文章编号:1000-0887(2012)06-0710-16

热交换多孔圆盘间三阶流体的 MHD 轴对称流动*

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摘要: 在两个具有热交换可渗透的多孔圆盘之间,研究三阶流体的磁流体动力学(MHD)流动. 通过适当变换,将偏微分的控制方程转换为常微分方程. 采用同伦分析法(HAM)求解转换后的方程. 定义了均方残余误差的表达式,并选择了最佳的、收敛的控制参数值. 检测了无量纲参数变化时的无量纲速度和温度场. 列表显示表面摩擦因数和 Nusselt 数,并分析了无量纲参数的影响.

关 键 词: 热交换; 轴对称流动; 三阶流体; 多孔圆盘; 表面摩擦因数; Nusselt 数
中图分类号: O357.1; O361.3 **文献标志码:** A
DOI: 10.3879/j.issn.1000-0887.2012.06.006

引 言

由于非 Newton 流体在工业和工程界各式各样的应用,近年来成为众多研究者十分感兴趣的课题. 与粘性流体不一样,自然界中的非 Newton 流体具有各种不同的特性,不能采用单一的应力-应变率本构关系来描述. 一般而言,相对于粘性流体,非 Newton 流体中的数学问题更复杂,非线性程度和方程的阶次也更高. 尽管其如此复杂,科学家和工程师们还是致力于非 Newton 流体动力学研究. 例如,Fetecau 等[1]就剪切应力依赖时间变化作用下的圆柱体,推导出粘弹性流体的精确解. Jamil 等[2-3]描述了 Oldroyd-B 流体非稳定的螺旋状流动问题,并计算了二阶流体问题. Tan 和 Masuoka[4]讨论了多孔介质中 Maxwell 流体的稳定性. 接着,Tan 和 Masuoka[5]又推导出 Oldroyd-B 流体 Stokes 第一问题的精确解. 三阶流体是一种微分型流体,有能力展示薄/厚剪切层的影响. 许多学者研究过三阶流体的流动问题,如 Sajid 和 Hayat[6]讨论了三阶流体流过拉伸薄膜时的二维边界层流动问题. Sajid 等[7]考虑过三阶导电流体中的热交换特点. Hayat 等[8]检验了三阶流体非稳定流动中传热传质的共同影响. Abbasbandy 和 Hayat[9]计算了三阶流体非稳定边界层流动的级数解. Sahoo[10]计算了三阶流体 Heimenz 流动时热交换的数值解. Sahoo 和 Do[11]在三阶流体流动中研究了表面拉伸引起滑移的影响.

在圆盘上的流动,或者在圆盘之间的流动,是广受研究者欢迎的课题,因为它们在工程界的广泛应用. 圆盘型物体十分受欢迎,如旋转的热交换器、生物流体生产中的旋转圆盘反应堆、

* **收稿日期:** 2011-06-02; **修订日期:** 2012-02-08
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本文原文为英文,黄锋 译,张禄坤 校.

燃气涡轮机或者船用气轮机以及汽车工业．自从 von Kármán[12] 的开创性工作（他考虑过无限圆盘上水动力学的流动）以来，由于兴趣所在，一些研究者从不同的角度分析了这种流动．Cochran[13] 提出了无限圆盘上稳定的水动力学流动的渐近解．Benton[14] 将 Cochran 的工作拓展到，圆盘突然启动所控制流动的初值问题．Takhar 等[15] 研究了磁场对流过旋转垂直圆锥体的、非稳定混合对流的影响．Maleque 等[16] 研究了粘性流体特性可变时完全发育的层状流动．Stuart[17] 检验了旋转圆盘均匀吸入对稳定流动的影响．Sparrow 等[18] 考虑了流过多孔圆盘的流动问题．Miklavcic 和 Wang[19] 研究过粘性流体在粗糙旋转圆盘上的流动．最近，Hayat 和 Nawaz[20] 讨论了旋转圆盘上非稳定的驻点流动．

迄今为止，有关此课题的现有文献检索表明，三阶流体在两个热交换多孔圆盘间对称流动的研究还没有．本文的主要兴趣在于，考虑 Joule 传热和粘性耗散对三阶流体热交换的影响．通过应用同伦分析法——一个新颖的并被众多学者所采用，求解呈现的非线性问题．用同伦分析法求解非线性问题的研究还不多，在文献［21-30］中有所提及．

1　公式化表示

考虑导电的三阶流体在位于 $z=\pm H$ 处的多孔圆盘之间，作轴对称流动及其热交换问题，圆盘并不导电但有渗透性，可以使流体以速度 V_0 被吸入或者喷出．流体的吸入/喷出现象导致流体的流动．而且，$V_0>0$ 时流体被吸入，但 $V_0<0$ 导致流体的喷出．一个恒定磁场 $\boldsymbol{B}_0$ 沿 z- 方向作用（参见图 1 所示的物理模型及其坐标系），没有外部电场作用．假定磁 Reynolds 数不大，感应磁场可以忽略不计；两圆盘保持恒温 T_{w}，考虑了 Joule 传热和粘性耗散；假定所有的材料特性参数恒定．流动的控制方程为

$$\nabla\cdot\boldsymbol{V}=0, \tag{1}$$

$$\rho\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}=\operatorname{div}\boldsymbol{\tau}+\boldsymbol{J}\times\boldsymbol{B}_0, \tag{2}$$

$$\rho c_p\frac{\mathrm{d}T}{\mathrm{d}t}=K\nabla^2T+\operatorname{tr}(\boldsymbol{\tau}\boldsymbol{L})+\frac{1}{\sigma}\boldsymbol{J}\cdot\boldsymbol{J}, \tag{3}$$

$$\boldsymbol{J}=\sigma(\boldsymbol{V}\times\boldsymbol{B}), \tag{4}$$

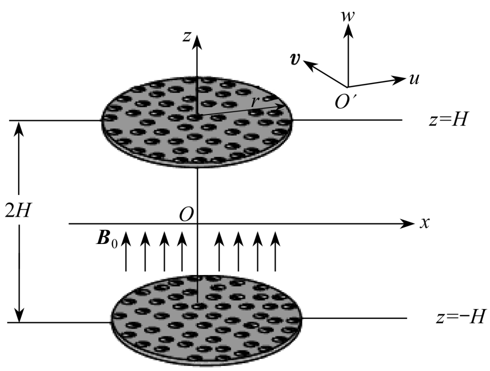

图 1　物理模型及其坐标系

Fig. 1　Schematic diagram and coordinate system

其中，$\boldsymbol{V}$ 为速度场，ρ 为流体密度，p 为压力，$\boldsymbol{J}$ 为电流密度，σ 为流体的电导率，K 为热导率，T 为温度，c_p 为比定压热容，$\mathrm{d}/\mathrm{d}t$ 为随体导数，$\boldsymbol{\tau}$ 为三阶流体的 Cauchy 应力张量：

$$\boldsymbol{\tau}=-p\boldsymbol{I}+\mu\boldsymbol{A}_1+\alpha_1\boldsymbol{A}_2+\alpha_2\boldsymbol{A}_1^2+\beta_1\boldsymbol{A}_3+\beta_2(\boldsymbol{A}_1\boldsymbol{A}_2+\boldsymbol{A}_2\boldsymbol{A}_1)+\beta_3(\operatorname{tr}\boldsymbol{A}_1^2)\boldsymbol{A}_1, \tag{5}$$

