Maxwell 流体在震荡的矩形
输送管道中的流动[*]

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摘要: 分析了不可压缩 Maxwell 流体在震荡矩形截面管道中的非稳定流动问题. 利用 Fourier 变换和 Laplace 变换作为数学工具,提出了问题的解,该解可以看成稳态解和暂态解之和. 大倍数时,暂态消失,解可以表示为稳态解. 在极限情况的案例中给出了 Newton 流体的解. 当震荡频率不存在时,得到了 Maxwell 流体在震荡矩形截面管道中流动问题的解. 最后,以图形形式给出不同参数时,矩形管道正弦震荡达到稳态所需要的时间. 同时,分别描绘了 x 和 y 变化时的速度曲线.

关 键 词: Maxwell 流体; 震荡的矩形输送管道; 速度场

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引　　言

在过去的数十年间,科学研究的一个重点课题聚焦于非 Newton 流体的研究. 非 Newton 流体由一组广泛的流体类型所组成,在这些流体中,剪应力和剪切率之间的关系是非线性的. 很难像 Newton 流体一样提出单一的本构模型,就能够清楚地展示出非 Newton 流体的全部特性. 通常,将非 Newton 流体划分为微分型流体、应变率型流体和积分型流体. 在这些流体当中,人们对应变率型流体尤感兴趣[1]. 应变率型流体需要考虑弹性和记忆的影响. 应变率型流体最简单的子类是 Maxwell 流体模型[2]. Maxwell 流体是更为普遍的流体,Newton 流体可以看成其一个特例. 典型的 Maxwell 流体特性包括极稀释的、极粘稠的,以及其它一般的.

精确解对于许多近似方法的检验,如数值法和试验法,十分重要. Fetecau 等[3] 得到了 Maxwell 流体流经无限平板时的精确解. Fetecau 等[4] 通过 Fourier 正弦变换,在一个无限伸展的平板上,内含无限震荡边缘的线性流动问题,确定了不可压缩 Maxwell 型流体流动的速度场. Nadeem 等[5] 采用 Laplace 变换,将边缘问题拓展到矩形管道的流动. Chen 等[6] 在一个震荡的圆形管道中,研究了 Maxwell 流体作非稳态单向流动时的速度分布和压力梯度,并将所得到结果应用于实际的梯形运动.

由于震荡的压力梯度作用,直输送管道中震荡的粘弹性直线流动分析有,Broer[7] 和 Thurs-

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ton[8-9]按线性 Maxwell 流体分析,Jones 和 Walters[10-11]及 Peev 等[12]按线性 Walter B 流体分析,Bhatnagar[13]按线性 Jeffrey(线性 Oldroyd)流体分析,Ramkissoon 等[14]按短记忆 Walter B 流体分析,Rahaman 和 Ramkissoon[15]按上部对流的 Maxwell 流体分析.进一步地,Walitza 等[16]按粘弹性流体在震荡管道中流动的分析,用来模拟人类的血液流动.

本文的目的是,为震荡的矩形输送管道中的 Maxwell 流体,提出新的精确解.通过双重有限的 Fourier 正弦变换和 Laplace 变换,确定速度场和相应剪切应力的解析表达式.为了问题的求解,需要一个附加的初始条件:在 $t = 0$ 时,速度关于时间的一阶导数为 $0^{[17-18]}$.对于 $\lambda \to 0$ 的特殊情形,所得到的全部结果简化为 Newton 流体的结果.

1   控 制 方 程

考虑不可压缩的 Maxwell 流体,在一个静止的、横截面为矩形的输送管道中,矩形的四边分别在 $x = 0$,$x = d$,$y = 0$ 和 $y = h$ 处.当 $t = 0^+$ 时刻,输送管道开始沿 z- 轴震荡,速度场为

$$\boldsymbol{v}(x,y,t) = w(x,y,t)\boldsymbol{k}, \tag{1}$$

速度场为[19]

$$\lambda \frac{\partial^2 w(x,y,t)}{\partial t^2} + \frac{\partial w(x,y,t)}{\partial t} = \nu \left[\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2} \right]. \tag{2}$$

考虑以下初始-边界条件:

$$w(x,y,0) = \frac{\partial w(x,y,0)}{\partial t} = 0, \tag{3a}$$

$$w(0,y,t) = w(d,y,t) = w(x,0,t) = w(x,h,t) = U_0 \cos(\omega t), \tag{3b}$$

或者

$$w(x,y,0) = \frac{\partial w(x,y,0)}{\partial t} = 0, \tag{4a}$$

$$w(0,y,t) = w(d,y,t) = w(x,0,t) = w(x,h,t) = U_0 \sin(\omega t). \tag{4b}$$

用 $u(x,y,t)$ 表示问题(1)、(3a)和(3b)的解,用 $v(x,y,t)$ 表示问题(1)、(4a)和(4b)的解.

引入函数

$$F(x,y,t) = u(x,y,t) + \mathrm{i}v(x,y,t), \tag{5}$$

得到如下问题:

$$\lambda \frac{\partial^2 F(x,y,t)}{\partial t^2} + \frac{\partial F(x,y,t)}{\partial t} = \nu \left[\frac{\partial^2 F(x,y,t)}{\partial x^2} + \frac{\partial^2 F(x,y,t)}{\partial y^2} \right], \tag{6}$$

$$\partial F(x,y,0) = \frac{\partial F(x,y,0)}{\partial t} = 0, \tag{7}$$

$$F(0,y,t) = F(d,y,t) = F(x,0,t) = F(x,h,t) = U_0 \mathrm{e}^{\mathrm{i}\omega t}. \tag{8}$$

问题(6) ~ (8)的解,将通过双重有限的 Fourier 正弦变换和 Laplace 变换得到.

记函数 $F(x,y,t)$ 的双重有限的 Fourier 正弦变换为

$$F_{mn}(t) = \int_0^d \int_0^h F(x,y,t) \sin\left(\frac{m\pi}{d} x\right) \sin\left(\frac{n\pi}{h} y\right) \mathrm{d}x\mathrm{d}y, \qquad m,n = 1,2,3,\cdots,$$

得到

$$I = \int_0^d \int_0^h \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) \sin(\alpha_m x) \sin(\beta_n y) \mathrm{d}x\mathrm{d}y =$$

$$- (\alpha_m^2 + \beta_n^2) F_{mn}(t) + \frac{[1-(-1)^m][1-(-1)^n]}{\alpha_m \beta_n}(\alpha_m^2 + \beta_n^2) U_0 \mathrm{e}^{\mathrm{i}\omega t}, \qquad (9)$$

