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轴向运动三参数黏弹性梁弱受迫振动的渐近分析*

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摘要: 研究了轴向运动三参数黏弹性梁的弱受迫振动. 建立了轴向运动三参数黏弹性梁受迫振动的控制方程. 使用多尺度法渐近分析了运动梁的稳态响应,导出了解稳定性边界方程、稳态振幅的表达式以及稳态响应非零解的存在条件. 依据 Routh-Hurwitz 定律决定了非线性稳态响应非零解的稳定性.

关 键 词: 轴向运动梁; 弱受迫振动; 三参数模型; 多尺度法; 稳态响应

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引　　言

轴向运动梁可作为多种工程装置的力学模型,如动力传送带、磁带、纸带、带锯、空中缆车索道和高楼升降机缆绳等[1]. 自 Pasin[2] 的首次研究以来,轴向变速弹性梁的横向参数振动得到了广泛研究. Öz 等[3] 使用多尺度法研究了小弯曲刚度轴向变速弹性梁的动态稳定性. Öz[4] 应用多尺度法近似计算了轴向变速梁在简支和固支下的稳定性. Suweken 等[5] 先用 Galerkin 离散后再使用多尺度法研究了轴向运动梁在简支条件下的稳定性. Pakdemirli 等[6] 使用多尺度法分析了四阶模态的共振问题. 除了弹性梁,近年来变速轴向运动黏弹性梁也已经被广泛关注. Chen 等[7] 使用平均法分析了固支条件下 Galerkin 离散的轴向变速黏弹性梁的稳定边界. Chen 等[8] 应用直接多尺度法近似分析并获得了固支和简支下运动梁的稳定性边界. Yang 等[9] 提出了梁的积分形式黏弹性本构关系,使用多尺度法分析了梁的振动和稳定性. Chen 等[10] 首次研究了在两端带有弹簧铰支的混杂边界下轴向运动黏弹性梁的振动和稳定性. 在 Chen 等[7-8,10] 的工作中,使用了 Kelvin 模型或在其基础上取物质时间导数来描述梁的黏弹性属性. 当较大横向位移发生时,一般由于梁的有限拉伸而引起的几何非线性不能被忽略. Maccari[11] 研究了梁主共振和亚共振的弱受迫振动,并获得了力响应曲线和频率响应曲线. Boertjens 等[12] 研究了带有几何非线性的弱受迫振动梁. Chen 等[13] 渐近分析了原子力显微镜中具有非线性边界的悬臂梁的非线性振动. 但这些研究都局限于受干扰的保守连续系统的静

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态梁. Chen 等[14]研究了轴向运动黏弹性梁的两种非线性模型的稳态响应和稳定性. Yang 等[15]使用 Galerkin 和摄动法研究了在多频激励下的轴向变速黏弹性的受迫振动. Ding 等[16]使用多尺度法分析了梁的稳定性并对其结果进行有限差分验证. Chen 等[17]使用有限差分法研究了受外载荷轴向运动黏弹性梁的稳态周期响应. 与 Kelvin 模型相比,三参数模型更典型和具代表性,同时它能退化为 Kevin 模型或 Maxwell 模型. 此外,黏弹性材料用 Botlzmann 叠加原理描述,其控制方程与用三参数模型的有相似形式[9]. Mockensturm 等[18]认为将 Kelvin 模型应用到轴向运动材料应当包含物质时间导数以说明稳态运动中的能量好散. 实际上,Chen 等[19]物质时间导数已经被应用在轴向运动梁的黏弹性属性中. 本文采用三参数模型来描述轴向运动梁的黏弹性属性并引入了物质时间导数,考虑了两种非线性模型. 使用多尺度法求解控制方程,导出稳态响应方程以及决定解的稳定性.

1 控制方程

考虑均匀轴向运动黏弹性梁,密度为 ρ,横截面积为 A,初始拉力为 P_0. 支承两端间长度为 l,轴向传输速度为 γ,运动梁受横向周期激励 $F\cos(\omega t)$. 这里由横向位移 $v(x,t)$ 描述梁的弯曲振动,t 为时间,x 为轴向坐标. 物理模型如图 1 所示,其控制方程表示为

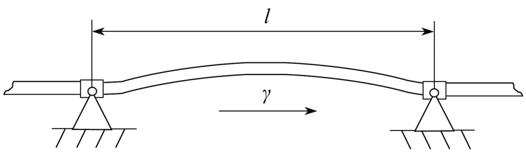


图 1 轴向运动梁的物理模型

Fig. 1 Physical model of axially moving beam

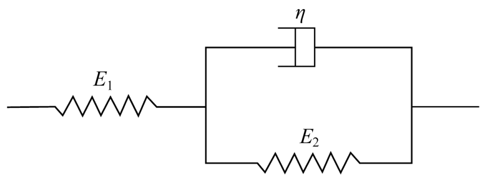


图 2 三参数黏弹性模型

Fig. 2 Standard linear solid model

$$\rho A\frac{\mathrm{d}^2 v}{\mathrm{d}t^2}+M_{,xx}=\frac{\partial}{\partial x}[(P_0+A\sigma)v_{,x}]+F\cos(\omega t),\tag{1}$$

