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一类四阶半线性方程的奇摄动解*

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摘要: 讨论了一类四阶半线性方程奇摄动边值问题. 利用上下解方法, 研究了边值问题解的存在性和渐近性态. 指出了在该文的情形下具有两参数的原奇摄动问题的解只有一个边界层.

关 键 词: 半线性; 两参数; 奇摄动; 上下解
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引 言

研究非线性奇摄动问题是数学界备受关注的一个对象. 在过去的十年中, 许多方法被发展和优化, 包括平均法、边界层法、匹配渐近展开法和多重尺度法. 近来许多学者做了大量的工作[1-5]. 利用微分不等式等方法, 莫嘉琪等也研究了一类奇摄动常微分方程非线性边值问题[6]、反应扩散方程[7-8]、催化反应系统[9]、生态环境[10]、激波[11]、孤立子[12-13]、激光[14]、海洋科学[15]和大气物理问题[16]. 本文是用上下解方法和特殊而简单的奇摄动理论, 研究一类带有两参数的半线性方程边值问题.

今讨论如下半线性问题:

$$\varepsilon\frac{d^4y}{dx^4}+\mu p(x)\frac{d^3y}{dx^3}+q(x)\frac{d^2y}{dx^2}=f(x,y),\qquad a<x<b,\tag{1}$$

$$y^{(r)}(a)=A_r,\qquad r=0,1,\tag{2}$$

$$y^{(r)}(b)=B_r,\qquad r=0,1,\tag{3}$$

其中 ε和 μ为正的小参数, A_r, B_r $(r=1,2)$为常数. 问题(1)~(3)是一个具有两参数的奇摄动问题.

假设

[H_1] 当 $\varepsilon\to 0$时, $\mu^2/\varepsilon\to 0$.

[H_2] 函数 $p(x)$, $q(x)$ 和 $f(x,y)$ 为关于其变量在对应的区域内为具有直到 $(m+1)$ 阶的

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偏导数, $\min(p, q, f_y) \geqslant \delta > 0$, 其中 δ为正常数.

由假设 [H_2],

$$s^4 + p(x)s^3 = 0,\quad p(x)t^3 + q(x)t^2 = 0$$

存在负实解 $s = -(p(x))^{1/3}$ 和 $t = -(q(x)/p(x))^{1/3}$. 于是由删除定理, 退化问题中(2)的 2 个边界条件应被删除.

方程(1)的退化方程为

$$q(x)\frac{d^2y}{dx^2} = f(x, y),\qquad a < x < b. \tag{4}$$

我们尚需假设:

[H_3] 问题(1)~(3)的退化问题存在解 $Y_0(x)$.

令

$$\xi = \varepsilon^{1/2},\quad \eta = \mu/\varepsilon^{1/2}, \tag{5}$$

即

$$\varepsilon = \xi^2,\quad \mu = \xi\eta.$$

这时问题(1)为

$$\xi^2\frac{d^4y}{dx^4} + \xi\eta p(x)\frac{d^3y}{dx^3} + q(x)\frac{d^2y}{dx^2} = f(x, y),\qquad a < x < b. \tag{6}$$

1 外 部 解

设问题(1)~(3)的外部解 Y为

$$Y(x, \xi) \sim \sum_{j,k=0}^{\infty} Y_{jk}(x)\xi^j\eta^k. \tag{7}$$

将方程(7)代入方程(6)和(3), 按 ξ和 η展开 f并合并方程的 ξ, η的同次幂系数有

$$q(x)\frac{d^2Y_{00}}{dx^2} = f(x, Y_{00}), \tag{8}$$

$$Y_{00}^{(r)}(b) = B_r,\qquad r = 0, 1, \tag{9}$$

$$q(x)\frac{d^2Y_{jk}}{dx^2} = f_y(x, Y_{00})Y_{jk} + F_{jk},\qquad j + k \neq 0, \tag{10}$$

$$Y_{jk}^{(r)}(b) = 0,\qquad r = 0, 1,\ j + k \neq 0, \tag{11}$$

其中 $F_{jk}(j + k \neq 0)$ 为已知函数.