其中 $\alpha_m = \dfrac{m\pi}{d}, \beta_n = \dfrac{n\pi}{h}.$

注意到边界条件(8),对方程(6)应用双重有限的 Fourier 正弦变换,并利用式(9),可以得到

$$\lambda \frac{\mathrm{d}^2 F_{mn}(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}F_{mn}(t)}{\mathrm{d}t} + a_{mn} F_{mn}(t) = \frac{[1-(-1)^m][1-(-1)^n]}{\alpha_m \beta_n} a_{mn} U_0 \mathrm{e}^{\mathrm{i}\omega t}, \quad (10)$$

其中

$$a_{mn} = \nu(\alpha_m^2 + \beta_n^2), \qquad m, n = 1, 2, 3, \cdots. \qquad (11)$$

双重有限的 Fourier 正弦变换 $F_{mn}(t)$ 必须满足初始条件:

$$F_{mn}(0) = 0, \ \frac{\mathrm{d}F_{mn}(0)}{\mathrm{d}t} = 0. \qquad (12)$$

对方程(10)应用 Laplace 变换,并利用式(12),可以得到

$$\bar{F}_{mn}(s) = \frac{[1-(-1)^m][1-(-1)^n]}{\alpha_m \beta_n} \frac{U_0 a_{mn}}{s - \mathrm{i}\omega} \frac{1}{\lambda s^2 + s + a_{mn}}, \qquad (13)$$

其中, $\bar{F}_{mn}(s) = \int_0^\infty F_{mn}(t) \mathrm{e}^{-st} \mathrm{d}t$ 为函数 $F_{mn}(t)$ 的 Laplace 变换, s 为 Laplace 变换的变量. 函数 $\bar{F}_{mn}(s)$ 改写为以下等式:

$$\bar{F}_{mn}(s) = \frac{[1-(-1)^m][1-(-1)^n]}{\alpha_m \beta_n} \frac{U_0}{s - \mathrm{i}\omega} - $$
$$\frac{U_0[1-(-1)^m][1-(-1)^n]}{\alpha_m \beta_n (s - \mathrm{i}\omega)} \frac{\lambda s^2 + s}{\lambda s^2 + s + a_{mn}}. \qquad (14)$$

取函数

$$\bar{H}_{mn}(s) = \frac{\lambda s^2 + s}{(s - \mathrm{i}\omega)(\lambda s^2 + s + a_{mn})} = \frac{A_{mn}}{s - \mathrm{i}\omega} + (1 - A_{mn}) \frac{\lambda s + (1 + \mathrm{i}\lambda\omega)}{\lambda^2 + s + a_{mn}} = $$
$$\frac{A_{mn}}{s - \mathrm{i}\omega} + (1 - A_{mn}) \frac{(s + 1/(2\lambda)) + (1/(2\lambda) + \mathrm{i}\omega)}{(s + 1/(2\lambda)^2) - (1 - 4\lambda a_{mn})/(4\lambda^2)} = $$
$$\frac{A_{mn}}{s - \mathrm{i}\omega} + (1 - A_{mn}) \left[\frac{s + 1/(2\lambda)}{(s + 1/(2\lambda))^2 - (b_{mn}/(2\lambda))^2} + \right.$$
$$\left. \frac{1 + 2\lambda\omega\mathrm{i}}{b_{mn}} \frac{b_{mn}/(2\lambda)}{(s + 1/(2\lambda))^2 - (b_{mn}/(2\lambda))^2} \right], \qquad (15)$$

其中

$$A_{mn} = \frac{\omega^2[1 - \lambda(a_{mn} - \lambda\omega^2)] + \mathrm{i}\omega a_{mn}}{(a_{mn} - \lambda\omega^2) + \omega^2}, \ b_{mn} = \sqrt{1 - 4\lambda a_{mn}}. \qquad (16)$$

函数 $\bar{H}_{mn}(s)$ 的 Laplace 逆变换为

$$H_{mn}(t) = A_{mn} \mathrm{e}^{\mathrm{i}\omega t} + (1 - A_{mn}) \mathrm{e}^{-t/(2\lambda)} \left[\cosh\left(\frac{b_{mn}}{2\lambda} t\right) + \frac{1 + 2\lambda\omega\mathrm{i}}{b_{mn}} \sinh\left(\frac{b_{mn}}{2\lambda} t\right) \right]. \quad (17)$$

简单计算表明,函数 $g(x, y, t) = 1$ 的双重有限的 Fourier 正弦变换为

$$G_{mn}(t) = \frac{[1-(-1)^m][1-(-1)^n]}{\alpha_m \beta_n}. \qquad (18)$$

对方程(14)应用 Laplace 逆变换和 Fourier 正弦逆变换,并利用式(17)和(18),可以得到

$$F(x,y,t) = U_0 \mathrm{e}^{\mathrm{i}\omega t} - \frac{4U_0}{dh} \sum_{m,n=1}^{\infty} \frac{[1-(-1)^m][1-(-1)^n]}{\alpha_m \beta_n} \sin(\alpha_m x)\sin(\beta_n y) \times$$

$$\left\{ A_{mn}\mathrm{e}^{\mathrm{i}\omega t} + (1 - A_{mn}\mathrm{e}^{-t/(2\lambda)}) \left[\cosh\left(\frac{b_{mn}}{2\lambda}t\right) + \frac{1+2\lambda\omega\mathrm{i}}{b_{mn}}\sinh\left(\frac{b_{mn}}{2\lambda}t\right) \right] \right\}, \quad (19)$$

或

$$F(x,y,t) = U_0 \mathrm{e}^{\mathrm{i}\omega t} - \frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p} \frac{\sin(\beta_q y)}{\beta_q} \left\{ A_{pq}\mathrm{e}^{\mathrm{i}\omega t} + (1-A_{pq})\mathrm{e}^{-t/(2\lambda)} \times \right.$$

$$\left. \left[\cosh\left(\frac{b_{pq}}{2\lambda}t\right) + \frac{1+2\lambda\omega\mathrm{i}}{b_{pq}}\sinh\left(\frac{b_{pq}}{2\lambda}t\right) \right] \right\}, \quad (20)$$

其中

$$\alpha_p = (2m+1)\frac{\pi}{d}, \ \beta_q = (2n+1)\frac{\pi}{h}, \ p = 2m+1, \ q = 2n+1.$$

当 $t \to \infty$,得到下面的稳态解:

$$F_{\mathrm{s}}(x,y,t) = U_0 \mathrm{e}^{\mathrm{i}\omega t} - \frac{16U_0 \mathrm{e}^{\mathrm{i}\omega t}}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p} \frac{\sin(\beta_q y)}{\beta_q} A_{pq}. \quad (21)$$

