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具有迅速振荡系数的非自共轭椭圆问题的二阶双尺度计算方法*

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摘要: 为了求解具有迅速振荡系数的非自共轭椭圆问题, 考虑了非自共轭椭圆问题二阶双尺度近似解的表示式, 推导了二阶双尺度近似解的误差估计式, 数值试验结果表明给出的近似解是有效的•

关 键 词: 非自共轭椭圆问题; 迅速振荡系数; 二阶双尺度有限元方法
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引　言

工程中经常遇到粘性或周期介质[1- 4], 经过这类介质的流的输运, 能够由具有迅速振荡系数的二阶非自共轭椭圆型方程的边值问题来表示, 数值上解这类问题主要困难之一是计算规模非常大• Lions, Bourget, Cioranescu 等从理论上给出了均匀化方法[1,5- 6], 主要的思想是通过构造相应的局部光滑算子得到平均场方程, 最终可在粗网格上数值地求解均匀化方程• Lions 对周期或拟周期结构的二阶椭圆方程给出了相应的均匀化方程和一阶矫正[1], Oleinik 用均匀化方法求解孔洞域上外边界部分为 Dirichlet 边界条件, 洞表面为 Neumann 边界条件的混合问题[3]• 均匀化方法和一阶矫正描述某些物理场的局部振荡不够充分, 对考虑的解寻找高阶渐进展开非常有必要• Cui 和 Cao 等对无孔洞整周期区域上的具有迅速振荡系数的弹性系统 Dirichlet 问题给出了高阶渐进展开[7- 11], Feng 等对小周期复合材料结构在耦合热弹性条件下的问题提出了高阶渐进展开[12], Li, Cui, Yu 等用这类高阶方法预测具有随机颗粒分布复合材料的热传导性能[13- 15]• 然而, 以上方法仅考虑了自共轭椭圆问题• Chen 等研究了具有迅速振荡系数的非自共轭椭圆问题[16], 在文献[16]中, 考虑了均匀化解和一阶双尺度近似解• 当系数相差 100 倍以上时, 近似解与原问题解的误差相差很大, 这类问题用通常的多尺度方法不

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能得到高阶渐进展开,本文通过构造两个矫正项给辅助函数消除干扰,引入了一种二阶双尺度近似解•

文章的内容安排如下:第1节引入了问题和一些记号;第2节给出了二阶双尺度近似解及其误差分析;第3节给出了一些数值实验,与本文理论结果非常吻合•

1 控制方程和新方法

假设有界区域 $\Omega\in R^2$ 包含完整的基本单元,即 $\Omega=\sum_{z\in I^\epsilon}\epsilon(z+Q)$,$\epsilon$ 是一小正整数,

$$I^\epsilon=\left\{z\in Z^2\mid \epsilon(z+Q)\subset\Omega\right\},$$
$$Q=\left\{\xi\mid 0<\xi_i<1,\ i=1,2\right\}•$$

考虑以下问题:

$$\begin{cases}L^\epsilon u^\epsilon=-\operatorname{div}(\boldsymbol{a}^\epsilon\nabla u^\epsilon+\alpha^\epsilon u^\epsilon)+\beta^\epsilon\cdot\nabla u^\epsilon+\gamma^\epsilon u^\epsilon+\lambda u^\epsilon=f & (\text{在 }\Omega\text{ 中}),\\ u^\epsilon=0 & (\text{在 }\partial\Omega\text{ 上}),\end{cases}\tag{1}$$

其中,f 是充分光滑的函数,$\boldsymbol{a}^\epsilon=(a^\epsilon_{ij}(x))$ 是具有周期 ϵ 的有界对称正定矩阵值函数,$\alpha^\epsilon=(\alpha^\epsilon_i(x))$ 和 $\beta^\epsilon=(\beta^\epsilon_i(x))$ 是具有周期 ϵ 的向量值函数,$\gamma^\epsilon(x)$ 是具有周期 ϵ 的函数,$\lambda>0$ 是足够大的实数,使得算子 L^ϵ 是强制的,即

$$(L^\epsilon v,v)\geqslant C\|v\|_1^2,\qquad \forall v\in H_0^1(\Omega)•$$

设

$$\xi=\frac{x}{\epsilon},\ \boldsymbol{a}=(a_{ij}(\xi))=(a^\epsilon_{ij}(x)),$$
$$\alpha=(\alpha_i(\xi))=(\alpha^\epsilon_i(x)),\ \beta=(\beta_i(\xi))=(\beta^\epsilon_i(x))\ \text{和}\ \gamma(\xi)=\gamma^\epsilon(x),$$

则 $a_{ij}(\xi)$, $\alpha_i(\xi)$, $\beta_i(\xi)$ 和 $\gamma(\xi)$ 是1周期函数,假设 $a_{ij}(\xi)$, $\alpha_i(\xi)\in H^1(Q)$, $\beta_i(\xi)$, $\gamma(\xi)\in L^\infty(Q)$• 引入周期函数 $M_0(\xi)$, $N_{\alpha_1}(\xi)$ $(\alpha_1=1,2)$,它们分别由以下方程定义:

$$\begin{cases}\dfrac{\partial}{\partial\xi_i}\left(a_{ij}(\xi)\dfrac{\partial M_0(\xi)}{\partial\xi_j}\right)=-\dfrac{\partial\alpha_i(\xi)}{\partial\xi_i} & (\text{在 }Q\text{ 中}),\\ M_0(\xi)=0 & (\text{在 }\partial Q\text{ 上}),\end{cases}\tag{2}$$

$$\begin{cases}\dfrac{\partial}{\partial\xi_i}\left(a_{ij}(\xi)\dfrac{\partial N_{\alpha_1}(\xi)}{\partial\xi_j}\right)=-\dfrac{\partial a_{i\alpha_1}(\xi)}{\partial\xi_i} & (\text{在 }Q\text{ 中}),\\ N_{\alpha_1}(\xi)=0 & (\text{在 }\partial Q\text{ 上})•\end{cases}\tag{3}$$

