ω_2 上恒为 0. $T = T(X, t)$ 是古温度函数,$\Phi(X, t)$ 是孔隙度函数满足方程

(37), s 和 P_n 分别是负荷重和静水柱压力, μ 是流体粘度, κ 是渗透率, $\boldsymbol{v}$ 是 Darcy 速度, κ_s 是沉积物的热导率, Q 是热源项, p, T, Φ 是需要寻求的基本未知函数.

研制成的软件系统已成功应用到胜利、辽河、大港等油田的油气资源评估.

3.2 本文所提出的数值方法还成功应用到油资源运移聚集模拟系统[①, 21], 其数学模型为

$$\nabla\cdot\left(K\frac{k_{ro}(s)}{\mu_o}\nabla\phi_o\right)+B_oq=-\Phi s'\left(\frac{\partial\phi_o}{\partial t}-\frac{\partial\phi_w}{\partial t}\right),$$

$$X=(x_1,x_2,x_3)^T\in\Omega(t),\quad t\in J\qquad(\text{油位势方程}),\tag{38a}$$

$$\nabla\cdot\left(K\frac{k_{rw}(s)}{\mu_w}\nabla\phi_w\right)+B_wq=\Phi s'\left(\frac{\partial\phi_o}{\partial t}-\frac{\partial\phi_w}{\partial t}\right),$$

$$X\in\Omega(t),\ t\in J\qquad(\text{水位势方程}),\tag{38b}$$

此处 ϕ_o, ϕ_w 是油相、水相流动位势, 是需求的基本未知函数. K 是地层的渗透率, μ_o, μ_w 分别是油相、水相粘度, κ_{ro}, κ_{rw} 分别是油相、水相的相对渗透率. $s'=\mathrm{d}s/\mathrm{d}p_c$, s 为含水饱和度, $p_c(s)$ 为毛细管压力函数, B_o, B_w 是流动函数, Φ 为地层孔隙度, q 为产量项函数.

研制成的软件系统已成功应用到胜利油田阳信凹陷地区的实际数值模拟和油资源评估.

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Upwind Finite Difference Method for Miscible (Oil and Water) Displacement Problem With Moving Boundary Values

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Abstract: The research of the miscible (oil and water) displacement problem with moving boundary values is of great value to the history of oil-gas transport and accumulation in basin evolution, as well as to the rational evaluation in prospecting and exploiting oil-gas resources. The mathematical model can be described as a coupled system of nonlinear partial differential equations with moving boundary values. For the two-dimensional bounded region, the upwind finite difference schemes were put forward. Some techniques, such as calculus of variations, change of variables, theory of a priori estimates and techniques were adopted. Optimal order estimates are derived for the errors in approximate solutions. The research is important both theoretically and practically for model analysis in the field, for model numerical method and for software development.

Key words: compressible displacement; moving boundary; upwind finite difference fraction steps; error estimate; application