其中，$\boldsymbol{I}$ 为单位张量，μ 为流体粘度，α_1，α_2，β_1，β_2，β_3 为材料常数．Fosdick 和 Rajagopal[31] 的研究表明，如果流体的所有运动全都热力学兼容，在这个意义上，这样的运动满足 Clausius-Duhem 不等式，同时假定，如果流体静止时，Helmholtz 自由比能达到最小，则 μ，α_1，α_2，β_1，β_2，β_3 受到如下约束：

$$\mu\geqslant0,\ \alpha_1\geqslant0,\ \beta_1=\beta_2=0,\ \beta_3\geqslant0,\ \alpha_1+\alpha_2\leqslant\sqrt{24\mu\beta_3}\,. \tag{6}$$

Rivlin-Ericksen 张量 $\boldsymbol{A}_1$ 和 $\boldsymbol{A}_2$ 分别为

$$\boldsymbol{A}_1=\nabla\boldsymbol{V}+(\nabla\boldsymbol{V})^{\mathrm{T}}, \tag{7}$$

$$\boldsymbol{A}_2 = \frac{\mathrm{d}\boldsymbol{A}_1}{\mathrm{d}t} + \boldsymbol{A}_1(\nabla \boldsymbol{V}) + (\nabla \boldsymbol{V})^{\mathrm{T}}\boldsymbol{A}_1, \tag{8}$$

定义速度场和温度场为

$$\boldsymbol{V} = [u(r,z), 0, w(r,z)],\ T = T(r,z). \tag{9}$$

控制方程为

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{10}$$

$$\begin{aligned}
\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) =& -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + \\
&\alpha_1\Big[-\frac{2u^2}{r^3} - 2\frac{w}{r^2}\frac{\partial u}{\partial z} + \frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} + w\frac{\partial^3 u}{\partial z^3} - 2\frac{u}{r^2}\frac{\partial u}{\partial r} + \\
&3\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2} + 3\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2} + \frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{2w}{r}\frac{\partial^2 u}{\partial r\partial z} + 5\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z} + \\
&4\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z} + \frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z} + u\frac{\partial^3 u}{\partial r\partial z^2} + w\frac{\partial^3 w}{\partial r\partial z^2} + \frac{2u}{r}\frac{\partial^2 u}{\partial r^2} + \\
&10\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2} + 3\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2} + 4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + 2w\frac{\partial^3 u}{\partial r^2\partial z} + u\frac{\partial^3 w}{\partial r^2\partial z} + 2u\frac{\partial^3 u}{\partial r^3} + \\
&\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + 2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2} + 4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\Big] + \alpha_2\Big[-\frac{4u^2}{r^3} + \frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2 + 2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} + \\
&2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2} + 2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2} + 4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z} + \frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2 + 2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z} + \\
&2\frac{\partial w}{\partial z} + 2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z} + 8\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2} + 2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2} + 2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \\
&\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + 2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2} + 4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\Big] + \beta_3\Big[-\frac{8u^3}{r^4} + \frac{8}{r}\left(\frac{\partial u}{\partial r}\right)^3 - \frac{8u^2}{r^3}\frac{\partial u}{\partial r} + \\
&\frac{8}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2 + \frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{4}{r}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{8}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + \\
&24\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2} + \frac{8u^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{16u}{r^2}\frac{\partial u}{\partial r} + 8\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2} + 16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z} + \\
&4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2} + 4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2} + 16\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z} + 8\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2} + \\
&16\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z} + 8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + 4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2} + 4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z} + \frac{4u^2}{r^2}\frac{\partial^2 u}{\partial z^2} + \\
&\frac{4u^2}{r^2}\frac{\partial^2 w}{\partial r\partial z} + \frac{4u}{r^2}\left(\frac{\partial u}{\partial z}\right)^2 + 8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} + 8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} + 4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2} + \\
&4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z} + 6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2} + 6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2} + 12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z} + \\
&12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2} - \frac{8u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2 - \frac{8u}{r^2}\left(\frac{\partial w}{\partial z}\right)^2 - \frac{4u}{r^2}\left(\frac{\partial w}{\partial r}\right)^2 + \\
&6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z} + 6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}\Big] - \sigma B_0^2 u,
\end{aligned} \tag{11}$$

$$\rho\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2}\right] +$$

$$\alpha_1\left[\frac{u}{r}\frac{\partial^2 w}{\partial r^2}+\frac{w}{r}\frac{\partial^2 u}{\partial z^2}+\frac{u}{r}\frac{\partial^2 u}{\partial r\partial z}+\frac{w}{r}\frac{\partial^2 w}{\partial r\partial z}+\frac{3}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\right.$$

$$\frac{3}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}+\frac{1}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r^2}+u\frac{\partial^3 w}{\partial r^3}+$$

$$3\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+w\frac{\partial^3 u}{\partial r\partial z^2}+4\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+u\frac{\partial^3 u}{\partial r^2\partial z}+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+$$

$$w\frac{\partial^3 w}{\partial r^2\partial z}+3\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+3\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+5\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+$$

$$\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+2u\frac{\partial^3 w}{\partial r\partial z^2}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+2w\frac{\partial^3 w}{\partial z^3}+8\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+$$

$$\left.4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right]+\alpha_2\left[\frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}+\right.$$

$$\frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial^2 u}{\partial r^2}\frac{\partial w}{\partial r}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r^2}+$$

$$4\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+8\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+$$

$$\left.2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}\right]+\beta_3\left[\frac{4}{r}\frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial r}\right)^2+\right.$$

$$\frac{4}{r}\frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial r}\right)^2+\frac{4u^2}{r^3}\frac{\partial u}{\partial z}+\frac{4u^2}{r^3}\frac{\partial w}{\partial r}+\frac{4}{r}\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial z}\right)^2+$$

$$\frac{4}{r}\frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+\frac{2}{r}\left(\frac{\partial u}{\partial z}\right)^3+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^3+\frac{6}{r}\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial r}+$$

$$\frac{6}{r}\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial r}\right)^2+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r\partial z}+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r^2}+8\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+$$

$$8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+\frac{8u}{r^2}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{8u}{r^2}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+\frac{4u^2}{r^2}\frac{\partial^2 u}{\partial r\partial z}+\frac{4u^2}{r^2}\frac{\partial^2 w}{\partial r^2}+$$

$$4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r\partial z}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+$$

$$6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial r\partial z}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial r\partial z}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r^2}+$$

$$12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+8\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial z^2}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+$$

$$\frac{8u^2}{r^2}\frac{\partial^2 w}{\partial z^2}+\frac{16u}{r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}+24\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial z^2}+4\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial z^2}+$$

$$4\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+$$

$$\left.8\frac{\partial w}{\partial r}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}\right],\tag{12}$$

$$\rho c_p\left(u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right)=K\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\sigma B_0^2 u^2+$$

$$\mu\left[2\frac{u^2}{r^2}+2\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+\left(\frac{\partial w}{\partial r}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2\right]+$$

$$\alpha_1\left[2\frac{u^3}{r^3}+2\frac{u^2}{r^2}\frac{\partial u}{\partial r}+4\left(\frac{\partial u}{\partial r}\right)^3+2u\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+2w\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+2\frac{uw}{r^2}\frac{\partial u}{\partial z}+\right.$$

$$u\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+w\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+u\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+$$

$$w\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+u\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+u\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+w\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+$$

$$w\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}+4\left(\frac{\partial w}{\partial z}\right)^3+6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+3\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial w}{\partial z}+$$