根据下式定义常数矩阵 $\boldsymbol{a}^0=(a^0_{ij})$:

$$a^0_{ij}=\frac{1}{|Q|}\int_Q\left(a_{ij}(\xi)+a_{ik}(\xi)\frac{\partial N_j(\xi)}{\partial\xi_k}\right)\mathrm{d}\xi•$$

由下式定义常数向量 $\alpha^0=(\alpha^0_i)$ 和 $\beta^0=(\beta^0_i)$:

$$\alpha^0_i=\frac{1}{|Q|}\int_Q\left(\alpha_i(\xi)+a_{ik}(\xi)\frac{\partial M_0(\xi)}{\partial\xi_k}\right)\mathrm{d}\xi,$$
$$\beta^0_i=\frac{1}{|Q|}\int_Q\left(\beta_i(\xi)+\beta_k(\xi)\frac{\partial N_i(\xi)}{\partial\xi_k}\right)\mathrm{d}\xi•$$

定义常数 γ^0 如下:

$$\gamma^0=\frac{1}{|Q|}\int_Q\left(\gamma(\xi)+\beta_k(\xi)\frac{\partial M_0(\xi)}{\partial\xi_k}\right)\mathrm{d}\xi•$$

现在引入问题(1)的以下均匀化方程:

$$\begin{cases} -\operatorname{div}(\boldsymbol{a}^0\nabla u^0+\alpha^0u^0)+\beta^0\cdot\nabla u^0+\gamma^0u^0+\lambda u^0=f & (\text{在 }\Omega\text{ 中}),\\ u^0=0 & (\text{在 }\partial\Omega\text{ 上})\cdot \end{cases} \tag{4}$$

为方便起见,给出记号 g_{ij}, p_i, q_i 和 r 的定义如下:

$$g_{ij}=a_{ij}+a_{ik}\frac{\partial N_j}{\partial\xi_k}-a^0_{ij},$$

$$p_i=\alpha_i+a_{ik}\frac{\partial M_0}{\partial\xi_k}-\alpha^0_i,$$

$$q_i=\beta_i+\beta_k\frac{\partial N_i}{\partial\xi_k}-\beta^0_i,$$

$$r=\gamma+\beta_k\frac{\partial M_0}{\partial\xi_k}-\gamma^0\cdot$$

由 L^ϵ 的定义知道

$$\begin{aligned} L^\epsilon=&-\left(\frac{\partial}{\partial x_i}+\frac{1}{\epsilon}\frac{\partial}{\partial\xi_i}\right)a_{ij}\left(\frac{\partial}{\partial x_j}+\frac{1}{\epsilon}\frac{\partial}{\partial\xi_j}\right)-\\ &\left(\frac{\partial}{\partial x_i}+\frac{1}{\epsilon}\frac{\partial}{\partial\xi_i}\right)\alpha_i+\beta_i\left(\frac{\partial}{\partial x_i}+\frac{1}{\epsilon}\frac{\partial}{\partial\xi_i}\right)+\gamma+\lambda=\\ &\epsilon^{-2}L_1+\epsilon^{-1}L_2+L_3, \end{aligned} \tag{5}$$

这里,变量 x 和 ξ 是相互独立的,并且

$$L_1=-\frac{\partial}{\partial\xi_i}\left(a_{ij}\frac{\partial}{\partial\xi_j}\right),$$

$$L_2=-\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial\xi_j}\right)-\frac{\partial}{\partial\xi_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)-\frac{\partial}{\partial\xi_i}\alpha_i+\beta_i\frac{\partial}{\partial\xi_i},$$

$$L_3=-\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)-\frac{\partial}{\partial x_i}\alpha_i+\beta_i\frac{\partial}{\partial x_i}+\gamma+\lambda\cdot$$

2 二阶双尺度近似解和误差估计

根据文献[16],定义问题(1)解 u^ϵ 的二阶双尺度近似表示式

$$\begin{aligned} u^\epsilon_2=&u^0(x)+\epsilon u^1(x,\xi)+\epsilon^2u^2(x,\xi)=\\ &u^0(x)+\epsilon\left(M_0(\xi)u^0+N_{\alpha_1}(\xi)\frac{\partial u^0}{\partial x_{\alpha_1}}\right)+\\ &\epsilon^2\left[C_0(\xi)u^0+M_{\alpha_1}(\xi)\frac{\partial u^0}{\partial x_{\alpha_1}}+D_{\alpha_1}(\xi)\frac{\partial u^0}{\partial x_{\alpha_1}}+N_{\alpha_1\alpha_2}(\xi)\frac{\partial^2u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\right], \end{aligned} \tag{6}$$