令 $d = 2a, h = 2b$,并改变坐标系的原点(取 $x = x^* + a, y = y^* + b$,并省去星号),复速度 $F(x,y,t)$ 可以写成

$$F(x,y,t) = U_0 \mathrm{e}^{\mathrm{i}\omega_0 t} - \frac{4U_0}{ab} \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\cos(\alpha_M x)}{\alpha_N} \frac{\cos(\beta_N y)}{\beta_N} \times$$

$$\left\{ A_{MN}\mathrm{e}^{\mathrm{i}\omega t} + (1-A_{MN})\mathrm{e}^{-t/(2\lambda)} \left[\cosh\left(\frac{b_{MN}}{2\lambda}t\right) + \frac{1+2\lambda\omega\mathrm{i}}{b_{MN}}\sinh\left(\frac{b_{MN}}{2\lambda}t\right) \right] \right\}, \quad (22)$$

其中　　$M = 2m+1, \ N = 2n+1, \ \alpha_M = \frac{(2m+1)\pi}{2a}, \ \beta_N = \frac{(2n+1)\pi}{2b}.$

记

$$\begin{cases} c_{mn} = \mathrm{Re}(A_{mn}) = \dfrac{\omega^2[1-\lambda(a_{mn}-\lambda\omega^2)]}{(a_{mn}-\lambda\omega^2)^2+\omega^2}, \\ d_{mn} = \mathrm{Im}(A_{mn}) = \dfrac{\omega a_{mn}}{(a_{mn}-\lambda\omega^2)^2+\omega^2}. \end{cases} \quad (23)$$

根据方程(20),分别得到输送管道余弦震荡时的速度场:

$$u(x,y,t) = U_0 \cos(\omega t) - \frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p} \frac{\sin(\beta_q y)}{\beta_q} \times$$

$$\left\{ c_{pq}\cos(\omega t) - d_{pq}\sin(\omega t) + \mathrm{e}^{-t/(2\lambda)} \left[(1-c_{pq})\cosh\left(\frac{b_{pq}}{2\lambda}t\right) + \right.\right.$$

$$\left.\left. \frac{1-c_{pq}+2\lambda\omega d_{pq}}{b_{pq}}\sinh\left(\frac{b_{pq}}{2\lambda}t\right) \right] \right\} \quad (24)$$

和正弦震荡时的速度场:

$$v(x,y,t) = U_0 \sin(\omega t) - \frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p} \frac{\sin(\beta_q y)}{\beta_q} \times$$

$$\left\{ c_{pq}\sin(\omega t) + d_{pq}\cos(\omega t) - \mathrm{e}^{-t/(2\lambda)} \left[d_{pq}\cosh\left(\frac{b_{pq}}{2\lambda}t\right) + \right.\right.$$

$$\frac{d_{pq} - 2\lambda\omega(1 - c_{pq})}{b_{pq}} \sinh\left(\frac{b_{pq}}{2\lambda}t\right)\Bigg]\Bigg\}\,. \tag{25}$$

利用方程(22),可以将方程(24)和(25)改写为下面形式:

$$u(x,y,t) = U_0\cos(\omega t) - \frac{4U_0}{ab}\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{\cos(\alpha_M x)}{\alpha_M}\frac{\cos(\beta_N y)}{\beta_N}\times$$

$$\left\{c_{MN}\cos(\omega t) - d_{MN}\sin(\omega t) + e^{-t/(2\lambda)}\left[(1 - c_{MN})\cosh\left(\frac{b_{MN}}{2\lambda}t\right) + \right.\right.$$

$$\left.\left.\frac{1 - c_{MN} + 2\lambda\omega d_{MN}}{b_{MN}}\sinh\left(\frac{b_{MN}}{2\lambda}t\right)\right]\right\}, \qquad x\in[-a,a],\ y\in[-b,b], \tag{26}$$

$$v(x,y,t) = U_0\sin(\omega t) - \frac{4U_0}{ab}\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{\cos(\alpha_M x)}{\alpha_M}\frac{\cos(\beta_N y)}{\beta_N}\times$$

$$\left\{c_{MN}\sin(\omega t) + d_{MN}\cos(\omega t) - e^{-t/(2\lambda)}\left[d_{MN}\cosh\left(\frac{b_{MN}}{2\lambda}t\right) + \right.\right.$$

$$\left.\left.\frac{d_{MN} - 2\lambda\omega(1 - c_{MN})}{b_{MN}}\sinh\left(\frac{b_{MN}}{2\lambda}t\right)\right]\right\}\,. \tag{27}$$

根据方程(24)和(25),分别得到余弦震荡和正弦震荡时的稳态解:

$$u_s(x,y,t) = U_0 D(x,y)\cos(\omega t) + U_0 E(x,y)\sin(\omega t)\,, \tag{28}$$

$$v_s(x,y,t) = U_0 D(x,y)\sin(\omega t) + U_0 E(x,y)\cos(\omega t)\,, \tag{29}$$

其中

$$D(x,y) = 1 - \frac{16}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\alpha_p x)}{\alpha_p}\frac{\sin(\beta_q y)}{\beta_q}c_{pq}\,, \tag{30}$$

$$E(x,y) = \frac{16}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\alpha_p x)}{\alpha_p}\frac{\sin(\beta_q y)}{\beta_q}d_{pq}\,. \tag{31}$$

方程(28)和(29)可以写成下面形式:

$$u_s(x,y,t) = U_0\sqrt[n]{D^2(x,y) + E^2(x,y)}\cos(\omega t - \phi)\,, \tag{32}$$

$$v_s(x,y,t) = U_0\sqrt[n]{D^2(x,y) + E^2(x,y)}\sin(\omega t - \phi)\,, \tag{33}$$

其中

$$\phi = \arctan\frac{E(x,y)}{D(x,y)}\,. \tag{34}$$

2   极 限 情 形

2.1   Newton 流体

当 $\lambda\to 0$,可以得到相应于 Newton 流体完成相同运动时的速度场.

在方程(23)中令 $\lambda\to 0$,有

$$c_{pq} = \frac{\omega^2}{\omega^2 + a_{pq}^2}\,,\ d_{pq} = \frac{\omega a_{pq}}{\omega^2 + a_{pq}^2}$$

以及

$$\lim_{\lambda\to 0}e^{-t/(2\lambda)}\cosh\left(\frac{b_{pq}}{2\lambda}t\right) =$$

$$\lim_{\lambda\to 0}\frac{1}{2}e^{-t/(2\lambda)}\left(e^{t\sqrt{1-4\lambda a_{pq}}/(2\lambda)} + e^{-t\sqrt{1-4\lambda a_{pq}}/(2\lambda)}\right) =$$

$$\frac{1}{2} \lim_{\lambda \to 0} \mathrm{e}^{t(\sqrt{1-4\lambda a_{pq}}-1)/(2\lambda)} + \frac{1}{2} \lim_{\lambda \to 0} \mathrm{e}^{-t(\sqrt{1-4\lambda a_{pq}}+1)/(2\lambda)} =$$

$$\frac{1}{2} \lim_{\lambda \to 0} \mathrm{e}^{(1-4\lambda a_{pq}-1)t/(2\lambda(\sqrt{1-4\lambda a_{pq}}+1))} =$$

$$\frac{1}{2} \lim_{\lambda \to 0} \mathrm{e}^{-2a_{pq}t/(\sqrt{1-4\lambda a_{pq}}+1)} = \frac{1}{2} \mathrm{e}^{-a_{pq}t} .$$