$$\left.2u\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+2w\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}\right]+\alpha_2\left[4\frac{u^3}{r^3}+4\left(\frac{\partial u}{\partial r}\right)^3+3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+\right.$$

$$6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+4\left(\frac{\partial w}{\partial z}\right)^3+3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}+6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+$$

$$\left.3\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial w}{\partial z}\right]+\beta_3\left[8\frac{u^4}{r^4}+8\left(\frac{\partial u}{\partial r}\right)^4+16\frac{u^2}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+8\frac{u^2}{r^2}\left(\frac{\partial u}{\partial z}\right)^2+\right.$$

$$8\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial u}{\partial z}\right)^2+2\left(\frac{\partial u}{\partial z}\right)^4+16\frac{u^2}{r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\left(\frac{\partial w}{\partial z}\right)^4+$$

$$16\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+8\left(\frac{\partial u}{\partial z}\right)^3\frac{\partial w}{\partial r}+8\frac{u^2}{r^2}\left(\frac{\partial w}{\partial r}\right)^2+8\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial w}{\partial r}\right)^2+$$

$$12\left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial w}{\partial r}\right)^2+8\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial r}\right)^3+2\left(\frac{\partial w}{\partial r}\right)^4+16\frac{u^2}{r^2}\left(\frac{\partial w}{\partial z}\right)^2+$$

$$16\left(\frac{\partial u}{\partial r}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+8\left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial w}{\partial z}\right)^2+16\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+$$

$$\left.8\left(\frac{\partial w}{\partial r}\right)^2\left(\frac{\partial w}{\partial z}\right)^2\right]. \tag{13}$$

相应的边界条件可以写成

$$u(r,H)=0,\ u(r,-H)=0,\ T(r,H)=T_{\mathrm{w}}, \tag{14}$$

$$w(r,H)=-V_0,\ w(r,-H)=V_0,\ T(r,-H)=T_{\mathrm{w}}, \tag{15}$$

其中，V_0 为恒定的吸入/喷出速度，T_{w} 为恒定的温度. $V_0>0$ 相应于吸入，而 $V_0<0$ 导致流体的喷出.

在方程(10)~(15)中应用如下变换：

$$u(r,z)=\frac{V_0 r}{2H}f'(\eta),\ w(r,z)=-V_0 f(\eta),\ \theta=\frac{T}{T_{\mathrm{w}}},\ \eta=\frac{z}{H}, \tag{16}$$

并消去压力项，得到

$$f^{(\mathrm{iv})}+Reff'''-\alpha[2f''f'''+f'f^{(\mathrm{iv})}+ff^{(\mathrm{v})}]-\gamma[2f''f'''+f'f^{(\mathrm{iv})}]+$$

$$\beta\left[7f''^3+24f'f''f'''+3f'^2f^{(\mathrm{iv})}+\frac{3}{2}\delta^2f''f'''^2+\frac{3}{4}\delta^2f''^2f^{(\mathrm{iv})}\right]-ReHa^2f''=0, \tag{17}$$

$$f'(1)=0,\ f'(-1)=0,\ f(1)=1,\ f(-1)=-1, \tag{18}$$

$$\theta''+PrRef\theta'+\frac{1}{4}EcPr\left[12f'^2+\delta^2f''^2-\right.$$

$$\alpha\{12f'^3+12ff'f''+\delta^2f'f''^2+\delta^2ff''f'''\}-\frac{3}{2}\gamma\{8f'^3+\delta^2f'f''^2\}+$$

$$\left.\frac{\beta}{4}\{144f'^4+24\delta^2f'^2f''^2+\delta^4f''^4\}+\frac{ReHa^2}{4}f'^2\right]=0, \tag{19}$$

$$\theta(1)=1,\ \theta(-1)=1, \tag{20}$$

其中，方程(10)得到自动满足，Reynolds 数(Re)、Prandtl 数(Pr)、Eckert 数(Ec)、Hartman 数(Ha)、三阶参数($\alpha,\ \beta,\ \gamma$)和无量纲径向位移(δ)分别用如下的无量纲变量定义：

$$\begin{cases} Re=\dfrac{V_0H}{\nu},\ Ha^2=\dfrac{\sigma B_0^2H}{\rho V_0},\ \upsilon=\dfrac{\mu}{\rho},\ \alpha=\dfrac{\alpha_1V_0}{\mu H},\ \delta=\dfrac{r}{H}, \\ \gamma=\dfrac{\alpha_2V_0}{\mu H},\ \beta=\dfrac{2\beta_3V_0^2}{\mu H^2},\ Pr=\dfrac{\mu c_p}{K},\ Ec=\dfrac{V_0^2}{c_pT_{\mathrm{w}}}. \end{cases} \tag{21}$$

因为 $V_0>0$ 相应于吸入，而 $V_0<0$ 导致流体的喷出。因此，吸入时有 $Re(=V_0H/\nu)>0$；喷出时有 $Re(=V_0H/\nu)<0$。

上、下圆盘的表面摩擦因数 $C_{1\mathrm{f}}$ 和 $C_{2\mathrm{f}}$ 分别为

$$C_{1\mathrm{f}}=\frac{\tau_{\mathrm{w}}}{\rho(V_0)^2/2}=\frac{\tau_{rz}|_{z=H}}{\rho(V_0)^2/2}=Re_r^{-1/2}\left[f''(1)-\alpha f'''(1)+\frac{\beta\delta^2}{4}f''^3(1)\right], \tag{22}$$

$$C_{2\mathrm{f}}=\frac{\tau_{\mathrm{w}}}{\rho(V_0)^2/2}=\frac{\tau_{rz}|_{z=-H}}{\rho(V_0)^2/2}=Re_r^{-1/2}\left[f''(-1)+\alpha f'''(-1)+\frac{\beta\delta^2}{4}f''^3(-1)\right], \tag{23}$$

其中，$Re_{\mathrm{r}}=V_0H^2/(\nu r)$ 为局部的 Reynolds 数。

物理上感兴趣的量还有上、下圆盘的 Nusselt 数，分别定义为

$$Nu_1=\frac{Hq_{\mathrm{w}}}{KT_{\mathrm{w}}}=-\frac{HK(\partial T/\partial z)|_{z=H}}{KT_{\mathrm{w}}}=-\theta'(1), \tag{24}$$

$$Nu_2=\frac{Hq_{\mathrm{w}}}{KT_{\mathrm{w}}}=-\frac{HK(\partial T/\partial z)|_{z=-H}}{KT_{\mathrm{w}}}=-\theta'(-1). \tag{25}$$

2　求 解 过 程

选择基函数

$$\{\eta^{2n+1};n\geqslant 0\},\ \{\eta^{2n};n\geqslant 0\}, \tag{26}$$

并记

$$f(\eta)=\sum_{n=0}^{\infty}a_n\eta^{2n+1},\ \theta(\eta)=\sum_{n=0}^{\infty}b_n\eta^{2n}, \tag{27}$$

其中，a_n 和 b_n 为待定系数。初始猜测和辅助线性算子为

$$f_0(\eta)=\frac{3}{2}\eta-\frac{1}{2}\eta^3,\ \theta_0(\eta)=1, \tag{28}$$

$$\mathcal{L}_2[\theta(\eta)]=\frac{\mathrm{d}^2f}{\mathrm{d}\eta^2},\ \mathcal{L}_1[f(\eta)]=\frac{\mathrm{d}^4f}{\mathrm{d}\eta^4}. \tag{29}$$

上述辅助线性算子有以下特性：

$$\mathcal{L}_1\left[\frac{C_1}{6}\eta^3+\frac{C_2}{4}\eta^2+C_3\eta+C_4\right]=0,\ \mathcal{L}_2[C_5+C_6\eta]=0, \tag{30}$$