其中, $u^0(x)$, $M_0(\xi)$ 和 $N_{\alpha_1}(\xi)$ 分别由式(4)、(2)和(3)定义,在这部分先给出 $C_0(\xi)$, $M_{\alpha_1}(\xi)$, $D_{\alpha_1}(\xi)$ 和 $N_{\alpha_1\alpha_2}(\xi)$ 的定义:

$$\begin{cases} \dfrac{\partial}{\partial\xi_i}\left(a_{ij}(\xi)\dfrac{\partial C_0(\xi)}{\partial\xi_j}\right)=\\ \qquad -\gamma^0+\gamma(\xi)+\beta_i(\xi)\dfrac{\partial M_0(\xi)}{\partial\xi_i}-\dfrac{\partial}{\partial\xi_i}(\alpha_i(\xi)M_0(\xi)) & (\text{在 }Q\text{ 中}),\\ C_0(\xi)=0 & (\text{在 }\partial Q\text{ 上}), \end{cases}$$

$$
\begin{cases}
\dfrac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\, \dfrac{\partial M_{\alpha_1}(\xi)}{\partial \xi_j}\right) = \\
\qquad \alpha^0_{\alpha_1} - \alpha_{\alpha_1}(\xi) - a_{\alpha_1 j}(\xi)\, \dfrac{\partial M_0(\xi)}{\partial \xi_j} - \dfrac{\partial}{\partial \xi_i}\left(a_{i\alpha_1}(\xi) M_0(\xi)\right) & (\text{在 } Q \text{ 中}), \\
M_{\alpha_1}(\xi) = 0 & (\text{在 } \partial Q \text{ 上}),
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\, \dfrac{\partial D_{\alpha_1}(\xi)}{\partial \xi_j}\right) = \\
\qquad -\beta^0_{\alpha_1} + \beta_{\alpha_1}(\xi) + \beta_i(\xi)\, \dfrac{\partial N_{\alpha_1}(\xi)}{\partial \xi_i} - \dfrac{\partial}{\partial \xi_i}\left(\alpha_i(\xi) N_{\alpha_1}(\xi)\right) & (\text{在 } Q \text{ 中}), \\
D_{\alpha_1}(\xi) = 0 & (\text{在 } \partial Q \text{ 上}),
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\, \dfrac{\partial N_{\alpha_1\alpha_2}(\xi)}{\partial \xi_j}\right) = \\
\qquad a^0_{\alpha_1\alpha_2} - a_{\alpha_1\alpha_2}(\xi) - a_{\alpha_2 j}(\xi)\, \dfrac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j} - \dfrac{\partial}{\partial \xi_i}\left(a_{i\alpha_2}(\xi) N_{\alpha_1}(\xi)\right) & (\text{在 } Q \text{ 中}), \\
N_{\alpha_1\alpha_2}(\xi) = 0 & (\text{在 } \partial Q \text{ 上})\cdot
\end{cases}
$$

现在已经给出了二阶双尺度近似解所有辅助函数的定义, 接下来估计近似解 $u_2^{\epsilon}(x)$ 的误差, 先给出以下结果•

引理 1　假设 $u^0 \in W^4_{\infty}(\Omega) \cap H^1_0(\Omega)$, 则 $u^{\epsilon} - u_2^{\epsilon}$ 满足以下变分方程:

$$
(L^{\epsilon}(u^{\epsilon} - u_2^{\epsilon}), v) = (f^*, v), \qquad \forall v \in H^1_0(\Omega), \tag{7}
$$

其中

$$
\begin{aligned}
f^* = {} & g_{ij}\, \frac{\partial^2 u^0}{\partial x_i \partial x_j} + p_i\, \frac{\partial u^0}{\partial x_i} - q_i\, \frac{\partial u^0}{\partial x_i} - r u^0 + \frac{\partial}{\partial \xi_i}(a_{ij} N_k)\, \frac{\partial^2 u^0}{\partial x_j \partial x_k} + \\
& \frac{\partial}{\partial \xi_i}(a_{ij} M_0)\, \frac{\partial u^0}{\partial x_j} + \frac{\partial u^0}{\partial x_k}\, \frac{\partial}{\partial \xi_i}(\alpha_i N_k) + u^0\, \frac{\partial}{\partial \xi_i}(\alpha_i M_0) + \\
& u^0\, \frac{\partial}{\partial \xi_i}\left(a_{ij}\, \frac{\partial C_0}{\partial \xi_j}\right) + \frac{\partial}{\partial \xi_i}\left(a_{ij}\, \frac{\partial M_{\alpha_1}}{\partial \xi_j}\right) \frac{\partial u^0}{\partial x_{\alpha_1}} + \\
& \frac{\partial}{\partial \xi_i}\left(a_{ij}\, \frac{\partial D_{\alpha_1}}{\partial \xi_j}\right) \frac{\partial u^0}{\partial x_{\alpha_1}} + \frac{\partial}{\partial \xi_i}\left(a_{ij}\, \frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\right) \frac{\partial^2 u^0}{\partial x_{\alpha_1} \partial x_{\alpha_2}} + \\
& \epsilon \left\{ \frac{\partial u^0}{\partial x_i} a_{ij}\, \frac{\partial C_0}{\partial \xi_j} + \frac{\partial u^0}{\partial x_j}\, \frac{\partial}{\partial \xi_i}(a_{ij} C_0) + u^0\, \frac{\partial}{\partial \xi_i}(\alpha_i C_0) - \beta_i u^0\, \frac{\partial C_0}{\partial \xi_i} \right\} + \\
& \epsilon \left\{ \frac{\partial^2 u^0}{\partial x_{\alpha_1} \partial x_i} a_{ij}\, \frac{\partial M_{\alpha_1}}{\partial \xi_j} + \frac{\partial^2 u^0}{\partial x_{\alpha_1} \partial x_j}\, \frac{\partial}{\partial \xi_i}(a_{ij} M_{\alpha_1}) + \frac{\partial u^0}{\partial x_{\alpha_1}}\, \frac{\partial}{\partial \xi_i}(\alpha_i M_{\alpha_1}) - \beta_i\, \frac{\partial u^0}{\partial x_{\alpha_1}}\, \frac{\partial M_{\alpha_1}}{\partial \xi_i} \right\} + \\
& \epsilon \left\{ \frac{\partial^2 u^0}{\partial x_{\alpha_1} \partial x_i} a_{ij}\, \frac{\partial D_{\alpha_1}}{\partial \xi_j} + \frac{\partial^2 u^0}{\partial x_{\alpha_1} \partial x_j}\, \frac{\partial}{\partial \xi_i}(a_{ij} D_{\alpha_1}) + \frac{\partial u^0}{\partial x_{\alpha_1}}\, \frac{\partial}{\partial \xi_i}(\alpha_i D_{\alpha_1}) - \beta_i\, \frac{\partial u^0}{\partial x_{\alpha_1}}\, \frac{\partial D_{\alpha_1}}{\partial \xi_i} \right\} + \\
& \epsilon \left\{ \frac{\partial^3 u^0}{\partial x_{\alpha_1} \partial x_{\alpha_2} \partial x_i} a_{ij}\, \frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j} + \frac{\partial^3 u^0}{\partial x_{\alpha_1} \partial x_{\alpha_2} \partial x_j}\, \frac{\partial}{\partial \xi_i}(a_{ij} N_{\alpha_1\alpha_2}) + \right. \\
& \left. \frac{\partial^2 u^0}{\partial x_{\alpha_1} \partial x_{\alpha_2}}\, \frac{\partial}{\partial \xi_i}(\alpha_i N_{\alpha_1\alpha_2}) - \beta_i\, \frac{\partial^2 u^0}{\partial x_{\alpha_1} \partial x_{\alpha_2}}\, \frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_i} \right\} +
\end{aligned}
$$