相似地

$$\lim_{\lambda \to 0} \mathrm{e}^{-t/(2\lambda)} \sinh\left(\frac{b_{pq}}{2\lambda} t\right) = \frac{1}{2} \mathrm{e}^{-a_{pq}t} .$$

将上述结果代入方程(24)和(25),得到

$$u(x,y,t) = U_0\cos(\omega t) - \frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p} \frac{\sin(\beta_q y)}{\beta_q} \times$$

$$\left\{ \frac{\omega^2\cos(\omega t) - \omega a_{pq}\sin(\omega t)}{\omega^2 + a_{pq}^2} + \frac{a_{pq}^2}{\omega^2 + a_{pq}^2} \mathrm{e}^{-a_{pq}t} \right\} , \tag{35}$$

$$v(x,y,t) = U_0\sin(\omega t) - \frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p} \frac{\sin(\beta_q y)}{\beta_q} \times$$

$$\left\{ \frac{\omega^2\sin(\omega t) + \omega a_{pq}\cos(\omega t)}{\omega^2 + a_{pq}^2} - \frac{\omega a_{pq}}{\omega^2 + a_{pq}^2} \mathrm{e}^{-a_{pq}t} \right\} , \tag{36}$$

这便是 Newton 流体的速度场.

2.2　矩形管道的 Rayleigh-Stokes 问题

2.2.1　Maxwell 流体

此时问题(2)、(3a)和(3b)中的 $\omega = 0$.

将 $\omega = 0$ 代入方程(24),得到 Maxwell 流体流过矩形管道时,Rayleigh-Stokes 问题的速度场:

$$u(x,y,t) = U_0 - \frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p x} \frac{\sin(\beta_q y)}{\beta_q} \times$$

$$\left\{ - \mathrm{e}^{-t/(2\lambda)} \left[\cosh\left(\frac{b_{pq}}{2\lambda} t\right) + \frac{1}{b_{pq}} \sinh\left(\frac{b_{pq}}{2\lambda} t\right) \right] \right\} . \tag{37}$$

2.2.2　Newton 流体

将 $\omega = 0$ 代入方程(35),得到

$$u(x,y,t) = U_0 - \frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p} \frac{\sin(\beta_q y)}{\beta_q} \left\{ \mathrm{e}^{-a_{pq}t} \right\} , \tag{38}$$

这便是 Newton 流体流过矩形管道时,Rayleigh-Stokes 问题的速度场.

3　外部应力张量 $\boldsymbol{\tau}$ 的计算

在所考虑的问题中,有 $S_{xx} = S_{xy} = S_{yy} = 0$, 且

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\tau_1(x,y,t) = \mu \frac{\partial\omega(x,y,t)}{\partial x}, \ \tau_1 = S_{xz}, \tag{39}$$

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\tau_2(x,y,t) = \mu \frac{\partial\omega(x,y,t)}{\partial y}, \ \tau_2 = S_{yz}, \tag{40}$$

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\sigma(x,y,t) = 2\lambda \left[\tau_1 \frac{\partial\omega}{\partial x} + \tau_2 \frac{\partial\omega}{\partial y}\right], \ \sigma = S_{zz}. \tag{41}$$

记 $\tau_{1c}(x,y,t)$, $\tau_{2c}(x,y,t)$ 表示输送管道余弦震荡时的切向张量, $\tau_{1s}(x,y,t)$, $\tau_{2s}(x,y,t)$ 表示输送管道正弦震荡时的切向张量.

如果对上述方程引入

$$T_1(x,y,t) = \tau_{1c}(x,y,t) + i\tau_{1s}(x,y,t), \tag{42}$$

$$T_2(x,y,t) = \tau_{2c}(x,y,t) + i\tau_{2s}(x,y,t), \tag{43}$$

意味着

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) T_1(x,y,t) = \mu \frac{\partial F(x,y,t)}{\partial x}, \tag{44}$$

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) T_2(x,y,t) = \mu \frac{\partial F(x,y,t)}{\partial y}. \tag{45}$$

对方程(44)和(45)应用 Laplace 变换,得到

$$\bar{T}_1(x,y,s) = \frac{\mu}{1 + \lambda s} \frac{\partial \bar{F}(x,y,s)}{\partial x}, \tag{46}$$

$$\bar{T}_2(x,y,s) = \frac{\mu}{1 + \lambda s} \frac{\partial \bar{F}(x,y,s)}{\partial y}. \tag{47}$$

对方程(14)应用 Fourier 变换,得到

$$\bar{F}(x,y,s) = \frac{U_0}{s - i\omega} -$$

$$\frac{4U_0}{dh} \sum_{m,n=1}^{\infty} \frac{[1 - (-1)^m][1 - (-1)^n]}{\alpha_m \beta_n} \frac{1}{s - i\omega} \frac{s(1 + \lambda s)}{\lambda s^2 + s + a_{mn}} \sin(\alpha_m x)(\beta_n y), \tag{48}$$

或

$$\bar{F}(x,y,s) = \frac{U_0}{s - i\omega} - \frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p} \frac{\sin(\beta_q y)}{\beta_q} \frac{s(1 + \lambda s)}{(s - i\omega)(\lambda s^2 + s + a_{mn})}, \tag{49}$$

其中, $p = 2m + 1$, $q = 2n + 1$.

