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任意厚度具有自由边叠层板的精确解析解*

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摘要: 自由边问题一直是三维弹性力学中的难题,通常很难满足自由边上一个正应力和两个剪应力都等于0. 基于三维弹性力学基本方程和状态空间方法,引入自由边界位移函数并考虑全部弹性常数,建立了正交异性具有自由边单层和叠层板的状态方程. 对状态方程中的变量以级数形式展开,通过边界条件的满足精确求解任意厚度具有自由边叠层板的位移和应力,此解满足层间应力和位移的连续条件. 算例计算表明,采用引入的位移函数形式,简化了计算过程并且采用较少的级数项可以获得收敛解. 与有限元方法计算结果进行了对比,可以得到较高精度的数值结果. 其解可以作为其它数值方法和半解析方法的参考解.

关　键　词: 叠层板; 自由边界; 状态方程; 精确解析解
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引　　言

随着新型复合材料的大量使用,在航空、机械和土木工程等领域,叠层结构构件得到了广泛应用. 由于复合材料通常具有各向异性、成层性和不均匀性等力学特点,使求解各向同性、均匀材料力学问题的理论和方法很难应用于复合材料叠层板、壳结构. 具有自由边的板、壳结构,自由边的应力分析通常比较困难,对于任意厚度具有自由边的叠层板则更为困难. 采用传统的三维弹性理论很难选择到合适的解函数满足自由边界应力条件,而且层间的连续条件也不易满足[1]. 经典的叠层板理论本质上是将三维问题简化为二维问题,忽略了横向剪切和法向变形的影响,只适用于薄层板. 尽管不同学者在此基础上进行了改进,但是仍然存在层间应力连续条件没有考虑,自由表面条件不满足等问题[2-3].

已有许多学者致力于板、壳结构三维弹性力学精确解的研究[4-6]. 状态空间方法从三维弹性力学基本方程出发,抛弃任何关于位移模式和应力分布的人为假设,可以直接从状态空间微分方程构造传递矩阵,可以得到叠层板上下表面状态空间变量,这些变量恰好是层间需要满足的位移和应力连续条件的物理量[7]. 因此,这种方法特别适用于求解叠层结构,并且不受构件厚度的限制,从而在该领域得到了广泛应用.

文献[7]基于弹性力学基本方程,引入状态空间理论,应用分离变量法建立了正交异性厚

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板的状态方程,精确求解了叠层正交异性板的静、动力问题,此解未知量少,只有 3 个,满足层间应力和位移的连续条件. 文献[8-11]采用状态空间方法对多种边界条件下任意厚度叠层板、壳的静、动力问题和稳定问题,给出了一系列精确解析解. 文献[12]则对任意厚度叠层板壳的精确理论、计算技巧和精度进行了系统的论述. 文献[13]通过引入位移和应力函数构造两类相互独立的状态空间方程,求解了四边简支横观各向同性矩形板的弯曲、振动和稳定问题. 文献[14]利用状态空间法分析了静力作用下四边简支压电叠层板的耦合特性. 文献[15]在厚度方向采用状态空间法,而在面内采用微分求积法给出了叠层板弯曲和振动的半解析解. 文献[16]则将有限条带法与状态空间方法结合,给出了简支叠层板静力作用下的半解析解.

具有自由边界的叠层板结构,其自由边的应力分析相对较为困难,通常很难满足自由边上一个正应力和两个剪应力都等于 0. 文献[12]在处理自由边时,所设边界函数中因含弹性常数,且所设函数较多,从而给处理层间连续条件带来诸多不便,而且推导繁琐,增加了求解的难度. 本文提出新的自由边界位移函数,不但克服了上述缺点,且大大加快了收敛速度. 通过数值结果对比,证明了本文的有效性.