其中 $C_i(i=1,2,\cdots,6)$ 为常数。

2.1　零-阶变形问题

零-阶变形问题如下给出：

$$(1-q)\mathcal{L}_1[\hat{f}(\eta,q)-f_0(\eta)]=q\hbar_f\mathcal{N}_1[\hat{f}(\eta,q)], \tag{31}$$

$$(1-q)\mathcal{L}_2[\hat{\theta}(\eta,q)-\theta_0(\eta)]=q\hbar_\theta\mathcal{N}_2[\hat{\theta}(\eta,q)], \tag{32}$$

$$\hat{f}'(1,q)=0,\ \hat{f}'(-1,q)=0,\ \hat{f}(1,q)=1,\ \hat{f}(-1,q)=-1,\tag{33}$$

$$\hat{\theta}(1,q)=1,\ \hat{\theta}(-1,q)=1,\tag{34}$$

其中，$\hbar_{f,\theta}\neq 0$ 和 $q\in[0,1]$ 分别为辅助参数和嵌入参数。当 q 从 0 变化到 1 时，则 $\hat{f}(\eta,q)$ 从初始猜测 $f_0(\eta)$ 变化到最终解 $f(\eta)$，同时，$\hat{\theta}(\eta,q)$ 从初始猜测 $\theta_0(\eta)$ 变化到最终解 $\theta(\eta)$；非线性算子成为

$$\begin{aligned}\mathcal{N}_1[f(\eta,q)]=&\frac{\partial^4\hat{f}}{\partial\eta^4}+Re\hat{f}\frac{\partial^3\hat{f}}{\partial\eta^3}-\\&\alpha\left[2\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}+\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^4\hat{f}}{\partial\eta^4}+f\frac{\partial^5\hat{f}}{\partial\eta^5}\right]-\gamma\left[2\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}+\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^4\hat{f}}{\partial\eta^4}\right]+\\&\beta\left[7\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^3+24\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}+3\left(\frac{\partial\hat{f}}{\partial\eta}\right)^2\frac{\partial^4\hat{f}}{\partial\eta^4}+\frac{3}{4}\delta^2\frac{\partial^2\hat{f}}{\partial\eta^2}\left(\frac{\partial^3\hat{f}}{\partial\eta^3}\right)^2+\right.\\&\left.\frac{3}{4}\delta^2\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2\frac{\partial^4\hat{f}}{\partial\eta^4}\right]-ReHa^2\frac{\partial^2\hat{f}}{\partial\eta^2},\end{aligned}\tag{35}$$

$$\begin{aligned}\mathcal{N}_2[\theta(\eta,q),f(\eta,q)]=&\frac{\partial^2\hat{\theta}}{\partial\eta^2}+PrRef\frac{\partial\hat{\theta}}{\partial\eta}+\\&\frac{1}{4}PrEc\left[12\left(\frac{\partial\hat{f}}{\partial\eta}\right)^2+\delta^2\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2-\right.\\&\alpha\left\{12\left(\frac{\partial\hat{f}}{\partial\eta}\right)^3+12f\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^2\hat{f}}{\partial\eta^2}+\delta^2\frac{\partial\hat{f}}{\partial\eta}\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2+\delta^2f\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}\right\}-\\&\frac{3}{2}\gamma\left\{8\left(\frac{\partial\hat{f}}{\partial\eta}\right)^3+\delta^2\frac{\partial\hat{f}}{\partial\eta}\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2\right\}+\frac{\beta}{4}\left\{144\left(\frac{\partial\hat{f}}{\partial\eta}\right)^4+\right.\\&\left.\left.24\delta^2\left(\frac{\partial\hat{f}}{\partial\eta}\right)^2\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2+\delta^2\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^4\right\}+\frac{ReHa^2\delta^2}{4}\left(\frac{\partial\hat{f}}{\partial\eta}\right)^2\right].\end{aligned}\tag{36}$$

应用 Taylor 级数展开，有

$$\hat{f}(\eta,q)=f_0(\eta)+\sum_{m=1}^{\infty}f_m(\eta)q^m,\ \hat{f}_m(\eta)=\frac{1}{m!}\frac{\partial^m f(\eta,q)}{\partial q^m}\bigg|_{q=0},\tag{37}$$

$$\hat{\theta}(\eta,q)=\theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta)q^m,\ \hat{\theta}_m(\eta)=\frac{1}{m!}\frac{\partial^m\theta(\eta,q)}{\partial q^m}\bigg|_{q=0},\tag{38}$$

当 $q=1$ 时，上述表达式简化为

$$\hat{f}(\eta)=f_0(\eta)+\sum_{m=1}^{\infty}f_m(\eta),\ \hat{\theta}(\eta)=\theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta).\tag{39}$$

2.2 *m*-阶变形问题

将方程(31)和(32)中的零-阶变形问题，对 q 求导 m 次，用 $m!$ 除，并令 $q=0$，立即得到

$$\mathcal{L}_1[\hat{f}_m(\eta)-\chi_m\hat{f}_{m-1}(\eta)]=\hbar_f\mathcal{R}_m^f(\eta),\tag{40}$$

$$\frac{\partial\hat{f}_m(\eta,q)}{\partial\eta}\bigg|_{\eta=1}=0,\ \frac{\partial\hat{f}_m(\eta,q)}{\partial\eta}\bigg|_{\eta=-1}=0,\ \hat{f}_m(1,q)=0,\ \hat{f}_m(-1,q)=0,\tag{41}$$

$$\mathcal{L}_2[\hat{\theta}_m(\eta)-\chi_m\hat{\theta}_{m-1}(\eta)]=\hbar_\theta\mathcal{R}_m^\theta(\eta),\tag{42}$$

$$\hat{\theta}_m(1,q)=0,\ \hat{\theta}(-1,q)=0, \tag{43}$$

$$\mathscr{R}_m^f(\eta)=f_{m-1}^{(\mathrm{iv})}(\eta)+\sum_{k=0}^{m-1}\Big[Ref_{m-1-k}f_k''' - \alpha(2f_{m-1-k}f_k'''+f_{m-1-k}'f_k^{(\mathrm{iv})}+f_{m-1-k}f_k^{(\mathrm{v})})-\gamma(2f_{m-1-k}''f_k'''+f_{m-1-k}f_k^{(\mathrm{iv})})+\beta\sum_{l=0}^{k}\Big\{7f_{m-1-k}''f_{k-l}''f_l''+24f_{m-1-k}'f_{k-l}''f_l'''+3f_{m-1-k}'f_{k-l}f_l'^{(\mathrm{iv})}+\frac{3}{2}\delta^2f_{m-1-k}''f_{k-l}'''f_l'''+\frac{3}{4}\delta^2f_{m-1-k}''f_{k-l}''f_l^{(\mathrm{iv})}\Big\}\Big]-ReHa^2f_{m-1}'', \tag{44}$$