其中,$(\cdot)_{,x}$ 表示函数对 x 的偏导数;激励振幅 F 和激励频率 ω 均为常数;$\sigma(x,t)$ 为梁的干扰应力,弯矩 $M(x,t)$ 被定义为

$$M(x,t)=-\int_A z\sigma(x,z,t)\mathrm{d}A,\tag{2}$$

其中,z-x 平面为弯曲主平面,$\sigma(x,z,t)$ 为正应力;梁的黏弹性由三参数模型本构关系描述,其物理模型由图 2 所示. 微分方程为[20-23]

$$(E_1+E_2)\sigma(x,z,t)+\eta\frac{\mathrm{d}}{\mathrm{d}t}\sigma(x,z,t)=E_1E_2\varepsilon(x,z,t)+E_1\eta\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(x,z,t),\tag{3}$$

其中,E_1 和 E_2 为梁的弹性模量,$\varepsilon(x,z,t)$ 为轴向应变,η 为黏性阻尼;三参数模型可退化为 Kelvin 模型($E_1\to\infty$ 和 $E_2\neq 0$)或 Maxwell 模型($E_1\neq 0$ 和 $E_2=0$). 对于小变形,应力-应变关系可表示为

$$\varepsilon(x,z,t)=-z\frac{\partial^2 v(x,t)}{\partial x^2}.\tag{4}$$

如果使用三参数模型描述方程(1)中的干扰应力 $\sigma(x,t)$,则其微分关系为

$$(E_1+E_2)\sigma(x,t)+\eta\frac{\mathrm{d}}{\mathrm{d}t}\sigma(x,t)=E_1E_2\varepsilon_{\mathrm{L}}(x,t)+E_1\eta\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_{\mathrm{L}}(x,t),\tag{5}$$

其中,用 Lagrange 应变 $\varepsilon_{\mathrm{L}}(x,t)$ 来衡量由于梁的有限拉伸而引起的几何非线性,其表达式为

$$\varepsilon_{\mathrm{L}}=\frac{1}{2}v_{,x}^{2}. \tag{6}$$

通过定义微分算子 d/dt 来引入物质时间导数,其表达式为[23]

$$\frac{\mathrm{d}}{\mathrm{d}t}\leftrightarrow\frac{\partial}{\partial t}+\gamma\frac{\partial}{\partial x}, \tag{7}$$

其中,速度 γ 等于 $\mathrm{d}x/\mathrm{d}t$. 显然,方程(7)就是全导数的微分算子.

将方程(7)代入方程(1)和(3)及(5),得

$$\rho A(v_{,tt}+2\gamma v_{,xt}+\gamma^{2}v_{,xx})+M_{,xx}=[(P_{0}+A\sigma)v_{,x}]_{,x}+F\cos(\omega t), \tag{8}$$

$$(E_{1}+E_{2})\sigma+\eta(\sigma_{,t}+\gamma\sigma_{,x})=E_{1}[E_{2}\varepsilon+\eta(\varepsilon_{,t}+\gamma\varepsilon_{,x})], \tag{9}$$

$$(E_{1}+E_{2})\sigma+\eta(\sigma_{,t}+\gamma\sigma_{,x})=E_{1}[E_{2}\varepsilon_{\mathrm{L}}+\eta(\varepsilon_{\mathrm{L},t}+\gamma\varepsilon_{\mathrm{L},x})]. \tag{10}$$

使用方程(4)和方程(6)分别消去方程(9)中的 ε 和方程(10) 中的 ε_{L}, 得

$$(E_{1}+E_{2})\sigma+\eta(\sigma_{,t}+\gamma\sigma_{,x})=-zE_{1}[E_{2}v_{,xx}+\eta(v_{,xxt}+\gamma v_{,xxx})], \tag{11}$$

$$(E_{1}+E_{2})\sigma+\eta(\sigma_{,t}+\gamma\sigma_{,x})=\frac{1}{2}E_{1}[E_{2}(v_{,x}^{2})+\eta(v_{,x}^{2})_{,t}+\gamma\eta(v_{,x}^{2})_{,x}]. \tag{12}$$

动力学方程(8),本构关系(11)和几何关系(12)构成了轴向运动黏弹性梁的控制方程. 如果轴向拉力远小于初始拉力,干扰拉力 $A\sigma$ 可用其平均值$\frac{1}{l}\int_{0}^{l}A\sigma\mathrm{d}x$ 来代替[14,23-25]. 则方程(8)变为

$$\rho A(v_{,tt}+2\gamma v_{,xt}+\gamma^{2}v_{,xx})+M_{,xx}=\left(P_{0}+\frac{1}{l}\int_{0}^{l}A\sigma\mathrm{d}x\right)v_{,xx}+F\cos(\omega t). \tag{13}$$