$$\epsilon M_0\left\{a_{ij}\frac{\partial^2 u^0}{\partial x_i\partial x_j}+(\alpha_i-\beta_i)\frac{\partial u^0}{\partial x_i}-(\gamma+\lambda)u^0\right\}+$$
$$\epsilon N_k\left\{a_{ij}\frac{\partial^3 u^0}{\partial x_i\partial x_j\partial x_k}+(\alpha_i-\beta_i)\frac{\partial^2 u^0}{\partial x_i\partial x_k}-(\gamma+\lambda)\frac{\partial u^0}{\partial x_k}\right\}+$$
$$\epsilon^2 C_0\left\{a_{ij}\frac{\partial^2 u^0}{\partial x_i\partial x_j}+(\alpha_i-\beta_i)\frac{\partial u^0}{\partial x_i}-(\gamma+\lambda)u^0\right\}+$$
$$\epsilon^2 M_k\left\{a_{ij}\frac{\partial^3 u^0}{\partial x_i\partial x_j\partial x_k}+(\alpha_i-\beta_i)\frac{\partial^2 u^0}{\partial x_i\partial x_k}-(\gamma+\lambda)\frac{\partial u^0}{\partial x_k}\right\}+$$
$$\epsilon^2 D_k\left\{a_{ij}\frac{\partial^3 u^0}{\partial x_i\partial x_j\partial x_k}+(\alpha_i-\beta_i)\frac{\partial^2 u^0}{\partial x_i\partial x_k}-(\gamma+\lambda)\frac{\partial u^0}{\partial x_k}\right\}+$$
$$\epsilon^2 N_{ks}\left\{a_{ij}\frac{\partial^4 u^0}{\partial x_i\partial x_j\partial x_k\partial x_s}+(\alpha_i-\beta_i)\frac{\partial^3 u^0}{\partial x_i\partial x_k\partial x_s}-(\gamma+\lambda)\frac{\partial^2 u^0}{\partial x_k\partial x_s}\right\}.$$

引理 2

$$\|M_0\|_0\leqslant C_1\epsilon\quad|M_0|_1\leqslant C_2\quad 和\quad|M_0|_2\leqslant C\epsilon^{-2},$$
$$\|N_{\alpha_1}\|_0\leqslant C_1\epsilon\quad|N_{\alpha_1}|_1\leqslant C_2\quad 和\quad|N_{\alpha_1}|_2\leqslant C\epsilon^{-2},$$
$$\|C_0\|_0\leqslant C_1\epsilon\quad|C_0|_1\leqslant C_2\quad 和\quad|C_0|_2\leqslant C\epsilon^{-2},$$
$$\|M_{\alpha_1}\|_0\leqslant C_1\epsilon\quad|M_{\alpha_1}|_1\leqslant C_2\quad 和\quad|M_{\alpha_1}|_2\leqslant C\epsilon^{-2},$$
$$\|D_{\alpha_1}\|_0\leqslant C_1\epsilon\quad|D_{\alpha_1}|_1\leqslant C_2\quad 和\quad|D_{\alpha_1}|_2\leqslant C\epsilon^{-2},$$
$$\|N_{\alpha_1\alpha_2}\|_0\leqslant C_1\epsilon\quad|N_{\alpha_1\alpha_2}|_1\leqslant C_2\quad 和\quad|N_{\alpha_1\alpha_2}|_2\leqslant C\epsilon^{-2}.$$

引理 3 对 $\forall v\in H^1(\Omega)$，以下不等式成立:

$$\left|\int_\Omega g_{ij}\,v\,\mathrm{d}x\right|\leqslant C_1\epsilon|v|_1,\quad\left|\int_\Omega p_i\,v\,\mathrm{d}x\right|\leqslant C_1\epsilon|v|_1,$$
$$\left|\int_\Omega q_i\,v\,\mathrm{d}x\right|\leqslant C_1\epsilon|v|_1,\quad\left|\int_\Omega r\,v\,\mathrm{d}x\right|\leqslant C_1\epsilon|v|_1.$$

引理 1、引理 2 和引理 3 的证明见文献[16]，现在证明这部分主要的结果.