$$\mathscr{R}_m^\theta(\eta)=\theta_{m-1}''(\eta)+PrRe\sum_{k=0}^{m-1}f_{m-1-k}\theta_k'+\frac{1}{2}EcPr\sum_{k=0}^{m-1}\Big[12f_{m-1-k}'f_k'+\delta^2f_{m-1-k}''f_k''-\alpha\sum_{l=0}^{k}\{12f_{m-1-k}'f_{k-l}'f_l'+12f_{m-1-k}f_{k-l}'f_l''+\delta^2f_{m-1-k}'f_{k-l}''f_l''+\delta^2f_{m-1-k}f_{k-l}''f_l'''\}-\frac{3}{2}\gamma\sum_{l=0}^{k}\{8f_{m-1-k}'f_{k-l}'f_l'+\delta^2f_{m-1-k}'f_{k-l}''f_l''\}+\frac{\beta}{4}\sum_{l=0}^{k}\Big\{144f_{m-1-k}'f_{k-l}'\sum_{j=0}^{l}f_{l-j}'f_j'+24\delta^2f_{m-1-k}'f_{k-l}'\sum_{j=0}^{l}f_{l-j}''f_j''+\delta^4f_{m-1-k}''f_{k-l}''\sum_{j=0}^{l}f_{l-j}''f_j''\Big\}+\frac{ReHa^2\delta^2}{4}f_{m-1-k}'f_k'\Big], \tag{45}$$

$$\chi_m=\begin{cases}0, & m\leqslant 1,\\ 1, & m>1.\end{cases} \tag{46}$$

由方程(40)和(42)得到问题的通解为

$$f(\eta)=f^*+\frac{1}{6}C_1^m\eta^3+\frac{1}{2}C_2^m\eta^2+C_3^m\eta+C_4^m, \tag{47}$$

$$\theta(\eta)=\theta^*+C_5^m+C_6^m\eta, \tag{48}$$

其中，f^* 和 θ^* 为特解，$C_i^m(i=1,2,\cdots,6)$ 为任意常数，可以根据边界条件(41)和(43)确定. 此外，定义 $f(\eta)$ 和 $\theta(\eta)$ 的均方残余误差 Δ_m 和 Δ_m^* 为[25]

$$\Delta_m=\left(\int_0^1\mathscr{N}_1\left[\sum_{n=0}^{m}f_m(\zeta)\right]\right)^2\mathrm{d}\zeta, \tag{49}$$

$$\Delta_m^*=\left(\int_0^1\mathscr{N}_2\left[\sum_{n=0}^{m}\theta_m(\zeta)\right]\right)^2\mathrm{d}\zeta. \tag{50}$$

3　解的收敛性

导出的级数解(47)和(48)包含了辅助参数 $\hbar_f$ 和 $\hbar_\theta$. 正如文献[21-25]指出的，级数解(47)和(48)的收敛性，强烈地依赖于 $\hbar_f$ 和 $\hbar_\theta$ 值是否适当. 为此目的，图2和图3分别绘出 $\hbar_f$ 和 $\hbar_\theta$ 曲线. 从这些图形可以发现，$\hbar_f$ 值的合适范围是 $-0.85\leqslant\hbar_f<-0.35$，$\hbar_\theta$ 值的合适范围是 $-0.9\leqslant\hbar_\theta<-0.49$. 根据文献[25]选取最佳的辅助参数值，获得更好的收敛性. 为此，图4和图5分别绘出 $\hbar_f$ 和 $\hbar_\theta$ 的均方残余误差曲线，且在表1中列出级数解的收敛性检测. 表1显示10阶的级数解，其6位小数位有效.

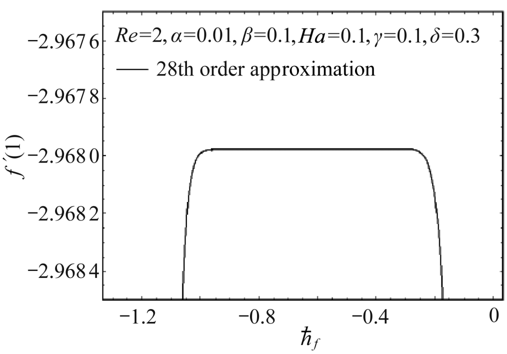

图 2 $f'(1)$-$\hbar_f$ 曲线图

Fig. 2 $\hbar_f$ - curve of $f'(1)$

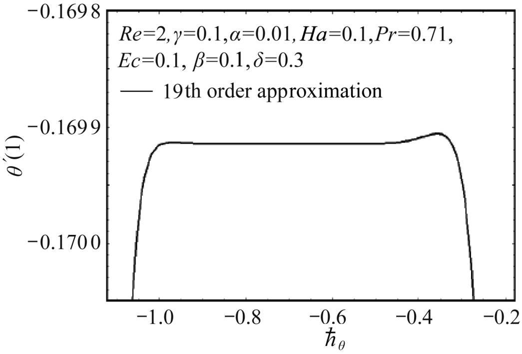

图 3 $\theta'(1)$-$\hbar_\theta$ 曲线图

Fig. 3 $\hbar_\theta$ - curve of $\theta'(1)$

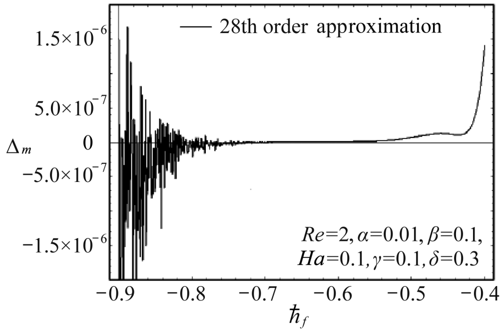

图 4 Δ_m-$\hbar_f$ 曲线图

Fig. 4 $\hbar_f$ - curve for square residual error of $f'(1)$

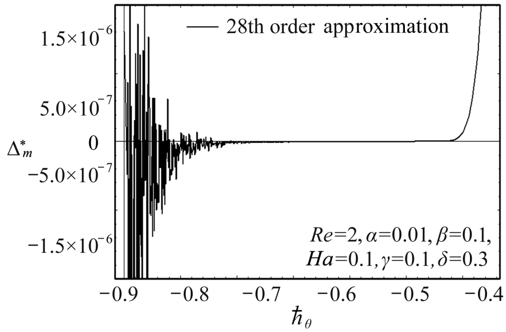

图 5 Δ_m^*-$\hbar_\theta$ 曲线图

Fig. 5 $\hbar_\theta$ - curve for square residual error of $\theta'(1)$

表 1 同伦解的收敛性(其中 $\alpha = 0.01$, $\beta = 0.1$, $\gamma = 0.1$, $\delta = 0.3$, $Ha = 0.1$, $Re = 2$, $Pr = 0.71$, $Ec = 0.1$, $\hbar_f = \hbar_\theta = -0.65$)

Table 1 Convergence of homotopy solutions when $\alpha = 0.01$, $\beta = 0.1$, $\gamma = 0.1$, $\delta = 0.3$, $Ha = 0.1$, $Re = 2$, $Pr = 0.71$, $Ec = 0.1$ and $\hbar_f = \hbar_\theta = -0.65$

order of approximation	$f''(1)$	$\theta'(1)$
1	−3.038 22	−0.250 999
2	−2.991 33	−0.188 300
5	−2.968 84	−0.169 592
8	−2.967 95	−0.169 908
10	−2.967 98	−0.169 914
15	−2.967 98	−0.169 914
20	−2.967 98	−0.169 914

4 结果及讨论

本节分析无量纲参数,对无量纲径向和轴向速度分量,以及无量纲温度的影响. 讨论无量纲参数取不同数值时,表面摩擦因数和 Nusselt 数的变化情况. 图 6 和图 7 描述了 $Re > 0$ 时,三阶参数 β 对 $f'(\eta)$ 和 $f(\eta)$ 的影响. 可以发现,对于 $Re > 0$ 和 $Re < 0$ 两种情况,β 对 $f'(\eta)$ 和 $f(\eta)$ 的影响都很类似. 由图 8 可以发现,当 $Re > 0$ 时,无量纲温度 $\theta(\eta)$ 是 β 的增函数. 当 $Re < 0$ 时也观察到类似的变化趋势. 从图 9 和图 10 可以看到,$\theta(\eta)$ 是 α 和 γ 的减函数. 比较图 8