为了无量纲化并消去 σ 和 z,方程(11) 两边乘 $z/(P_{0}l)$ 再积分,得

$$\frac{1}{P_{0}l}\left[(E_{1}+E_{2})\int_{A}z\sigma\mathrm{d}A+\eta\left(\int_{A}z\sigma_{,t}\mathrm{d}A+\gamma\int_{A}z\sigma_{,x}\mathrm{d}A\right)\right]=$$
$$-\frac{E_{1}I}{P_{0}l}[E_{2}v_{,xx}+\eta(v_{,xxt}+\gamma v_{,xxx})], \tag{14}$$

其中, I 为梁的惯性矩,其表达式为

$$I=\int_{A}z^{2}\mathrm{d}A. \tag{15}$$

方程(12)两边乘 A/P_{0} 导出

$$\frac{A}{P_{0}}[(E_{1}+E_{2})\sigma+\eta(\sigma_{,t}+\gamma\sigma_{,x})]=\frac{1}{2}\frac{AE_{1}}{P_{0}}[E_{2}(v_{,x}^{2})+\eta(v_{,x}^{2})_{,t}+\gamma\eta(v_{,x}^{2})_{,x}]. \tag{16}$$

引入无量纲变量

$$\begin{cases}v\leftrightarrow\dfrac{v}{\sqrt{\varepsilon}\,l},\ x\leftrightarrow\dfrac{x}{l},\ t\leftrightarrow t\sqrt{\dfrac{P_{0}}{\rho Al^{2}}},\ c=\gamma\sqrt{\dfrac{\rho A}{P_{0}}},\ \omega\leftrightarrow\omega\sqrt{\dfrac{\rho Al^{2}}{P_{0}}},\ f=\dfrac{Fl}{\varepsilon P_{0}},\\ \varsigma(x,t)=\dfrac{1}{P_{0}l}\displaystyle\int_{\mathrm{A}}z\sigma(x,z,t)\mathrm{d}A,\ \zeta(x,t)=\dfrac{A\sigma}{P_{0}},\ \alpha=\dfrac{\eta}{\varepsilon(E_{1}+E_{2})}\sqrt{\dfrac{P_{0}}{\rho Al^{2}}},\\ E_{\mathrm{a}}=\dfrac{IE_{1}E_{2}}{P_{0}l^{2}(E_{1}+E_{2})},\ E_{\mathrm{b}}=\dfrac{IE_{1}}{P_{0}l^{2}},\ E_{\mathrm{c}}=\dfrac{E_{1}E_{2}A}{P_{0}(E_{1}+E_{2})},\ E_{\mathrm{d}}=\dfrac{E_{1}A}{P_{0}},\end{cases} \tag{17}$$

其中, ε 为无量纲小参数,且振幅 F 为 ε 的同阶. 因此,方程(8)和(14)及(16)分别转化为无量纲形式

$$v_{,tt}+2cv_{,xt}+(c^2-1)v_{,xx}-\varsigma_{,xx}=\varepsilon(\zeta v_{,x})_{,x}+\varepsilon f\cos(\omega t), \tag{18}$$

$$\varsigma+\varepsilon\alpha\varsigma_{,t}+\varepsilon\alpha c\varsigma_{,x}=-E_{\mathrm{a}}v_{,xx}-\varepsilon\alpha E_{\mathrm{b}}(v_{,xxt}+cv_{,xxx}), \tag{19}$$

$$\zeta+\varepsilon\alpha\zeta_{,t}+\varepsilon\alpha c\zeta_{,x}=\frac{1}{2}E_{\mathrm{c}}v_{,x}^2+\frac{1}{2}\varepsilon\alpha E_{\mathrm{d}}[(v_{,x}^2)_{,t}+c(v_{,x}^2)_{,x}]. \tag{20}$$

假设梁的两端均受简支约束,则其边界条件的无量纲形式为

$$v(0,t)=0,\ v(1,t)=0;\ v_{,xx}(0,t)=0,\ v_{,xx}(1,t)=0. \tag{21}$$

2 第 n 阶模态主共振的近似解

在简支边界条件(21)下, 使用多尺度法求解方程(18)和(19)及(20), 并定义时间尺度 $T=\varepsilon t$, 得

$$v(x,t;\varepsilon)=v_0(x,t,T)+\varepsilon v_1(x,t,T)+O(\varepsilon^2), \tag{22}$$

$$\varsigma(x,t;\varepsilon)=\varsigma_0(x,t,T)+\varepsilon\varsigma_1(x,t,T)+O(\varepsilon^2), \tag{23}$$

$$\zeta(x,t;\varepsilon)=\zeta_1(x,t,T)+O(\varepsilon). \tag{24}$$

将方程(22)代入方程(18),并令 ε^0 和 ε^1 的系数等于0,分别在 ε^0 和 ε^1 处,得

$$v_{0,tt}+2cv_{0,xt}+(c^2-1)v_{0,xx}-\varsigma_{0,xx}=0, \tag{25}$$

$$v_{1,tt}+2cv_{1,xt}+(c^2-1)v_{1,xx}-\varsigma_{1,xx}= \\ f\cos(\omega t)-2v_{0,tT}+v_{0,x}\zeta_{1,x}+2\gamma v_{1,xt}+v_{0,xx}\zeta_1. \tag{26}$$