显然, 问题(8)和(9)的解 Y_{00}就是退化问题(4)和(3)的解 $Y_0(x)$. 由问题(10)和(11), 我们能依次得到 $Y_{jk}(j, k = 0, 1, \cdots)$. 于是我们能够决定外部解(7). 但它未必满足边界条件(2), 所以我们还需在 $x = a$ 邻近构造边界层校正项.

2 边界层校正

引入伸长变量[17] $\tau = (x - a)/\xi$, 并设方程(6)和(2)的解 z为

$$z = Y(x, \xi, \eta) + u(\tau, \xi, \eta). \tag{12}$$

将方程(12)代入方程(6)有

$$\frac{d^4(Y + u)}{d\tau^4} + \eta p(\xi\tau + a)\frac{d^3(Y + u)}{d\tau^3} + q(\xi\tau + a)\frac{d^2(Y + u)}{d\tau^2} =$$

$$\xi^2 f(\xi\tau + a, Y + u), \qquad a < x < b, \tag{13}$$

令

$$u \sim \sum_{j,k=0}^{\infty} u_{jk}(\tau)\xi^j \eta^k. \tag{14}$$

将方程(7)和(14)代入方程(13)和条件(2),按 ξ,η展开 f并合并方程的 ξ,η的同次幂系数有

$$\frac{d^4 u_{00}}{d\tau^4} + q(a)\frac{d^2 u_{00}}{d\tau^2} = 0, \quad u_{00}(0) = A_0, \quad \frac{du_{00}}{d\tau}(0) = 0, \tag{15}$$

$$\frac{d^4 u_{10}}{d\tau^4} + q(a)\frac{d^2 u_{10}}{d\tau^2} = G_{10}, \quad u_{10}(0) = 0, \quad \frac{du_{10}}{d\tau}(0) = A_1, \tag{16}$$

$$\frac{d^4 u_{jk}}{d\tau^4} + q(a)\frac{d^2 u_{jk}}{d\tau^2} = G_{jk}, \quad u_{jk}(0) = 0, \quad \frac{du_{jk}}{d\tau}(0) = 0, \qquad jk \neq 00, 10, \tag{17}$$

其中 G_{10}和 $G_{jk}(jk \neq 00, 10)$为依次已知函数,其结构从略.

由假设[H_2],问题(15)~(17),我们能得到 $u_{jk}(j, k = 0, 1, \cdots)$,并具有性态

$$u_{jk} = O(\exp(-k_{jk}\tau)) = O\left(\exp\left(-k_{jk}\frac{x-a}{\xi}\right)\right) = O\left(\exp\left(-k_{jk}\frac{x-a}{\varepsilon^{1/2}}\right)\right),$$
$$j, k = 0, 1, \cdots, \tag{18}$$

其中 k_{jk}为正常数. 将 u_{jk}代入式(14),我们得到 $x = a$在附近的边界层校正项 u.

于是我们得到如下原问题(1)~(3)解 y的渐近展开式:

$$y = \sum_{j=0}^{n}\sum_{k=0}^{l}[Y_{jk} + u_{jk}]\varepsilon^{j/2}(\mu/\varepsilon^{1/2})^k + O(\max_{n,l}(\varepsilon^{n/2}(\mu/\varepsilon^{1/2})^l),$$
$$0 < \varepsilon, \mu \ll 1, \quad 0 < \xi = \varepsilon^{1/2}, \quad \eta = \mu/\varepsilon^{1/2} \ll 1. \tag{19}$$

3 一致有效性

现有如下定理:

定理 在假设[H_1]~[H_3]下,存在两参数奇摄动问题(1)~(3)的一个解 y,并且解在 $x \in [a, b]$上成立一致有效的渐近展开式(19).