定理 1 假设 $u^0(x)\in W^4_\infty(\Omega)\cap H^1_0(\Omega)$，则有以下成立:

$$\|u^\epsilon(x)-u_2^\epsilon(x)\|_1\leqslant C\epsilon\|u^0\|_{W^4_\infty(\Omega)},\tag{8}$$

其中 $C>0$，且与 ϵ, u^ϵ 和 u^0 无关.

证明 因为算子 L^ϵ 是强制的，$u^\epsilon-u_2^\epsilon\in H^1_0(\Omega)$，由式(7)，得

$$C\|u^\epsilon-u_2^\epsilon\|_1^2\leqslant(L^\epsilon(u^\epsilon-u_2^\epsilon),u^\epsilon-u_2^\epsilon)=(f^*,u^\epsilon-u_2^\epsilon).\tag{9}$$

用分部积分和文献[17]中的一些定理，有

$$(f^*,u^\epsilon-u_2^\epsilon)\leqslant$$
$$\left|\int_\Omega g_{ij}\frac{\partial^2 u^0}{\partial x_i\partial x_j}(u^\epsilon-u_2^\epsilon)\,\mathrm{d}x\right|+\left|\int_\Omega p_i\frac{\partial u^0}{\partial x_i}(u^\epsilon-u_2^\epsilon)\,\mathrm{d}x\right|+$$
$$\left|\int_\Omega q_i\frac{\partial u^0}{\partial x_i}(u^\epsilon-u_2^\epsilon)\,\mathrm{d}x\right|+\left|\int_\Omega r\,u^0(u^\epsilon-u_2^\epsilon)\,\mathrm{d}x\right|+$$
$$\epsilon\left|\int_\Omega a_{ij}N_k\frac{\partial}{\partial x_i}\left(\frac{\partial^2 u^0}{\partial x_j\partial x_k}(u^\epsilon-u_2^\epsilon)\right)\mathrm{d}x\right|+$$
$$\epsilon\left|\int_\Omega a_{ij}M_0\frac{\partial}{\partial x_i}\left(\frac{\partial u^0}{\partial x_j}(u^\epsilon-u_2^\epsilon)\right)\mathrm{d}x\right|+$$

$$\varepsilon\left|\int_\Omega \alpha_i N_k \frac{\partial}{\partial x_i}\left(\frac{\partial u^0}{\partial x_k}(u^\varepsilon - u_2^\varepsilon)\right)\mathrm{d}x\right| + \varepsilon\left|\int_\Omega \alpha_i M_0 \frac{\partial}{\partial x_i}(u^0(u^\varepsilon - u_2^\varepsilon))\,\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega a_{ij} \frac{\partial C_0}{\partial \xi_j} \frac{\partial}{\partial x_i}(u^0(u^\varepsilon - u_2^\varepsilon))\,\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega a_{ij} \frac{\partial M_{\alpha_1}}{\partial \xi_j} \frac{\partial}{\partial x_i}\left(\frac{\partial u^0}{\partial x_{\alpha_1}}(u^\varepsilon - u_2^\varepsilon)\right)\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega a_{ij} \frac{\partial D_{\alpha_1}}{\partial \xi_j} \frac{\partial}{\partial x_i}\left(\frac{\partial u^0}{\partial x_{\alpha_1}}(u^\varepsilon - u_2^\varepsilon)\right)\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega a_{ij} \frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j} \frac{\partial}{\partial x_i}\left(\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}(u^\varepsilon - u_2^\varepsilon)\right)\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega \left\{\frac{\partial u^0}{\partial x_i} a_{ij} \frac{\partial C_0}{\partial \xi_j} + \frac{\partial u^0}{\partial x_j}\frac{\partial}{\partial \xi_i}(a_{ij} C_0) + u^0 \frac{\partial}{\partial \xi_i}(\alpha_i C_0) - \right.\right.$$

$$\left.\left.\beta_i u^0 \frac{\partial C_0}{\partial \xi_i}\right\}(u^\varepsilon - u_2^\varepsilon)\,\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega \left\{\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_i} a_{ij} \frac{\partial M_{\alpha_1}}{\partial \xi_j} + \frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_j}\frac{\partial}{\partial \xi_i}(a_{ij} M_{\alpha_1}) + \right.\right.$$

$$\left.\left.\frac{\partial u^0}{\partial x_{\alpha_1}} \frac{\partial}{\partial \xi_i}(\alpha_i M_{\alpha_1}) - \beta_i \frac{\partial u^0}{\partial x_{\alpha_1}} \frac{\partial M_{\alpha_1}}{\partial \xi_i}\right\}(u^\varepsilon - u_2^\varepsilon)\,\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega \left\{\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_i} a_{ij} \frac{\partial D_{\alpha_1}}{\partial \xi_j} + \frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_j}\frac{\partial}{\partial \xi_i}(a_{ij} D_{\alpha_1}) + \right.\right.$$

$$\left.\left.\frac{\partial u^0}{\partial x_{\alpha_1}} \frac{\partial}{\partial \xi_i}(\alpha_i D_{\alpha_1}) - \beta_i \frac{\partial u^0}{\partial x_{\alpha_1}} \frac{\partial D_{\alpha_1}}{\partial \xi_i}\right\}(u^\varepsilon - u_2^\varepsilon)\,\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega \left\{\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i} a_{ij} \frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j} + \frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j}\frac{\partial}{\partial \xi_i}(a_{ij} N_{\alpha_1\alpha_2}) + \right.\right.$$