同样,将方程(23)代入方程(19),分别在 ε^0 和 ε^1 处,得

$$\varsigma_0(x,t,T)=-E_{\mathrm{a}}v_{0,xx}, \tag{27}$$

$$\varsigma_1+\alpha\varsigma_{0,t}+\alpha\gamma\varsigma_{0,x}=-E_{\mathrm{a}}v_{1,xx}-\alpha E_{\mathrm{b}}v_{0,xxt}-\alpha\gamma E_{\mathrm{b}}v_{0,xxx}. \tag{28}$$

将方程(24)代入方程(20),在 ε^0 处,得

$$\zeta_1(x,t,T)=E_{\mathrm{c}}(v_{0,x})^2. \tag{29}$$

将方程(27)代入方程(25),并将方程(27)至方程(29)代入方程(26),导出

$$Mv_{0,tt}+Gv_{0,t}+Kv_0=0, \tag{30}$$

$$Mv_{1,tt}+Gv_{1,t}+Kv_1=-2v_{0,tT}-2cv_{0,xT}+f\cos(\omega t)+ \\ \frac{3}{2}E_{\mathrm{c}}(v_{0,x})^2v_{0,xx}+\alpha(E_{\mathrm{a}}-E_{\mathrm{b}})(v_{0,xxxxt}+cv_{0,xxxxx}), \tag{31}$$

其中,质量算子 M、陀螺算子 G 以及刚度算子 K 分别被定义为

$$M=I,\ G=2c\frac{\partial}{\partial x},\ K=(c^2-1)\frac{\partial^2}{\partial x^2}+E_{\mathrm{a}}\frac{\partial^4}{\partial x^4}. \tag{32}$$

系统在外激励作用下发生主共振,为了描述激励频率 ω 与固有频率 ω_n 的接近程度,引进解谐参数 σ, 并定义

$$\omega=\omega_n+\varepsilon\sigma, \tag{33}$$

其中, ω_n 为第 n 阶固有频率. 方程(30)的解可表示为

$$v_0(x,t,T)=\phi_n(x)A_n(T)\mathrm{e}^{\mathrm{i}\omega_n t}+\mathrm{cc}, \tag{34}$$

其中,cc 表示方程(34)右边所有项的复数共轭, $\phi_n(x)$ 为模态函数[10], $A_n(T)$ 为任意函数,

$$\phi_n(x)=\mathrm{e}^{\mathrm{i}r_{1n}x}-\frac{(r_{4n}^2-r_{1n}^2)(\mathrm{e}^{\mathrm{i}r_{3n}}-\mathrm{e}^{\mathrm{i}r_{1n}})}{(r_{4n}^2-r_{2n}^2)(\mathrm{e}^{\mathrm{i}r_{3n}}-\mathrm{e}^{\mathrm{i}r_{2n}})}\mathrm{e}^{\mathrm{i}r_{2n}x}-\frac{(r_{4n}^2-r_{1n}^2)(\mathrm{e}^{\mathrm{i}r_{2n}}-\mathrm{e}^{\mathrm{i}r_{1n}})}{(r_{4n}^2-r_{3n}^2)(\mathrm{e}^{\mathrm{i}r_{2n}}-\mathrm{e}^{\mathrm{i}r_{3n}})}\mathrm{e}^{\mathrm{i}r_{3n}x}- \\ \left[1-\frac{(r_{4n}^2-r_{1n}^2)(\mathrm{e}^{\mathrm{i}r_{3n}}-\mathrm{e}^{\mathrm{i}r_{1n}})}{(r_{4n}^2-r_{2n}^2)(\mathrm{e}^{\mathrm{i}r_{3n}}-\mathrm{e}^{\mathrm{i}r_{2n}})}-\frac{(r_{4n}^2-r_{1n}^2)(\mathrm{e}^{\mathrm{i}r_{2n}}-\mathrm{e}^{\mathrm{i}r_{1n}})}{(r_{4n}^2-r_{3n}^2)(\mathrm{e}^{\mathrm{i}r_{2n}}-\mathrm{e}^{\mathrm{i}r_{3n}})}\right]\mathrm{e}^{\mathrm{i}r_{4n}x}, \tag{35}$$

其中，$r_{in}(i=1,2,3,4)$ 是如下 4 次代数方程的根：

$$E_a r_{in}^4 - (c^2-1)r_{in}^2 - 2c\omega_n r_{in} - \omega_n^2 = 0. \tag{36}$$