另外

$$\frac{\partial \bar{F}}{\partial x}(x,y,s) = -\frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \cos(\alpha_p x) \frac{\sin(\beta_q y)}{\beta_q} \frac{s(1 + \lambda s)}{(s - i\omega)(\lambda s^2 + s + a_{mn})}, \tag{50}$$

$$\frac{\partial \bar{F}}{\partial y}(x,y,s) = -\frac{16U_0}{dh} \sum_{m,n=0}^{\infty} \frac{\sin(\alpha_p x)}{\alpha_p} \cos(\beta_q y) \frac{s(1 + \lambda s)}{(s - i\omega)(\lambda s^2 + s + a_{pq})}. \tag{51}$$

将方程(50)和(51)代入方程(46)和(47),得到

$$\bar{T}_1(x,y,s) = -\frac{16\mu U_0}{dh} \sum_{m,n=0}^{\infty} \cos(\alpha_p x) \frac{\sin(\beta_q y)}{\beta_q} \frac{s}{(s - i\omega)(\lambda s^2 + s + a_{pq})}, \tag{52}$$

$$\bar{T}_2(x,y,s) = -\frac{16\mu U_0}{dh} \sum_{m,n=0}^{\infty} \cos(\beta_q y) \frac{\sin(\alpha_p x)}{\beta_p} \frac{s}{(s - i\omega)(\lambda s^2 + s + a_{pq})}. \tag{53}$$

取

$$\bar{G}_{pq}(s) = \frac{s}{(s - i\omega)(\lambda s^2 + s + a_{pq})} =$$

$$\frac{B}{s - i\omega} - B \frac{\lambda s + i a_{pq}/\omega}{\lambda s^2 + s + a_{pq}} =$$

$$\frac{B}{s-\mathrm{i}\omega} - B\frac{(s+1/(2\lambda)) + (\mathrm{i}a_{pq}/(\lambda\omega) - 1/(2\lambda))}{(s+1/(2\lambda))^2 - (1-4\lambda a_{pq})/(4\lambda^2)} =$$

$$\frac{B}{s-\mathrm{i}\omega} - B\frac{s+1/(2\lambda)}{(s+1/(2\lambda))^2 - (b_{pq}/(2\lambda))^2} +$$

$$B\frac{\omega - 2\mathrm{i}a_{pq}}{\omega b_{pq}}\frac{b_{pq}/(2\lambda)}{(s+1/(2\lambda))^2 - (b_{pq}/(2\lambda))^2}.$$

对上述方程应用 Laplace 逆变换，可以得到

$$G_{pq}(t) = Be^{-\mathrm{i}\omega t} - Be^{-t/(2\lambda)}\left[\cosh\left(\frac{b_{pq}}{2\lambda}t\right) - \frac{\omega - 2\mathrm{i}a_{pq}}{\omega b_{pq}}\sinh\left(\frac{b_{pq}}{2\lambda}t\right)\right], \tag{54}$$

其中

$$B = \frac{\omega^2 + \mathrm{i}\omega(a_{pq} - \lambda\omega^2)}{\omega^2 + (a_{pq} - \lambda\omega^2)^2}. \tag{55}$$

对方程(52)和(53)应用 Laplace 逆变换，并利用式(54)，可以得到

$$\tau_1(x,y,t) = -\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\alpha_p x)\frac{\sin(\beta_q y)}{\beta_q} \times$$

$$\left\{Be^{\mathrm{i}\omega t} - Be^{-t/(2\lambda)}\left[\cosh\left(\frac{b_{pq}}{2\lambda}t\right) - \frac{\omega - 2\mathrm{i}a_{pq}}{\omega b_{pq}}\sinh\left(\frac{b_{pq}}{2\lambda}t\right)\right]\right\}, \tag{56}$$

$$\tau_1(x,y,t) = -\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\beta_q x)\frac{\sin(\alpha_p x)}{\alpha_p} \times$$

$$\left\{Be^{\mathrm{i}\omega t} - Be^{-t/(2\lambda)}\left[\cosh\left(\frac{b_{pq}}{2\lambda}t\right) - \frac{\omega - 2\mathrm{i}a_{pq}}{\omega b_{pq}}\sinh\left(\frac{b_{pq}}{2\lambda}t\right)\right]\right\}. \tag{57}$$

定义

$$e_{pq} = \mathrm{Re}(B) = \frac{\omega^2}{\omega^2 + (a_{pq} - \lambda\omega^2)^2}, f_{pq} = \mathrm{Im}(B) = \frac{\omega(a_{pq} - \lambda\omega^2)}{\omega^2 + (a_{pq} - \lambda\omega^2)^2}. \tag{58}$$

由方程(56)和(57)，分别得到 Maxwell 流体相应于余弦震荡时的剪切应力

$$\tau_{1c}(x,y,t) = \mathrm{Re}[T_1(x,y,t)] =$$

$$-\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\alpha_p x)\frac{\sin(\beta_q y)}{\beta_q}\left\{e_{pq}\cos(\omega t) - f_{pq}\sin(\omega t) +\right.$$

$$\left.e^{-t/(2\lambda)}\left[\frac{\omega e_{pq} + 2a_{pq}f_{pq}}{\omega b_{pq}}\sinh\left(\frac{b_{pq}}{2\lambda}t\right) - e_{pq}\cosh\left(\frac{b_{pq}}{2\lambda}t\right)\right]\right\}, \tag{59}$$

$$\tau_{2c}(x,y,t) = \mathrm{Re}[T_2(x,y,t)] =$$

$$-\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\beta_q y)\frac{\sin(\alpha_p x)}{\alpha_p}\left\{e_{pq}\cos(\omega t) - f_{pq}\sin(\omega t) +\right.$$

$$\left.e^{-t/(2\lambda)}\left[\frac{\omega e_{pq} + 2a_{pq}f_{pq}}{\omega b_{pq}}\sinh\left(\frac{b_{pq}}{2\lambda}t\right) - e_{pq}\cosh\left(\frac{b_{pq}}{2\lambda}t\right)\right]\right\} \tag{60}$$

和正弦震荡时的剪切应力

$$\tau_{1s}(x,y,t) = \mathrm{Im}[T_1(x,y,t)] =$$

$$-\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\alpha_p x)\frac{\sin(\beta_q y)}{\beta_q}\left\{f_{pq}\cos(\omega t) + e_{pq}\sin(\omega t) +\right.$$

$$\left.e^{-t/(2\lambda)}\left[\frac{\omega f_{pq} - 2a_{pq}e_{pq}}{\omega b_{pq}}\sinh\left(\frac{b_{pq}}{2\lambda}t\right) - f_{pq}\cosh\left(\frac{b_{pq}}{2\lambda}t\right)\right]\right\}, \tag{61}$$

$$\tau_{2s}(x,y,t) = \text{Im}\big[\,T_2(x,y,t)\,\big] =$$

$$-\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\beta_q y)\,\frac{\sin(\alpha_p x)}{\alpha_p}\Big\{f_{pq}\cos(\omega t)+e_{pq}\sin(\omega t)+$$

$$\mathrm{e}^{-t/(2\lambda)}\Big[\frac{\omega e_{pq}-2a_{pq}e_{pq}}{\omega b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)-f_{pq}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big]\Big\}. \tag{62}$$