$$\left.\left.\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \frac{\partial}{\partial \xi_i}(\alpha_i N_{\alpha_1\alpha_2}) - \beta_i \frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_i}\right\}(u^\varepsilon - u_2^\varepsilon)\,\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega M_0\left\{a_{ij}\frac{\partial^2 u^0}{\partial x_i\partial x_j} + (\alpha_i - \beta_i)\frac{\partial u^0}{\partial x_i} - (\gamma + \lambda)u^0\right\}(u^\varepsilon - u_2^\varepsilon)\,\mathrm{d}x\right| +$$

$$\varepsilon\left|\int_\Omega N_{\alpha_1}\left\{a_{ij}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_i\partial x_j} + (\alpha_i - \beta_i)\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_i} - (\gamma + \lambda)\frac{\partial u^0}{\partial x_{\alpha_1}}\right\}(u^\varepsilon - u_2^\varepsilon)\,\mathrm{d}x\right| +$$

$$\varepsilon^2\left|\int_\Omega C_0\left\{a_{ij}\frac{\partial^2 u^0}{\partial x_i\partial x_j} + (\alpha_i - \beta_i)\frac{\partial u^0}{\partial x_i} - (\gamma + \lambda)u^0\right\}(u^\varepsilon - u_2^\varepsilon)\,\mathrm{d}x\right| +$$

$$\varepsilon^2\left|\int_\Omega M_{\alpha_1}\left\{a_{ij}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_i\partial x_j} + (\alpha_i - \beta_i)\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_i} - (\gamma + \lambda)\frac{\partial u^0}{\partial x_{\alpha_1}}\right\}(u^\varepsilon - u_2^\varepsilon)\,\mathrm{d}x\right| +$$

$$\varepsilon^2\left|\int_\Omega D_{\alpha_1}\left\{a_{ij}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_i\partial x_j} + (\alpha_i - \beta_i)\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_i} - (\gamma + \lambda)\frac{\partial u^0}{\partial x_{\alpha_1}}\right\}(u^\varepsilon - u_2^\varepsilon)\,\mathrm{d}x\right| +$$

$$\varepsilon^2\left|\int_\Omega N_{\alpha_1\alpha_2}\left\{a_{ij}\frac{\partial^4 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i\partial x_j} + (\alpha_i - \beta_i)\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i} - \right.\right.$$

$$(\gamma + \lambda)\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\Big\}(u^\epsilon - u_2^\epsilon)\mathrm{d}x\Bigg| \leqslant$$

$$C\epsilon\Big\{\|u^0\|_{W_\infty^2}|u^\epsilon - u_2^\epsilon|_1 + \|N_k\|_0\|u^0\|_{W_\infty^3}\|u^\epsilon - u_2^\epsilon\|_1 +$$

$$\|M_0\|_0\|u^0\|_{W_\infty^2}\|u^\epsilon - u_2^\epsilon\|_1 + \epsilon\|C_0\|_1\|u^0\|_{W_\infty^2}\|u^\epsilon - u_2^\epsilon\|_1 +$$

$$\epsilon\|M_{\alpha_1}\|_1\|u^0\|_{W_\infty^3}\|u^\epsilon - u_2^\epsilon\|_1 + \epsilon\|D_{\alpha_1}\|_1\|u^0\|_{W_\infty^3}\|u^\epsilon - u_2^\epsilon\|_1 +$$

$$\epsilon\|N_{\alpha_1\alpha_2}\|_1\|u^0\|_{W_\infty^4}\|u^\epsilon - u_2^\epsilon\|_1\Big\} \leqslant$$

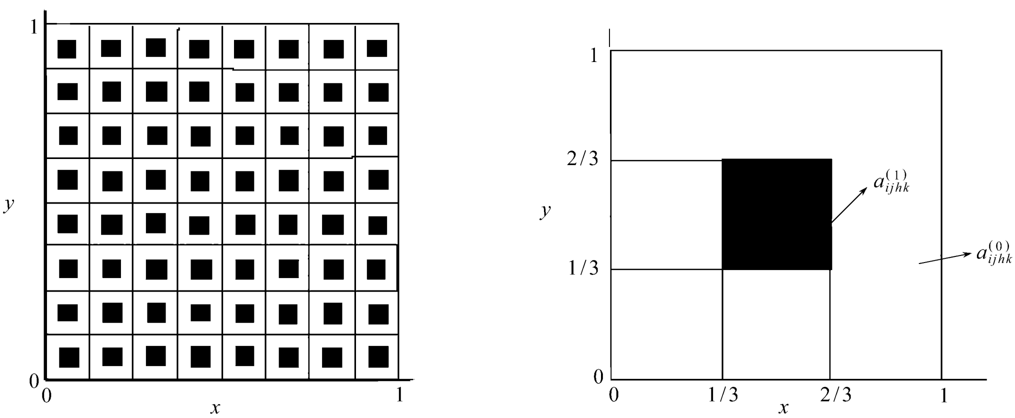
$$C\epsilon\|u^0\|_{W_\infty^4}\|u^\epsilon - u_2^\epsilon\|_1 .$$

根据引理 2 和引理 3, 结合式(9) 和以上不等式, 有

$$\|u^\epsilon - u_2^\epsilon\|_1 \leqslant C\epsilon\|u^0\|_{W_\infty^4} .$$

3 数 值 结 果

考虑式(1), 其中 Ω 如图 1(a) 所示, 周期单胞 Q 如图 1(b) 所示, $\epsilon = 1/8$. 因为很难找到式(1) 的解析解, 用非常细网格上的有限元解代替 $u^\epsilon(x)$, 对 Ω 进行矩形剖分, 使得系数 a_{ij} 的不连续与剖分矩形边一致, 矩形单元数和节点数如表 1 所示.