和图 9 发现,β 对 $\theta(\eta)$ 的影响,与 α 对 $\theta(\eta)$ 的影响正相反. 图 11 描述了 $Re > 0$ 时,Hartman 数 Ha 对无量纲温度 $\theta(\eta)$ 的影响. 无量纲温度 $\theta(\eta)$ 随着 Ha 的增大而增大. 事实上,Hartman 数的增大相当于应用磁场的增大. 外部磁场的增大导致 Joule 传热的增大,从而温度也会升高. 当 $Re < 0$ 时,情况类似. Eckert 数 Ec 为动能与焓之比. 因此,Ec 的增大,相当于流体颗粒动能的增大,引起温度 $\theta(\eta)$ 的升高(参看图 12). 对于 $Re > 0$ 和 $Re < 0$ 两种情况,无量纲温度 $\theta(\eta)$ 都是 Prandtl 数 Pr 的增函数. 因此热边界层厚度在减小,如图 13 所示.

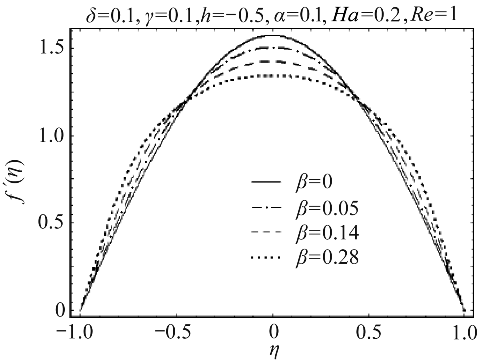

图 6　当 $Re > 0$ 时, β 对 $f'(\eta)$ 的影响

Fig. 6　Influence of β on $f'(\eta)$ when $Re > 0$

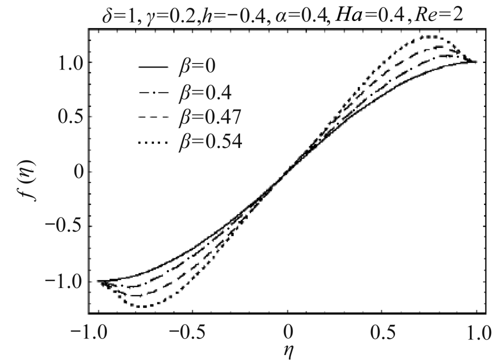

图 7　当 $Re > 0$ 时, β 对 $f'(\eta)$ 的影响

Fig. 7　Influence of β on $f'(\eta)$ when $Re > 0$

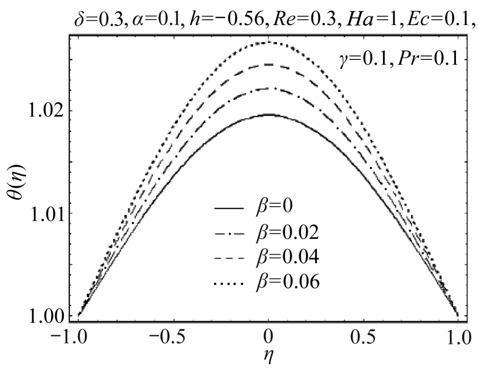

图 8　当 $Re > 0$ 时, β 对 $\theta(\eta)$ 的影响

Fig. 8　Influence of β on $\theta(\eta)$ when $Re > 0$

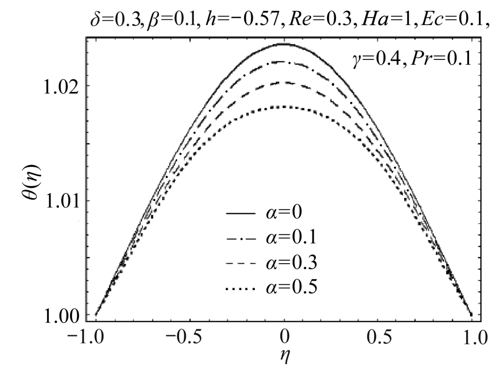

图 9　当 $Re > 0$ 时, α 对 $\theta(\eta)$ 的影响

Fig. 9　Influence of α on $\theta(\eta)$ when $Re > 0$

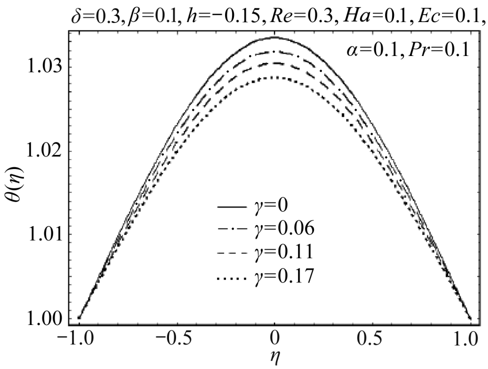

图 10　当 $Re > 0$ 时, γ 对 $\theta(\eta)$ 的影响

Fig. 10　Influence of γ on $\theta(\eta)$ when $Re > 0$

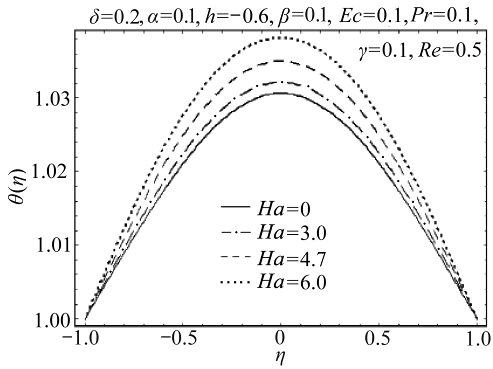

图 11　当 $Re > 0$ 时, Ha 对 $\theta(\eta)$ 的影响

Fig. 11　Influence of Ha on $\theta(\eta)$ when $Re > 0$

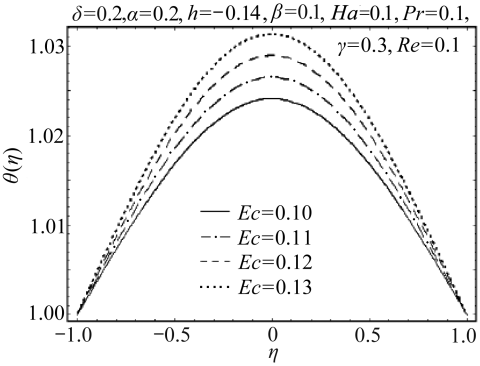

图 12 当 $Re > 0$ 时, Ec 对 $\theta(\eta)$ 的影响

Fig. 12 Influence of Ec on $\theta(\eta)$ when $Re > 0$

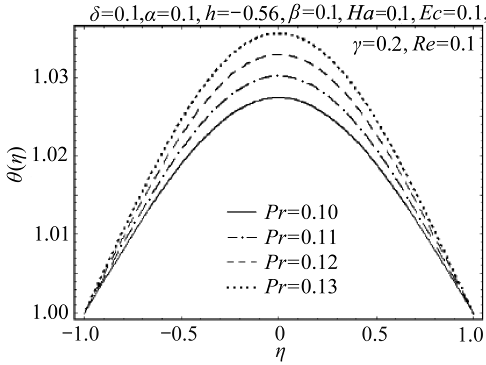

图 13 当 $Re > 0$ 时, Pr 对 $\theta(\eta)$ 的影响

Fig. 13 Influence of Pr on $\theta(\eta)$ when $Re > 0$

表 2 就 $Re > 0$ 时显示了上、下圆盘的表面摩擦因数 $Re_r C_{1f}$ 和 $Re_r C_{2f}$ 的变化. 表 3 描述了 $Re < 0$ 时, $Re_r C_{1f}$ 和 $Re_r C_{2f}$ 的变化情况. 从表 2 可以发现, $Re_r C_{1f}$ 和 $Re_r C_{2f}$ 是 β 和 Ha 的增函数, 而是 α, γ 和 Re 的减函数. 这就意味着, 上、下圆盘的切向应力都是 β 和 Ha 的增函数. 但是, 这些应力也随着 α, γ 和 Re 的增大而增大. 表 2 还显示, 二阶流体($\beta = 0$)中的 $Re_r C_{1f}$ 和 $Re_r C_{2f}$, 小于三阶流体($\beta \neq 0$)中的 $Re_r C_{1f}$ 和 $Re_r C_{2f}$. 而且, 外加磁场强度的增大, 会导致圆盘表面应力的增大. 表 4 和表 5 检验了无量纲参数对 Nusselt 数 Nu_1 和 Nu_2 的影响. 这些表显示, Nu_1 和 Nu_2 是 Ha, Pr, β 和 Ec 的增函数, 是 α 和 γ 的减函数. 还可以发现, 当 $Re > 0$ 时, 热交换率随着 Ha, Pr, β 和 Ec 的增大而增大, 而随着 α 和 γ 的增大而减小.

表 2 当 $Re > 0$ 时, 上圆盘的表面摩擦因数 $Re_r C_{1f}$ 和下圆盘的表面摩擦因数 $Re_r C_{2f}$ 的数值解

Table 2 Numerical values of skin friction coefficients $Re_r C_{1f}$ at upper disk and $Re_r C_{2f}$ at lower disk when $Re > 0$

α	γ	β	Re	Ha	$-ReC_{1f} = ReC_{2f}$
0.00	0.01	0.1	2	0.1	3.067 26
0.01					3.055 66
0.02					3.021 73
0.03					2.978 96
0.01	0.0	0.1	2	0.1	3.053 25
	0.1				2.953 80
	0.2				2.856 05
	0.3				2.759 56
0.01	0.1	0.0	2	0.1	2.502 11
		0.1			2.973 27
		0.2			3.429 07
		0.3			3.972 99
0.01	0.01	0.1	0.0	0.3	3.51253
			0.1		3.489 55
			0.2		3.460 19
			0.3		3.434 28
0.01	0.1	0.1	2	0.0	2.961 28
				0.1	2.964 70
				0.2	2.974 96
				0.3	2.992 11

表 3　当 $Re < 0$ 时，上圆盘的表面摩擦因数 $Re_r C_{1f}$ 和下圆盘的表面摩擦因数 $Re_r C_{2f}$ 的数值解