将 $\lambda\to0$ 代入方程(59)~(62),分别得到 Newton 流体相应于余弦震荡的剪切应力

$$\tau_{1c}(x,y,t) = -\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\alpha_p x)\,\frac{\sin(\beta_q y)}{\beta_q}\times$$

$$\Big\{\frac{\omega^2\cos(\omega t)-\omega a_{pq}\sin(\omega t)}{\omega^2+a_{pq}^2}+\frac{a_{pq}^2}{\omega^2+a_{pq}^2}\,\mathrm{e}^{-a_{pq}t}\Big\}, \tag{63}$$

$$\tau_{2c}(x,y,t) = -\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\beta_q y)\,\frac{\sin(\alpha_p x)}{\alpha_p}\times$$

$$\Big\{\frac{\omega^2\cos(\omega t)-\omega a_{pq}\sin(\omega t)}{\omega^2+a_{pq}^2}+\frac{a_{pq}^2}{\omega^2+a_{pq}^2}\,\mathrm{e}^{-a_{pq}t}\Big\} \tag{64}$$

和正弦震荡时的剪切应力

$$\tau_{1s}(x,y,t) = -\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\alpha_p x)\,\frac{\sin(\beta_q y)}{\beta_q}\times$$

$$\Big\{\frac{\omega a_{pq}\cos(\omega t)+\omega^2\sin(\omega t)}{\omega^2+a_{pq}^2}-\frac{\omega a_{pq}}{\omega^2+a_{pq}^2}\,\mathrm{e}^{-a_{pq}t}\Big\}, \tag{65}$$

$$\tau_{1s}(x,y,t) = -\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\beta_q y)\,\frac{\sin(\alpha_p x)}{\alpha_p}\times$$

$$\Big\{\frac{\omega a_{pq}\cos(\omega t)+\omega^2\sin(\omega t)}{\omega^2+a_{pq}^2}-\frac{\omega a_{pq}}{\omega^2+a_{pq}^2}\,\mathrm{e}^{-a_{pq}t}\Big\}. \tag{66}$$

4　结　　论

　　本文的目的是在一个振荡的矩形输送管道中,作非稳定的正弦和余弦振荡时,建立起 Maxwell 流体速度场及其相关剪切应力的初始解. 以双重有限的 Fourier 正弦变换和 Laplace 变换数学工具求解该问题. 所得解呈现为稳态解和暂态解之和. Newton 流体作同样运动时的解, 是本文所得解的极限情况. 余弦振荡解中频率 ω 不存在时,相应地简化为矩形输送管道中 Maxwell 和 Newton 流体的 Rayleigh-Stokes 问题解. 而且,所得到的解满足控制方程及其相应的初始-边界条件.
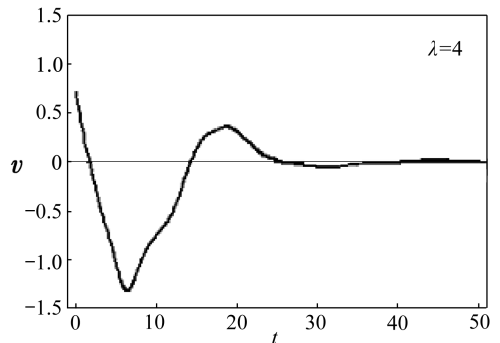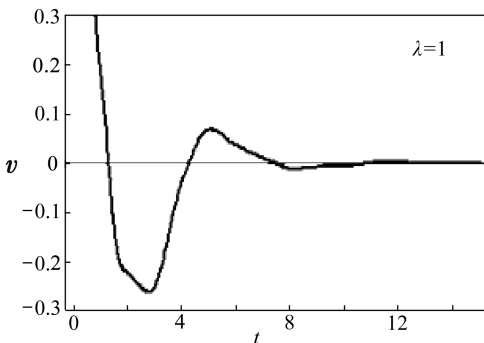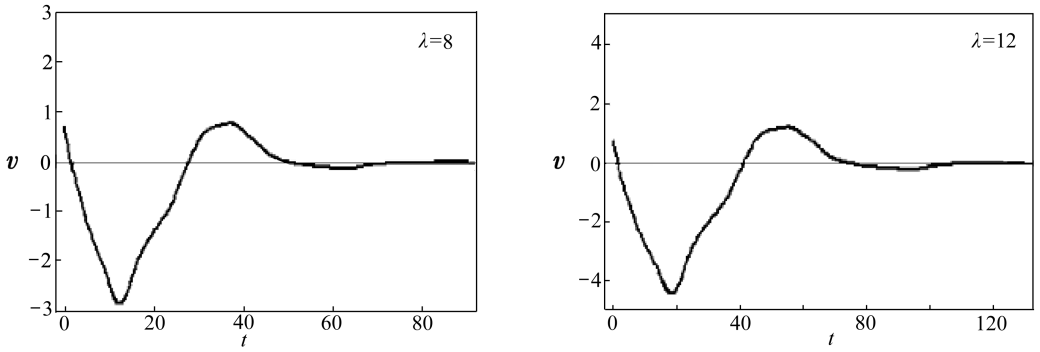
图1　松弛时间 λ 取不同数值时,矩形输送管道中 Maxwell 流体作正弦振荡
　　（即方程(25)中的瞬时部分）时的瞬时速度分布曲线.其他参数取值为
　　$U_0 = 1,\ d = 1,\ h = 2,\ \omega = 0.5,\ x = 0.5,\ \nu = 0.1$ 和 $y = 1$

Fig. 1　Transient velocity profiles of sine oscillation（i. e. , transient part of Eq. （25）） of Maxwell
　　fluid in rectangular duct for different values of relaxation time λ . Other parameters and values
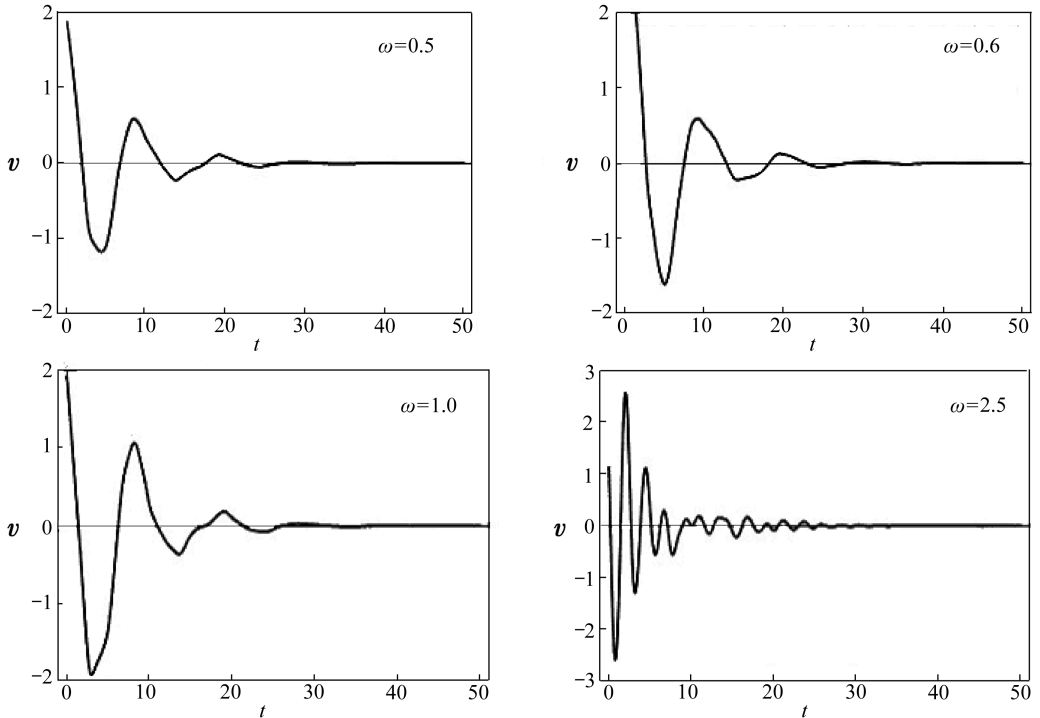　　are taken as $U_0 = 1,\ d = 1,\ h = 2,\ \omega = 0.5,\ x = 0.5,\ \nu = 0.1$ and $y = 1$