(a) 区域 $\Omega = [0, 1]^2$　　(b) 周期单胞 $Q = [0, \epsilon]^2$

图 1 区域和单胞图

表 1 单元数和节点数比较

	原始方程	单胞方程	均匀化方程
单元数	9 216	576	2 304
节点数	37 249	2 401	9 409

情况 1

$$\epsilon = \frac{1}{8},\ a_{ij0} = \delta_{ij},\ a_{ij1} = \frac{1}{1\ 000}\delta_{ij},$$

$$f(x) = 10^5(x_1x_2(1 - x_1)(1 - x_2))^3;$$

情况 2

$$\epsilon = \frac{1}{8},\ a_{ij0} = \delta_{ij},\ a_{ij1} = \frac{1}{500}\delta_{ij},$$

$$f(x) = 10\sin(\pi x_1)\sin(\pi x_2);$$

其中

$$\delta_{ij} = 1(i = j),\ \delta_{ij} = 0(i \neq j),$$

$$u_2^{\epsilon} = u^0(x) + \epsilon u^1(x,\xi) + \epsilon^2 u^2(x,\xi) =$$

$$u^0(x) + \epsilon\left(M_0(\xi)u^0 + N_{\alpha_1}(\xi)\frac{\partial u^0}{\partial x_{\alpha_1}}\right) +$$

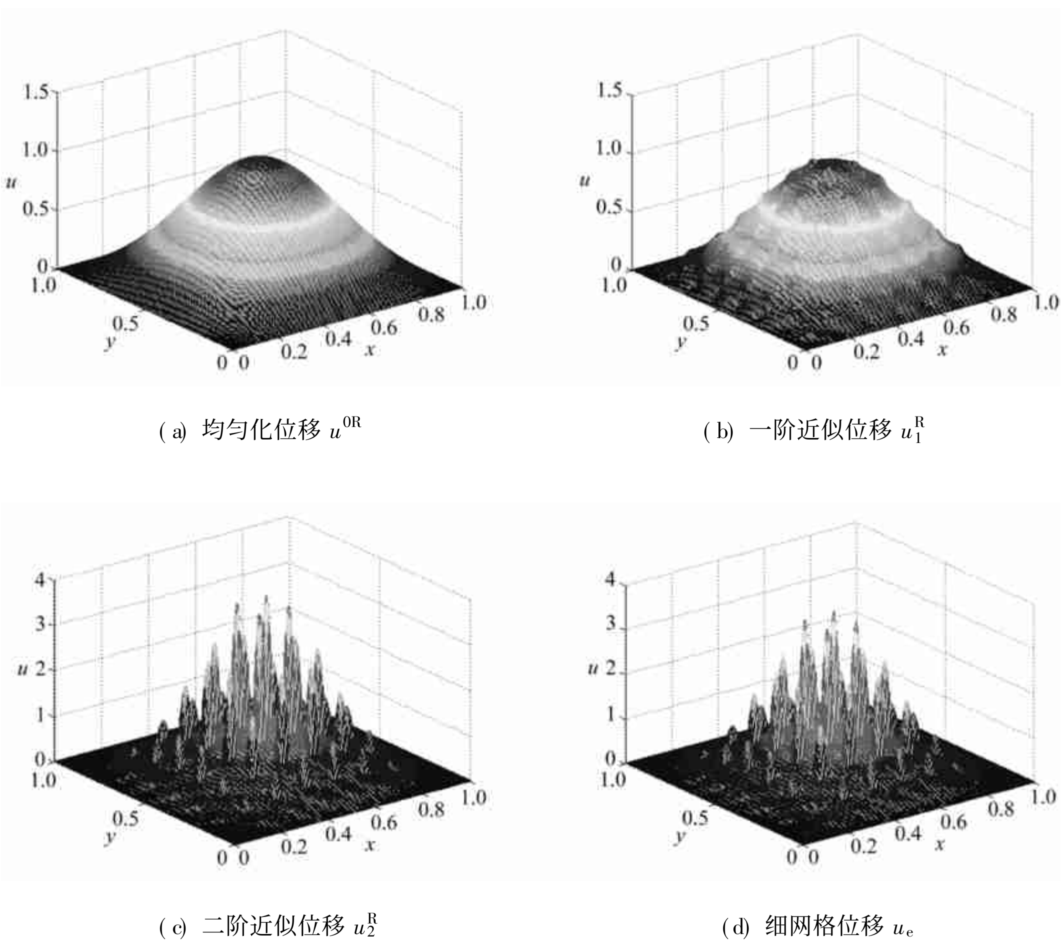
$$\epsilon^2\left[C_0(\xi)u^0 + M_{\alpha_1}(\xi)\frac{\partial u^0}{\partial x_{\alpha_1}} + D_{\alpha_1}(\xi)\frac{\partial u^0}{\partial x_{\alpha_1}} + N_{\alpha_1\alpha_2}(\xi)\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\right]$$

$$(\alpha_1, \alpha_2 = 1, 2), \tag{10}$$

这里

$$e_0 = u_e - u^0,\ e_1 = u_e - u_1^{\epsilon},\ e_2 = u_e - u_2^{\epsilon},$$

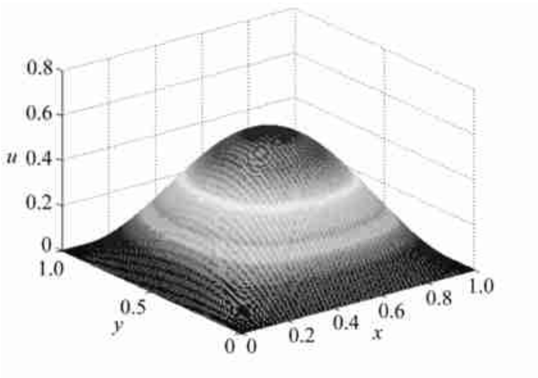
$u^0(x)$ 是均匀化方程的有限元解, $u_e(x)$ 是式(1)在非常细网格上的有限元解, $u_1^{\epsilon}(x)$, $u_2^{\epsilon}(x)$ 分别是一阶、二阶多尺度有限元解, $T_{u_2^{\epsilon}}$ 是二阶多尺度有限元解的计算时间, T_{u_e} 是非常细网格上有限元解的计算时间•