将方程(33)和方程(34)代入方程(31)，得

$$\begin{aligned}&Mv_{1,tt} + Gv_{1,t} + Kv_1 = \\&\quad\Big[-2(\mathrm{i}\omega_n\phi_n + c\phi_n')\dot{A}_n + \alpha A_n(E_a - E_b)(\mathrm{i}\omega_n\phi_n^{(4)} + c\phi_n^{(5)}) + \\&\quad\frac{3}{2}E_c A_n^2\bar{A}_n(2\phi_n'\bar{\phi}_n'\phi_n'' + \phi_n'^2\bar{\phi}_n'') + \frac{1}{2}\mathrm{e}^{\mathrm{i}\sigma T}f\Big]\mathrm{e}^{\mathrm{i}\omega_n t} + \mathrm{NST} + \mathrm{cc},\end{aligned} \tag{37}$$

其中，“ · ”和“′”分别表示对 T 和 x 的导数；方程(37)必须满足如下可解性条件才有有界解[26]：

$$\begin{aligned}&\Big\langle -2\dot{A}_n(\mathrm{i}\omega_n\phi_n + c\phi_n') + \frac{1}{2}\mathrm{e}^{\mathrm{i}\sigma T}f + \frac{3}{2}E_c A_n^2\bar{A}_n(2\phi_n'\bar{\phi}_n'\phi_n'' + \phi_n'^2\bar{\phi}_n'') + \\&\quad A_n\alpha(E_a - E_b)(\mathrm{i}\omega_n\phi_n^{(4)} + c\phi_n^{(5)}),\ \phi_n\Big\rangle = 0,\end{aligned} \tag{38}$$

其中，引入复函数 g_1 和 g_2 在$[0,1]$上的内积定义$\langle g_1,g_2\rangle$：

$$\langle g_1,g_2\rangle = \int_0^1 g_1(x)\bar{g}_2(x)\,\mathrm{d}x. \tag{39}$$

对方程(38)应用内积分配律，导出

$$\dot{A}_n + \mu_n\alpha(E_b - E_a)A_n + \kappa_n E_c\bar{A}_n A_n^2 + \chi_n\mathrm{e}^{\mathrm{i}\sigma T}f = 0, \tag{40}$$

其中

$$\begin{cases}\mu_n = \dfrac{c\int_0^1\bar{\phi}_n\phi_n^{(5)}\mathrm{d}x + \mathrm{i}\omega_n\int_0^1\bar{\phi}_n\phi_n^{(4)}\mathrm{d}x}{2\int_0^1(\mathrm{i}\omega_n\phi_n + c\phi_n')\bar{\phi}_n\mathrm{d}x},\ \chi_n = \dfrac{-\frac{1}{2}\int_0^1\bar{\phi}_n\mathrm{d}x}{2\int_0^1(\mathrm{i}\omega_n\phi_n + c\phi_n')\bar{\phi}_n\mathrm{d}x},\\[2ex]\kappa_n = \dfrac{-\frac{3}{2}\int_0^1\bar{\phi}_n\phi_n'^2\bar{\phi}_n''\mathrm{d}x - 3\int_0^1\bar{\phi}_n\phi_n'\bar{\phi}_n'\phi_n''\mathrm{d}x}{2\int_0^1(\mathrm{i}\omega_n\phi_n + c\phi_n')\bar{\phi}_n\mathrm{d}x}.\end{cases} \tag{41}$$

显然，表达式(41)由边界条件(21)下的线性系统(30)的固有频率和模态函数决定，它们与平均速度 c 和刚度系数 E_a 有关，而与黏性阻尼 α 和非线性 E_c 无关。利用数值方法容易证明

$$\mathrm{Re}(\mu_n) > 0,\ \mathrm{Im}(\mu_n) = 0;\ \mathrm{Re}(\kappa_n) = 0,\ \mathrm{Im}(\kappa_n) < 0. \tag{42}$$

控制方程(13)的主共振问题也可以用同样的方法分析。可解性条件依然满足方程(40)，且系数由方程(41)定义，但 κ_n 由如下表达式定义：

$$\kappa_n = \frac{-\frac{1}{2}\int_0^1\phi_n'^2\mathrm{d}x\int_0^1\bar{\phi}_n\bar{\phi}_n''\mathrm{d}x - \int_0^1\phi_n'\bar{\phi}_n'\mathrm{d}x\int_0^1\bar{\phi}_n\phi_n''\mathrm{d}x}{2\int_0^1(\mathrm{i}\omega_n\phi_n + c\phi_n')\bar{\phi}_n\mathrm{d}x}. \tag{43}$$

3 稳态响应及稳定性的数值算例

将复函数 $A_n(T)$ 表示成极坐标形式：

$$A_n(T) = a_n(T)\mathrm{e}^{\mathrm{i}\beta_n(T)}, \tag{44}$$

其中，$\alpha_n(T)$ 和 $\beta_n(T)$ 分别是稳态响应振幅和相角；将方程(44)代入方程(40)并分离实部和虚部，得

$$\begin{cases}\dot{a}_n = f[\operatorname{Im}(\chi_n)\sin(\theta_n) - \operatorname{Re}(\chi_n)\cos(\theta_n)] + \alpha\operatorname{Re}(\mu_n)(E_a - E_b)a_n = \Phi(a_n,\theta_n),\\ \dot{\theta}_n = f[\operatorname{Im}(\chi_n)\cos(\theta_n) + \operatorname{Re}(\chi_n)\sin(\theta_n)]/a_n + \\ \qquad \sigma + \operatorname{Im}(\kappa_n)E_c a_n^2 = \Psi(a_n,\theta_n),\end{cases} \tag{45}$$