Table 3　Numerical values of skin friction coefficients $Re_r C_{1f}$ at upper disk and $Re_r C_{2f}$ at lower disk when $Re < 0$

α	γ	β	Re	Ha	$-ReC_{1f} = ReC_{2f}$
0.00	0.01	0.1	−0.2	0.1	3.665 67
0.01					3.818 43
0.02					3.800 93
0.03					3.671 06
0.01	0.0	0.1	−0.2	0.1	3.750 11
	0.1				3.637 75
	0.2				3.521 78
	0.3				3.403 17
0.01	0.1	0.0	−0.2	0.1	2.974 50
		0.1			3.745 36
		0.2			4.324 52
		0.21			4.377 40
0.01	0.01	0.1	0.0	0.3	3.512 53
			−0.1		3.535 71
			−0.2		3.565 54
			−0.3		3.592 27
0.01	0.1	0.1	−0.2	0.0	3.701 03
				0.1	3.700 57
				0.2	3.699 16
				0.3	3.696 81

表 4　当 $Re > 0$ 时，上圆盘的 Nusselt 数 Nu_1 和下圆盘的 Nusselt 数 Nu_2 的数值解

Table 4　Numerical values of Nusselt number Nu_1 at upper disk and Nu_2 at lower disk when $Re > 0$

α	γ	β	Re	Ha	Pr	Ec	$Nu_1 = -Nu_2$
0.0	0.1	0.1	1	0.1	0.71	0.1	0.265 683
0.01							0.264 562
0.02							0.263 478
0.03							0.262 428
0.01	0.0	0.1	2	0.1	0.71	0.1	0.207 198
	0.1						0.190 906
	0.2						0.174 529
	0.3						0.158 050
0.01	0.01	0.0	2	0.1	0.71	0.1	0.139 697
		0.1					0.204 784
		0.11					0.210 884
		0.12					0.216 943
0.01	0.01	0.1	0.0	0.1	0.71	0.1	0.397 815
			0.1				0.384 897
			0.2				0.372 395
			0.3				0.360 298
0.01	0.01	0.1	2	0.0	0.71	0.1	0.204 695

续表 4

α	γ	β	Re	Ha	Pr	Ec	$Nu_1 = -Nu_2$
				0.1			0.204 784
				0.2			0.205 049
				0.3			0.205 491
0.01	0.01	0.1	2	0.1	0.1	0.1	0.052 214 8
					0.2		0.094 539 2
					0.3		0.128 483
					0.4		0.155 345
0.01	0.71	0.1	0.1	0.1	0.71	0.1	0.164 350
						0.2	0.328 700
						0.3	0.493 050
						0.4	0.657 400

表 5 当 $Re < 0$ 时,上圆盘的 Nusselt 数 Nu_1 和下圆盘的 Nusselt 数 Nu_2 的数值解

Table 5 Numerical values of Nusselt number Nu_1 at upper disk and Nu_2 at lower disk when $Re < 0$

α	γ	β	Re	Ha	Pr	Ec	$Nu_1 = -Nu_2$
0.0	0.1	0.1	−1	0.1	0.71	0.1	0.514 328
0.01							0.511 209
0.02							0.508 202
0.03							0.505 320
0.01	0.0	0.1	−2	0.1	0.71	0.1	0.774 095
	0.1						0.710 417
	0.2						0.646 485
	0.3						0.582 276
0.01	0.01	0.0	−2	0.1	0.71	0.1	0.539 884
		0.1					0.765 731
		0.11					0.787 620
		0.12					0.809 448
0.01	0.01	0.1	0.0	0.1	0.71	0.1	0.397 815
			−0.1				0.411 164
			−0.2				0.424 956
			−0.3				0.439 207
0.01	0.01	0.1	−2	0.0	0.71	0.1	0.765 957
				0.1			0.765 731
				0.2			0.765 054
				0.3			0.763 930
0.01	0.01	0.1	−2	0.1	0.1	0.1	0.059 543 1
					0.2		0.131 010
					0.3		0.216 368
					0.4		0.317 881
0.01	0.71	0.1	−0.1	0.1	0.71	0.1	0.175 163
						0.2	0.350 327
						0.3	0.525 490
						0.4	0.700 653

5 结　论

本文研究了多孔圆盘之间导电三阶流体的热交换问题. 分析了无量纲参数对圆盘表面摩擦因数和 Nusselt 数的影响. 主要结论归纳如下:

1) 就 $Re > 0$ 和 $Re < 0$ 而言,三阶参数 β 对径向速度 $f'(\eta)$ 的影响是类似的;

2) 表面摩擦因数随着 Re 的增大而减小;

3) 数量上,β 对 $\theta(\eta)$ 的影响与 γ 对 $\theta(\eta)$ 的影响正相反;

4) 二阶流体 ($\beta = 0$) 中圆盘的表面应力,小于三阶流体($\beta \neq 0$) 中圆盘的表面应力;

5) 随着 Ha,Pr,β 和 Ec 的增大,圆盘向流体的热交换率也在增大;

6) 无量纲温度 $\theta(\eta)$ 是 Ha 的增函数,是 α 和 γ 的减函数.