1　单层板的状态方程

图 1 所示为正交异性板,坐标轴沿弹性主方向,z 轴向下,$x=0$,a 两边简支,$y=0$,b 两边自由. $V^{(0)}(x,z,t)$ 和 $V^{(b)}(x,z,t)$ 分别是两自由边沿 y 方向的位移函数. 把物理方程中的应变分量通过几何方程置换成位移分量后,得到的方程(1)称为弹性方程:

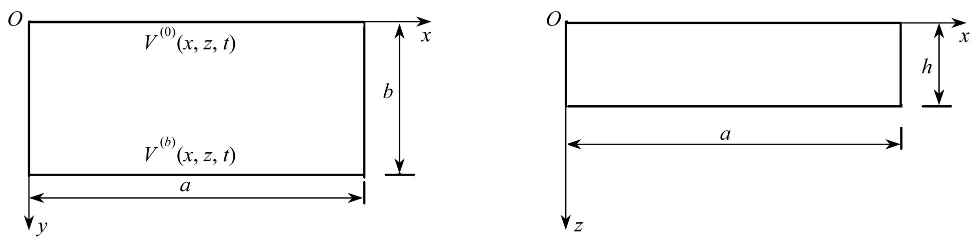


图 1　坐标系和位移函数

Fig. 1　Coordinate system and displacement function

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&0&0&0\\ C_{21}&C_{22}&C_{23}&0&0&0\\ C_{31}&C_{32}&C_{33}&0&0&0\\ 0&0&0&C_{44}&0&0\\ 0&0&0&0&C_{55}&0\\ 0&0&0&0&0&C_{66}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial U}{\partial x}\\ \dfrac{\partial V}{\partial y}\\ \dfrac{\partial W}{\partial z}\\ \dfrac{\partial V}{\partial z}+\dfrac{\partial W}{\partial y}\\ \dfrac{\partial U}{\partial z}+\dfrac{\partial W}{\partial x}\\ \dfrac{\partial U}{\partial y}+\dfrac{\partial V}{\partial x}\end{Bmatrix},\tag{1}$$

式中,C_{11},C_{12},C_{13},… 称为刚度系数. 把式(1)与三维弹性力学动力学平衡方程联立,消去平面

应力分量 $\sigma_x, \sigma_y, \tau_{xy}$，并记

$$X=\tau_{xz},\ Y=\tau_{yz},\ Z=\sigma_z,\ \alpha=\frac{\partial}{\partial x},\ \beta=\frac{\partial}{\partial y},\ \xi^2=\rho\frac{\partial^2}{\partial t^2},$$

得到

$$\frac{\partial}{\partial z}[U\quad V\quad Z\quad X\quad Y\quad W]^{\mathrm{T}}=\bar{\boldsymbol{D}}[U\quad V\quad Z\quad X\quad Y\quad W]^{\mathrm{T}},\tag{2}$$

其中

$$\bar{\boldsymbol{D}}=\begin{bmatrix}0 & 0 & 0 & C_8 & 0 & -\alpha\\ 0 & 0 & 0 & 0 & C_9 & -\beta\\ 0 & 0 & 0 & -\alpha & -\beta & \xi^2\\ \xi^2-C_2\alpha^2-C_6\beta^2 & -(C_3+C_6)\alpha\beta & C_1\alpha & 0 & 0 & 0\\ -(C_3+C_6)\alpha\beta & \xi^2-C_6\alpha^2-C_4\beta^2 & C_5\beta & 0 & 0 & 0\\ C_1\alpha & C_5\beta & C_7 & 0 & 0 & 0\end{bmatrix},\tag{3}$$

其中

$$\begin{cases}C_1=-\dfrac{C_{13}}{C_{33}},\ C_2=C_{11}-\dfrac{C_{13}^2}{C_{33}},\ C_3=C_{12}-\dfrac{C_{13}C_{33}}{C_{33}},\\ C_4=C_{22}-\dfrac{C_{23}^2}{C_{33}},\ C_5=-\dfrac{C_{23}}{C_{33}},\ C_6=C_{66},\\ C_7=\dfrac{1}{C_{33}},\ C_8=\dfrac{1}{C_{55}},\ C_9=\dfrac{1}{C_{44}}.\end{cases}\tag{4}$$