图2　角频率 ω 取不同数值时,矩形输送管道中 Maxwell 流体作正弦振荡
　　（即方程(25)中的瞬态部分）时的瞬时速度分布曲线.其他参数取值为
　　$U_0 = 1,\ d = 1,\ h = 2,\ x = 0.5,\ y = 1$ 和 $\lambda = 3$

Fig. 2　Transient velocity profiles of sine oscillation （i. e. , transient part of Eq. （25）） of Maxwell
　　fluid in rectangular duct for different values of angular frequency ω . Other parameters and values
　　are taken as $U_0 = 1,\ d = 1,\ h = 2,\ x = 0.5,\ y = 1$ and $\lambda = 3$

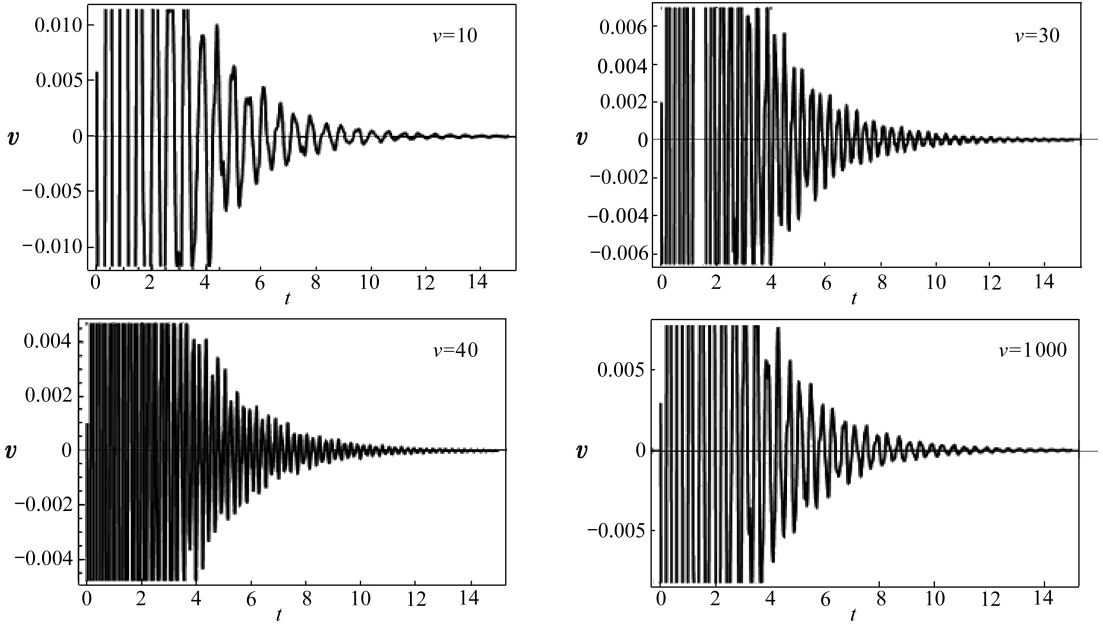　　为了检验不同材料参数对时间的依赖关系,以图形形式给出振荡的稳态区域.图1、图2和图3就材料常数 λ , ω 和 ν 分别取不同数值时,对正弦振荡边界条件绘出瞬时(方程(25)的

图 3　ν 取不同数值时，矩形输送管道中 Maxwell 流体作正弦振荡
（即方程（25）中的瞬态部分）时的瞬时速度分布曲线．其他参数取值为
$U_0 = 1,\ d = 1,\ h = 2,\ \omega = 0.5,\ x = 0.5,\ y = 1$ 和 $\lambda = 1$

Fig. 3　Transient velocity profiles of sine oscillation（i. e.，transient part of Eq.（25））of Maxwell
fluid in rectangular duct for different values of ν. Other parameters and values
are taken as $U_0 = 1,\ d = 1,\ h = 2,\ \omega = 0.5,\ x = 0.5,\ y = 1$ and $\lambda = 1$