致谢　Alsaedi 博士对沙特阿拉伯阿卜杜勒阿齐兹国王大学科学研究院(DSR)的资助表示感谢. 同时,作者对审稿人提出的宝贵意见表示衷心感谢.

参考文献(References):

[1] Fetecau C, Mahmood A, Jamil M. Exact solutions for the flow of a viscoelastic fluid induced by a circular cylinder subject to a time dependent shear stress[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2010, **15**(12): 3931-3938.

[2] Jamil M, Fetecau C, Imran M. Unsteady helical flows of Oldroyd-B fluids[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2011, **16**(3): 1378-1386.

[3] Jamil M, Rauf A, Fetecau C, Khan N A. Helical flows of second grade fluid to constantly accelerated shear stresses[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2011, **16**(4): 1959 -1969.

[4] Tan W C, Masuoka T. Stability analysis of a Maxwell fluid in a porous medium heated from below[J]. *Physics Letters A*, 2007, **360**(3): 454-460.

[5] Tan W C, Masuoka T. Stokes' first problem for an Oldroyd-B fluid in a porous half space [J]. *Physics of Fluids*, 2005, **17**(2): 023101-7.

[6] Sajid M, Hayat T. Non-similar series solution for boundary layer flow of a third-order fluid over a stretching sheet[J]. *Applied Mathematics and Computation*, 2007, **189**(2): 1576-1585.

[7] Sajid M, Hayat T, Asghar S. Non-similar analytic solution for MHD flow and heat transfer in a third order fluid over a stretching sheet[J]. *International Journal of Heat and Mass Transfer*, 2007, **50**(9/10): 1723-1736.

[8] Hayat T, Mustafa M, Asghar S. Unsteady flow with heat and mass transfer of a third grade fluid over a stretching surface in the presence of chemical reaction[J]. *Nonlinear Analysis: Real World Applications*, 2010, **11**(4): 3186-3197.

[9] Abbasbandy S, Hayat T. On series solution for unsteady boundary layer equations in a special third grade fluid[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2011, **16**(8): 3140-3146.

[10] Sahoo B. Hiemenz flow and heat transfer of a third grade fluid[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2009, **14**(3): 811-826.

[11] Sahoo B, Do Y. Effects of slip on sheet driven flow and heat transfer of a third-grade fluid past a stretching sheet[J]. *International Communication in Heat and Mass Transfer*, 2010,

37(8): 1064-1071.

[12] von Kármán T. Über laminare and turbulente reibung[J]. *Zeitschrift für Angewandte Mathematik und Mechanik*, 1921, **1**(4): 233-255.

[13] Cochran W G. The flow due to a rotating disk[J]. *Mathematical Proceedings of the Cambridge Philosophical Society*, 1934, **30**(3): 365-375.

[14] Benton E R. On the flow due to a rotating disk[J]. *Journal of Fluid Mechanics*, 1966, **24**(781/800): 133-137.

[15] Takhar H S, Chamkha A J, Nath G. Unsteady mixed convection flow from a rotating verticle cone with a magnetic field[J]. *Heat and Mass Transfer*, 2003, **39**(4): 297-304.

[16] Maleque K A, Sattar M A. The effects of variable properties and Hall current on steady MHD laminar convective fluid flow due to a porous rotating disk[J]. *International Journal of Heat and Mass Transfer*, 2005, **48**(23/24): 4963-4972.

[17] Stuart J T. On the effect of uniform suction on the steady flow due to a rotating disk[J]. *The Quarterly Journal of Mechanics and Applied Mathematics*, 1954, **7**(4): 446-457.

[18] Sparrow E M, Beavers G S, Hung L Y. Flow about a porous-surface rotating disk[J]. *International Journal of Heat and Mass Transfer*, 1971, **14**: 993-996.

[19] Miklavcic M, Wang C Y. The flow due to a rough rotating disk[J]. *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, 2004, **55**(2): 235-246.

[20] Hayat T, Nawaz M. Unsteady stagnation point flow of viscous fluid caused by an impulsively rotating disk[J]. *Journal of the Taiwan Institute of Chemical Engineers*, 2011, **42**(1): 41-49.

[21] Liao S J. *Beyond Perturbation: Introduction to Homotopy Analysis Method*[M]. Boca Raton: Chapman and Hall, CRC Press, 2003.

[22] Liao S J. On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet[J]. *Journal of Fluid Mechanics*, 2003, **488**: 189-212.

[23] Liao S J. On the homotopy analysis method for nonlinear problems[J]. *Applied Mathematics and Computation*, 2004, **147**(12): 499-513.

[24] Liao S J. Notes on the homotopy analysis method: some definitions and theorems[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2009, **14**(4): 983-997.

[25] Liao S J. An optimal homotopy analysis approach for strongly nonlinear differential equations [J]. *Communications in Nonlinear Science and Numerical Simulation*, 2010, **15**(8): 2003-2016.

[26] Rashidi M M, Domairry G, Dinarvand S. Approximate solutions for the Burger and regularized long wave equations by means of the homotopy analysis method[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2009, **14**(3): 708-717.

[27] Abbasbandy S, Shirzadi A. A new application of the homotopy analysis method: solving the Sturm-Liouville problems[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2011, **16**(1): 112-126.

[28] Bataineh A S, Noorani M S M, Hashim I. Approximate analytical solutions of systems of PDEs by homotopy analysis method[J]. *Computer and Mathematics With Applications*, 2008, **55**(12): 2913-2923.

[29] Hashim I, Abdulaziz O, Momani S. Homotopy analysis method for fractional IVPs[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2009, **14**(3): 674-684.

[30] 哈亚特 T, 纳瓦兹 M, 奥拜达特 S. 微极流体在两个伸展平面之间的不稳定轴对称 MHD 流动

[J]. 应用数学和力学, 2011, **32**(3): 344-356 (Hayat T, Nawaz M, Obaidat S. Axisymmetric magnetohydrodynamic flow of a micropolar fluid between unsteady stretching surfaces[J]. *Applied Mathematics and Mechanics* (*English Edition*), 2011, **32**(3): 361-374).

[31] Fosdick R L, Rajagopal K R. Thermodynamics and stability of fluids of third grade[J]. *Proc Roy Soc London*, 1980, **369**(1738): 351-377.

MHD Axisymmetric Flow of a Third-Grade Fluid Between Porous Disks With Heat Transfer

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Abstract: The magnetohydrodynamic (MHD) flow of third-grade fluid between two permeable disks with heat transfer was investigated. The governing partial differential equations were converted into the ordinary differential equations by using suitable transformations. Transformed equations were solved by using homotopy analysis method (HAM). The expressions for square residual errors were defined and optimal values of convergencecontrol parameters were selected. The dimensionless velocity and temperature fields were examined for various dimensionless parameters. Skin friction coefficient and Nusselt number were tabulated to analyze the effects of dimensionless parameters.

Key words: heat transfer; axisymmetric flow; third-grade fluid; porous disks; skin friction coefficient; Nusselt number