其中

$$\theta_n = \sigma T - \beta_n. \tag{46}$$

对于稳态响应,方程(45)中的振幅 α_n 和新相角 θ_n 是常数;设 $\alpha_n = \alpha_{0n}$ 和 $\theta_n = \theta_{0n}$, 得

$$\begin{cases}0 = f[\operatorname{Im}(\chi_n)\sin(\theta_{0n}) - \operatorname{Re}(\chi_n)\cos(\theta_{0n})] + \alpha\operatorname{Re}(\mu_n)(E_a - E_b)a_{0n},\\ 0 = f[\operatorname{Im}(\chi_n)\cos(\theta_{0n}) + \operatorname{Re}(\chi_n)\sin(\theta_{0n})] + \sigma a_{0n} + \operatorname{Im}(\kappa_n)E_c a_{0n}^3.\end{cases} \tag{47}$$

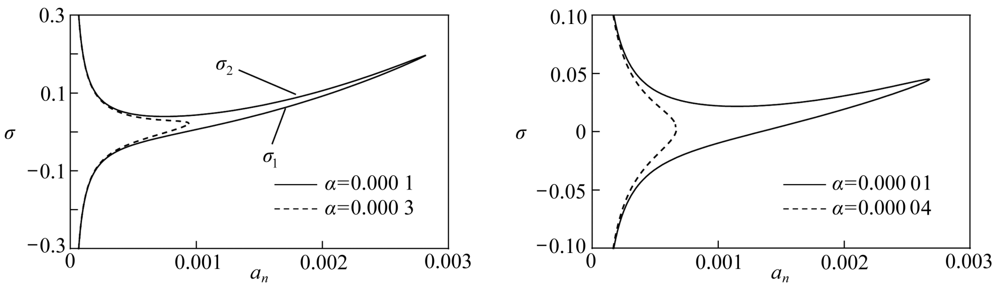
显然,方程(47)不存在平凡解,消去 θ_{0n} 导出

$$W(a_{0n},\sigma) = [\alpha(E_b - E_a)\operatorname{Re}(\mu_n)a_{0n}]^2 + [\operatorname{Im}(\kappa_n)E_c a_{0n}^3 - \sigma a_{0n}]^2 - f^2|\chi_n|^2 = 0. \tag{48}$$

此表达式称为稳态响应振幅和相角的模态方程,它的两个非平凡解可表示为

$$\begin{cases}\sigma_1 = -\operatorname{Im}(\kappa_n)E_c a_{0n}^2 - \dfrac{1}{a_{0n}}\sqrt{f^2|\chi_n|^2 - [\alpha\operatorname{Re}(\mu_n)(E_b - E_a)a_{0n}]^2},\\ \sigma_2 = -\operatorname{Im}(\kappa_n)E_c a_{0n}^2 + \dfrac{1}{a_{0n}}\sqrt{f^2|\chi_n|^2 - [\alpha\operatorname{Re}(\mu_n)(E_b - E_a)a_{0n}]^2}.\end{cases} \tag{49}$$

用梁中心的横向位移来描述运动梁周期稳态响应。考虑一轴向运动梁 $E_a = 0.64$ 和 $c = 2.0$ 受边界条件(21)约束。线性方程(30)的前两阶固有频率 $\omega_1 = 5.3692$ 和 $\omega_2 = 30.12$。方程(45)给出 $\mu_1 = 110.73$, $\chi_1 = 0.0182 + 0.0082\mathrm{i}$ 和 $\mu_2 = 1006.48$, $\chi_2 = 0.0009 + 0.0014\mathrm{i}$。对非线性偏微分模型(8),由方程(41)得到 $\kappa_1 = -61.8985\mathrm{i}$ 和 $\kappa_2 = -156.837\mathrm{i}$;对非线性积分-偏微分模型(13),可由方程(43),得到 $\kappa_1 = -40.9617\mathrm{i}$ 和 $\kappa_2 = -94.4142\mathrm{i}$。在所有图中,$E_1 = E_2 = 3$,且由方程(17)得 $E_b = E_a(E_1 + E_2)/E_2 = 1.28$。在图 3 至 6 中,给出了方程(49)中主共振的第 1 阶和第 2 阶模态频响曲线,其中响应 a_n 为解谐参数 σ 的函数。