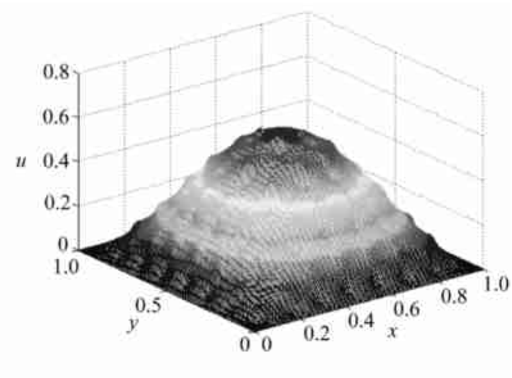
(a) 均匀化位移 u^{0R}　　(b) 一阶近似位移 u_1^R

(c) 二阶近似位移 u_2^R　　(d) 细网格位移 u_e

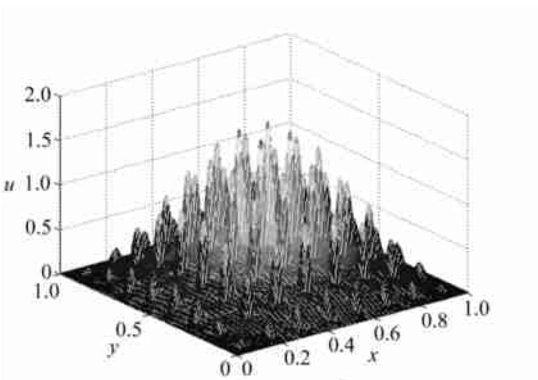
图2　情况1的有限元位移图



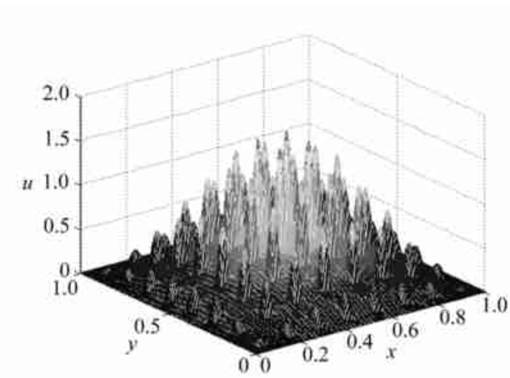
(a) 均匀化位移 u^{0R}



(b) 一阶近似位移 u_1^R



(c) 二阶近似位移 u_2^R



(d) 细网格位移 u_e

图 3 情况 2 的有限元位移图

表 2 L^2 范数、H^1 范数、时间计算结果比较

	$\frac{\Vert e_0\Vert_{L^2}}{\Vert u_e\Vert_{L^2}}$	$\frac{\Vert e_1\Vert_{L^2}}{\Vert u_e\Vert_{L^2}}$	$\frac{\Vert e_2\Vert_{L^2}}{\Vert u_e\Vert_{L^2}}$	$\frac{\Vert e_0\Vert_{H^1}}{\Vert u_e\Vert_{H^1}}$	$\frac{\Vert e_1\Vert_{H^1}}{\Vert u_e\Vert_{H^1}}$	$\frac{\Vert e_2\Vert_{H^1}}{\Vert u_e\Vert_{H^1}}$	$T_{u_2^\epsilon}(s)$	$T_{u_e}(s)$
情况 1	0.330 89	0.330 13	0.035 69	0.330 93	0.330 16	0.035 31	19.671	57.516 0
情况 2	0.309 45	0.308 62	0.033 27	0.309 48	0.308 64	0.032 91	16.031	47.187 0

u^{0R}, u_1^R 和 u_2^R 表示 u^0, u_1^ϵ 和 u_2^ϵ 在细网格上的插值图. 从图 2 和图 3 可以看出, 二阶双尺度有限元解与细网格上的有限元解吻合很好, 而均匀化解和一阶近似解与细网格有限元解相比效果差一些. 从表 1 和表 2 可以看到, 二阶双尺度近似解的网格剖分数比细网格有限元解的网格剖分数少得多, 计算时间也比细网格有限元解的计算时间少得多. 这说明本文给出的近似解能大大节省计算机存储空间和 CPU 运行时间, 在工程计算中, 这一点非常重要. 表 1 和表 2 还给出了不同阶近似解与细网格有限元解之间的各种相对误差, 所有信息表明, 这种二阶双尺度近似解近似具有迅速振荡系数的非自共轭椭圆问题是有效的.

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Two- Order and Two- Scale Computation Method for Nonselfadjoint Elliptic Problems With Rapidly Oscillatory Coefficients

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Abstract: The purpose was to solve nonselfadjoint elliptic problems with rapidly oscillatory coefficients. A two- order and two- scale approximate solution expression for nonselfadjoint elliptic problems was considered, and the error estimation of the two- order and two- scale approximate solution was derived. The numerical result shows that the approximation solution is effective.

Key words: nonselfadjoint elliptic problems; rapidly oscillatory coefficients; two- order and two- scale finite element method