证明 首先构造辅助函数 α和 β:

$$\alpha = W - rR, \quad \beta = W + rR, \tag{20}$$

其中 $R = \max_{n,l}(\varepsilon^{n/2}(\mu/\varepsilon^{1/2})^l)$,$r$为足够大的正常数,它将在下面选定,而

$$W = \sum_{j=0}^{n}\sum_{k=0}^{l}[Y_{jk} + u_{jk}]\varepsilon^{j/2}(\mu/\varepsilon^{1/2})^k.$$

显然,

$$\alpha \leqslant \beta, \qquad x \in [a, b], \tag{21}$$

且

$$\alpha^{(i)}|_{x=a} \leqslant A_i \leqslant \beta^{(i)}|_{x=a}, \quad \alpha^{(i)}|_{x=b} \leqslant B_i \leqslant \beta^{(i)}|_{x=b}, \qquad i = 0, 1. \tag{22}$$

现证

$$\varepsilon\frac{d^4\alpha}{dx^4} + \mu p(x)\frac{d^3\alpha}{dx^3} + q(x)\frac{d^2\alpha}{dx^2} - f(x, \alpha) \geqslant 0, \qquad a < x < b, \tag{23}$$

$$\varepsilon\frac{d^4\beta}{dx^4} + \mu p(x)\frac{d^3\beta}{dx^3} + q(x)\frac{d^2\beta}{dx^2} - f(x, \beta) \leqslant 0, \qquad a < x < b. \tag{24}$$

由假设[H_2], 对于足够小的 ε, μ, 存在一个正常数 M, 使得

$$\varepsilon\frac{d^4\alpha}{dx^4} + \mu p(x)\frac{d^3\alpha}{dx^3} + q(x)\frac{d^2\alpha}{dx^2} - f(x, \alpha) =$$

$$\varepsilon\frac{d^4(W - rR)}{dx^4} + \mu p(x)\frac{d^3(W - rR)}{dx^3} + q(x)\frac{d^2(W - rR)}{dx^2} -$$

$$f(x, W) + [f(x, W) - f(x, (W - rR))] \geqslant$$

$$\left[q(x)\frac{d^2Y_{00}}{dx^2} - f(x, Y_{00})\right] +$$

$$\sum_{j=0, j+k\neq 0}^{n}\sum_{k=0}^{l}\left[q(x)\frac{d^2Y_{jk}}{dx^2} - f_y(x, Y_{00})Y_{jk} - F_{jk}\right]\xi^j\eta^k +$$

$$\left[\frac{d^4u_{00}}{d\tau^4} + q(a)\frac{d^2u_{00}}{d\tau^2}\right] + \left[\frac{d^4u_{10}}{d\tau^4} + q(a)\frac{d^2u_{10}}{d\tau^2} - G_{10}\right]\xi +$$

$$\sum_{j=0, jk\neq 00, 10}^{n}\sum_{k=0}^{l}\left[\frac{d^4u_{jk}}{d\tau^4} + q(a)\frac{d^2u_{jk}}{d\tau^2} - G_{jk}\right]\xi^j\eta^k - MR + r\delta R =$$

$$(r\delta - M)R.$$

选择 $r \geqslant M/\delta$, 我们证明了不等式(23). 同理也可证不等式(24). 于是由式(21)~(24)和微分不等式定理[18], 我们得到

$$\alpha \leqslant y \leqslant \beta, \qquad x \in [a, b].$$

再由式(20), 我们给出了最后的结果:

$$y = \sum_{j=0}^{n}\sum_{k=0}^{l}[Y_{jk} + u_{jk}]\varepsilon^{j/2}(\mu/\varepsilon^{1/2})^k + O(R),$$

$$0 < \varepsilon \ll 1, \quad 0 < \mu \ll 1, \quad 0 < \mu^2/\varepsilon \ll 1,$$

其中 $R = \max_{n, l}(\varepsilon^{n/2}(\mu/\varepsilon^{1/2})^l)$, 定理证毕.

附注 注意到渐近展开式(19), 可以看出具有两参数的原问题(1)~(3), 当 $\mu^2/\varepsilon \to 0, \mu \to 0$时, 在 $x = a$ 邻近只有一个边界层, 它与文献[19]中, 当 $\varepsilon/\mu^2 \to 0, \mu \to 0$的情形不同. 后者在 $x = a$ 具有两个边界层.

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On a Class of Singular Perturbation Solution for Semilinear Equtions of Fourth Order

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Abstract: A class of singularly perturbed boundary value problem for semilinear equation of fourth order with two parameters is considered. Under suitable conditions, using lower and upper solutions method, the existence and asymptotic behavior of solution for boundary value problem were studied. It is pointed out that the solution for original singularly perturbed problem with two parameters has only one boundary layer.

Key words: semilinear; two parameters; singular perturbation; lower and upper solutions