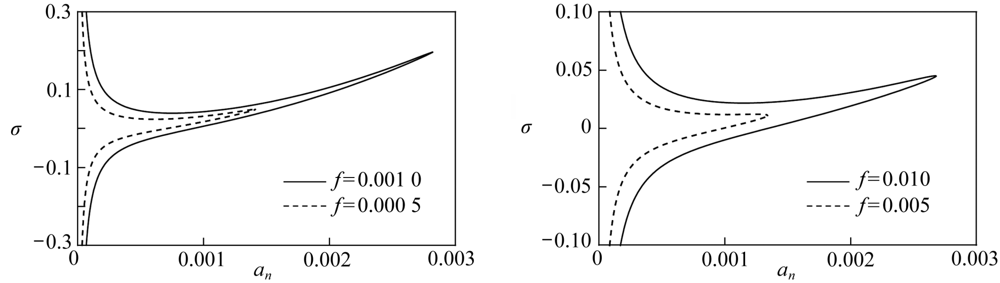
(a) 第 1 阶主共振($E_c = 400$, $f = 0.001$) (b) 第 2 阶主共振($E_c = 40$, $f = 0.01$)

(a) First modes with $E_c = 400$, $f = 0.001$ (b) Second modes with $E_c = 40$, $f = 0.01$

图 3 黏性阻尼影响

Fig. 3 Effect of viscous damping

当主共振发生时,由图 3 显示稳态响应的振幅随黏性阻尼的增大而变小,在固有频率 $\omega_n(\sigma = 0)$ 这种变化明显;图 4 显示增大外部激励,稳态响应的振幅将增大,包括远离固有频率 ω_n 处;图 5 显示增大非线性系数 E_c,稳态响应的振幅将减小;图 6 显示对于两种非线性模型,当指定相同的参数时,积分-偏微分模型的稳态振幅比偏微分模型更大。很明显,图 6 和图 5 具有



（a）第 1 阶主共振（$\alpha = 0.000\,1$，$E_c = 400$）

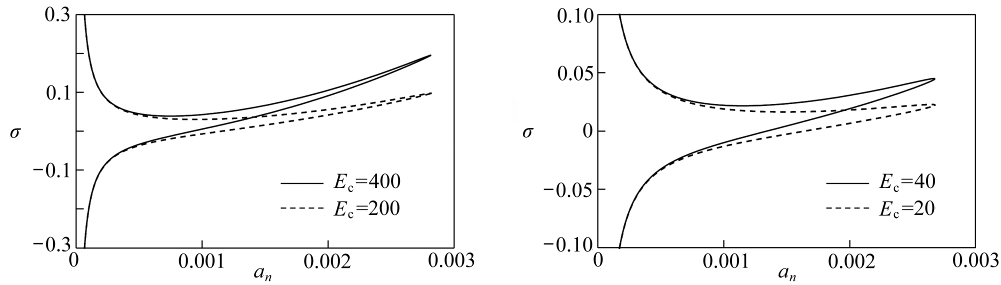
（a）First modes with $\alpha = 0.000\,1$，$E_c = 400$

（b）第 2 阶主共振（$\alpha = 0.000\,01$，$E_c = 40$）

（b）Second modes with $\alpha = 0.000\,01$，$E_c = 40$

图 4　外部激励影响

Fig. 4　Effect of force



（a）第 1 阶主共振（$\alpha = 0.000\,1$，$f = 0.001$）

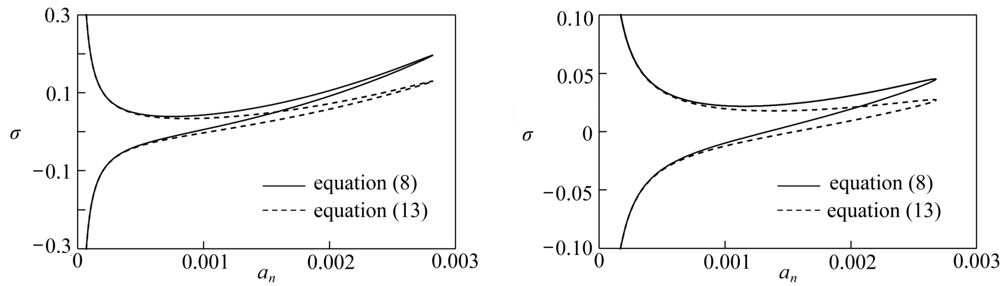
（a）First modes with $\alpha = 0.000\,1$，$f = 0.001$

（b）第 2 阶主共振（$\alpha = 0.000\,01$，$f = 0.01$）

（b）Second modes with $\alpha = 0.000\,01$，$f = 0.01$

图 5　非线性系数影响

Fig. 5　Effect of nonlinear coefficients



（a）第 1 阶主共振（$E_c = 400$，$\alpha = 0.000\,1$ 和 $f = 0.001$）

（a）First modes with $E_c = 400$，$\alpha = 0.000\,1$ and $f = 0.001$

（b）第 2 阶主共振（$E_c = 40$，$\alpha = 0.000\,01$ 和 $f = 0.01$）

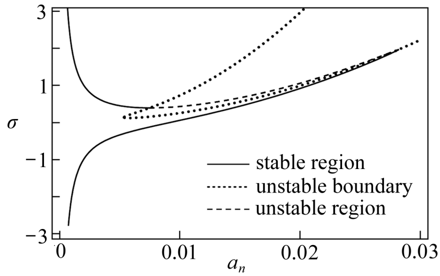
（b）Second modes with $E_c = 40$，$\alpha = 0.000\,01$ and $f = 0.01$

图 6　不同非线性模型

Fig. 6　Effect of nonlinear model

相似情况，两种模型的不同在于非线性项都存在方程（8）和方程（13）中，非线性系数 E_c 也存在它们中.