瞬态部分）的衰减图形．从图 1 可以清楚地看到速度振荡的振幅，及其达到稳定状态所需要的时间，且松弛时间随着 λ 的增大而增大．图 2 表明达到稳定状态所需要的时间，也随着角频率 ω 的增大而增大．图 3 显示达到稳定状态所需要的时间，并不随着动粘性系数 ν 的改变而出现明显变化．最后，图 4 和图 5 就 x 和 y 取不同数值时，分别显示速度分布曲线的三维影响图．图 4 就 $0 < y < 2$ 时，速度分布曲线振荡的振幅，随着 x 的增大而增大；但是在端点 $y = 0$ 和 $y = 2$ 处，振幅保持不变；当 x 增大，$0 \leqslant y \leqslant 2$，其他所有参数保持不变时，振荡周期也保持不变．图 5 可以发现，y 变化带来的影响，与图 4 中 x 变化所带来的影响相类似．
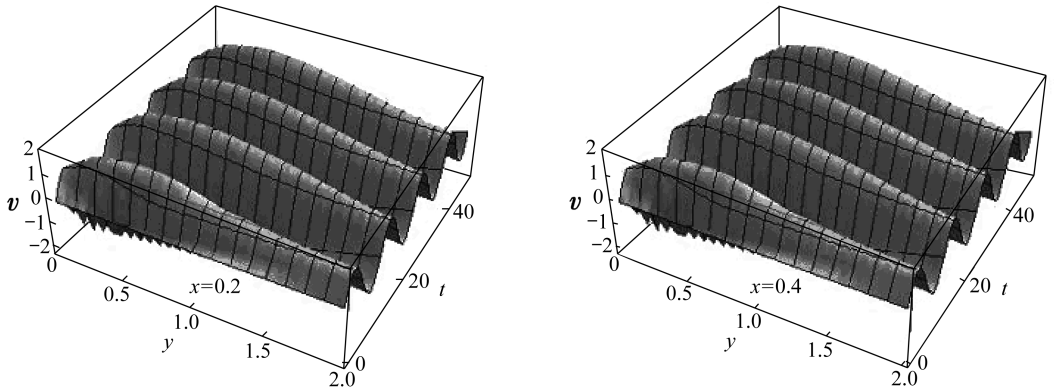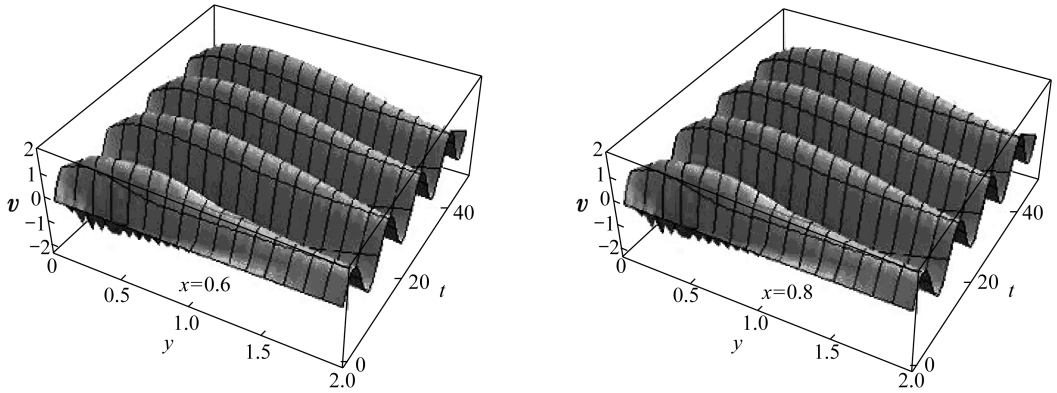
图4　x 取不同数值时,矩形输送管道中 Maxwell 流体作正弦振荡(方程(25))时的三维速度分布曲线.
其他参数取值为 $U_0 = 1$, $d = 1$, $h = 2$, $\omega = 0.5$, $\nu = 0.1$ 和 $\lambda = 3$

Fig. 4　Three dimensional velocity profiles of sine oscillation (Eq. (25)) of Maxwell
fluid in rectangular duct for different values of x. Other parameters and values
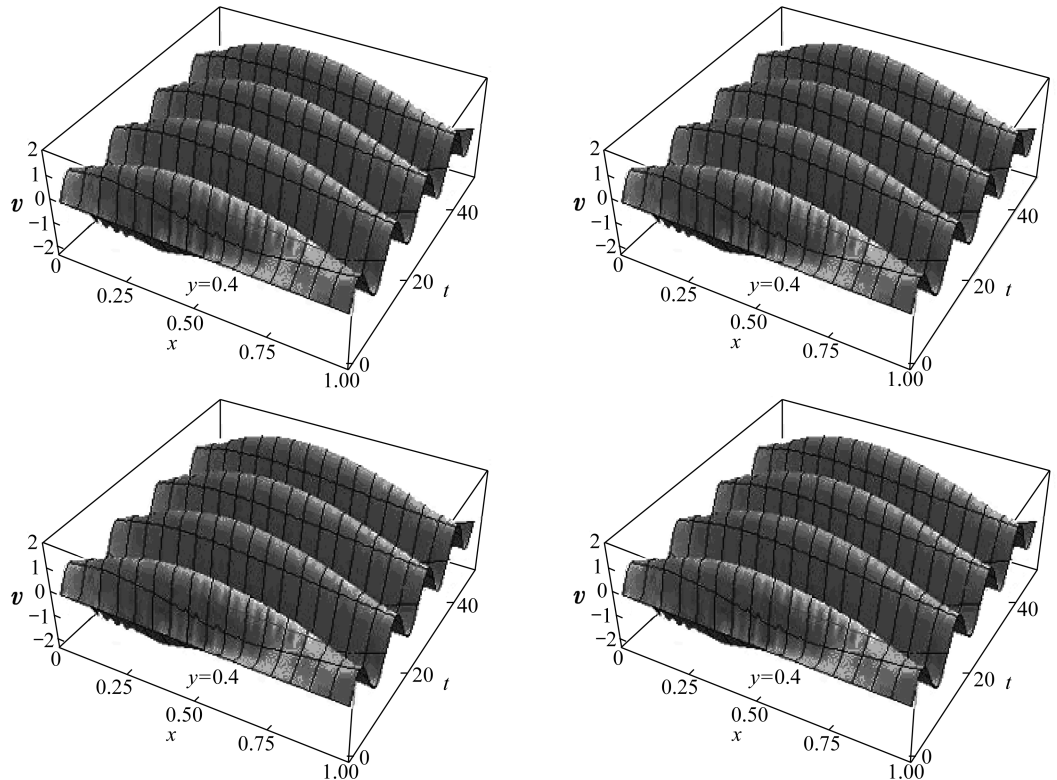are taken as $U_0 = 1$, $d = 1$, $h = 2$, $\omega = 0.5$, $\nu = 0.1$ and $\lambda = 3$

图5　y 取不同数值时,矩形输送管道中 Maxwell 流体作正弦振荡(方程(25))时的三维速度分布曲线.
其他参数取值为 $U_0 = 1$, $d = 1$, $h = 2$, $\omega = 0.5$, $\nu = 0.1$ 和 $\lambda = 3$

Fig. 5　Three dimensional velocity profiles of sine oscillation (Eq. (25)) of Maxwell
fluid in rectangular duct for different values of y. Other parameters and values are
taken as $U_0 = 1$, $d = 1$, $h = 2$, $\omega = 0.5$, $\nu = 0.1$ and $\lambda = 3$

贵意见.

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Flow on Oscillating Rectangular Duct for Maxwell Fluid

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Abstract: An analysis for the unsteady flow of an incompressible Maxwell fluid in an oscillating rectangular cross section was presented. Using the Fourier and Laplace transforms as mathematical tool, the solutions were presented as sum of steady-state and transient solutions. For large times, when the transients disappear, the solution was represented by the steady-state solution. Solutions for Newtonian fluids appear as limiting cases of the solutions obtained here. In the absence of frequency of oscillation, the problem for flow of Maxwell fluid in a duct of rectangular cross-section moving parallel to its length was obtained. Finally, the required time to reach the steady-state for sine oscillations of the rectangular duct is obtained by graphical illustrations for different parameters. Moreover, the graphs are sketched for velocity for the variations of x and y.

Key words: Maxwell fluid; oscillating rectangular duct; velocity field