接下来，我们来研究稳态响应的稳定性问题，并构建如下形式的 Jacobi 矩阵：



（a）第 1 阶主共振及失稳边界（$E_c = 40$，$\alpha = 0.000\ 1$ 和 $f = 0.001$）

（a）First modes with unstable boundary $E_c = 40$，$\alpha = 0.000\ 1$ and $f = 0.001$

$$\begin{bmatrix} \partial\Phi/\partial a_n & \partial\Psi/\partial\theta_n \\ \partial\Phi/\partial a_n & \partial\Psi/\partial\theta_n \end{bmatrix}_{a_n = a_{0n},\ \theta_n = \theta_{0n}} . \qquad (50)$$

根据方程(47)，此矩阵的特征方程可表示为

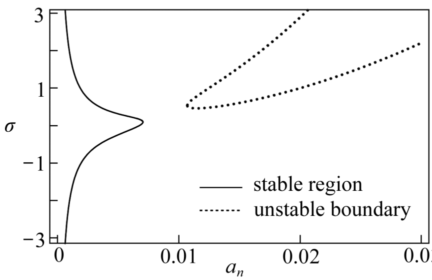
$$\lambda^2 + s_1\lambda + s_2 = 0, \qquad (51)$$

其中

$$s_1 = 2\mathrm{Re}(\mu_n)\alpha(E_b - E_a), \qquad (52)$$

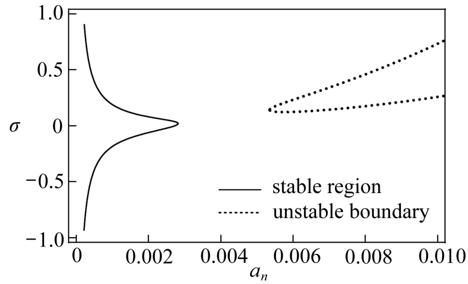
$$s_2 = \mathrm{Re}(\mu_n)^2\alpha^2(E_b - E_a)^2 + (\sigma + \mathrm{Im}(\kappa_n)E_c a_{0n}^2)(\sigma + 3\mathrm{Im}(\kappa_n)E_c a_{0n}^2). \qquad (53)$$

数值结果证明：在 σ_1 条件下，s_2 总是正数，因此依据 Routh-Hurwitz 定律，σ_1 曲线总是稳定的；但在 σ_2 条件下，σ_1 存在不稳定区域. 数值例子如图 7 所示：实线代表频响曲线的稳定区域，点线代表失稳边界（$s_2 = 0$），虚线代表失稳区域. 如果增大黏性阻尼和减小外激励，σ_2 曲线可以始终稳定.



（b）第 1 阶主共振及失稳边界（$E_c = 40$，$\alpha = 0.004$ 和 $f = 0.1$）

（b）First modes with unstable boundary $E_c = 40$，$\alpha = 0.004$ and $f = 0.1$



（c）第 1 阶主共振及失稳边界（$E_c = 40$，$\alpha = 0.001$ 和 $f = 0.01$）

（c）First modes with unstable boundary $E_c = 40$，$\alpha = 0.001$ and $f = 0.01$

图 7 第 1 阶模态的频响曲线和失稳曲线

Fig. 7 Frequency-response curve and stability for case of primary resonance

4 结 论

使用多尺度法研究了轴向运动三参数黏弹性梁弱受迫振动的稳态响应和稳定性. 数值结果显示：(i) 增大黏性阻尼或非线性系数将使稳态振幅变小；它们能约束主共振发生时横向位移发生大幅度波动；(ii) 增大外力将使稳态振幅在整个频域内都变大；(iii) 在相同条件下，非线性积分-偏微分模型的振幅比偏微分模型更大；(iv) 增大黏性阻尼和减小外力能使稳态响应更稳定.

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Asymptotic Analysis on Weakly Forced Vibration of an Axially Moving Viscoelastic Beam Constituted by Standard Linear Solid Model

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Abstract: The weakly forced vibration of an axially moving viscoelastic beam was investigated. The viscoelastic material of beams was constituted by the standard linear solid model with the material time derivative involved. The nonlinear equations governing the transverse vibration were derived from dynamical, constitutive, and geometrical relations. The method of multiple scales was applied to determine the steady-state response. The modulation equation was derived from the solvability condition of eliminating secular terms. Closed-form expressions of the amplitude and existence condition of nontrivial steady-state response were derived from the modulation equation. The stability of nontrivial steady-state response was examined via Routh-Hurwitz criterion.

Key words: axially moving beam; weakly forced vibration; standard linear solid model; method of multiple scales; steady-state response