被消去的 $\sigma_x, \sigma_y, \tau_{xy}$ 由下式求出：

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}C_2\alpha & C_3\beta & -C_1\\ C_3\alpha & C_4\beta & -C_5\\ C_6\beta & C_6\alpha & 0\end{bmatrix}\begin{Bmatrix}U\\ V\\ Z\end{Bmatrix}.\tag{5}$$

由上面的矩阵方程可以证明各力学量的真解不可能是坐标变量的多项式。例如，若 X, Y, W 是坐标 z 的 l 次多项式，则由式(2)前3行和式(3)可以看出 U, V, Z 必为坐标 z 的 $l+1$ 次多项式。若如此，再由同一式的后3行可知 X, Y, W 又必是坐标 z 的 $l+2$ 次多项式，从而与原设矛盾。若把某些力学量设成是某坐标变量的多项式，会导致弹性力学基本方程之间的互不相容。这种理论上的近似造成的误差，随板壳厚度增加而剧增，从而使可解问题的范围受到很大限制。特别是对叠层板壳，随着层数增加未知量剧增，而且很难满足层间位移和应力连续条件。

为了满足边界条件，令

$$\begin{cases}V=\bar{V}+\left(1-\dfrac{y}{b}\right)^2V^{(0)}(x,z,t)+\left(\dfrac{y}{b}\right)^2V^{(b)}(x,z,t),\\ U=\bar{U}+\dfrac{b}{3}\left(1-\dfrac{y}{b}\right)^3\alpha V^{(0)}(x,z,t)-\dfrac{b}{3}\left(\dfrac{y}{b}\right)^3\alpha V^{(b)}(x,z,t).\end{cases}\tag{6}$$

上式代入式(2)，得到

$$\frac{\partial}{\partial z}[\bar{U}\quad \bar{V}\quad Z\quad X\quad Y\quad W]^{\mathrm{T}}=\bar{\boldsymbol{D}}[\bar{U}\quad \bar{V}\quad Z\quad X\quad Y\quad W]^{\mathrm{T}}+\boldsymbol{B},\tag{7}$$

列阵

$$\boldsymbol{B}=[B_1\quad B_2\quad 0\quad B_4\quad B_5\quad B_6]^{\mathrm{T}},$$

其中

$$\left\{\begin{aligned}
B_1&=-\frac{b}{3}\left(1-\frac{y}{b}\right)^3\frac{\partial}{\partial z}\alpha V^{(0)}(x,z,t)+\frac{b}{3}\left(\frac{y}{b}\right)^3\frac{\partial}{\partial z}V^{(b)}(x,z,t),\\
B_2&=-\left(1-\frac{y}{b}\right)^2\frac{\partial}{\partial z}V^{(0)}(x,z,t)-\left(\frac{y}{b}\right)^2\frac{\partial}{\partial z}V^{(b)}(x,z,t),\\
B_4&=-\frac{C_2b}{3}\left[\left(1-\frac{y}{b}\right)^3\alpha^3V^{(0)}(x,z,t)-\left(\frac{y}{b}\right)^3\alpha^3V^{(b)}(x,z,t)\right]+\\
&\quad\frac{2C_3}{b}\left[\left(1-\frac{y}{b}\right)\alpha V^{(0)}(x,z,t)-\left(\frac{y}{b}\right)\alpha V^{(b)}(x,z,t)\right]+\\
&\quad\zeta^2\frac{b}{3}\left[\left(1-\frac{y}{b}\right)^3\alpha V^{(0)}(x,z,t)-\left(\frac{y}{b}\right)^3\alpha V^{(b)}(x,z,t)\right],\\
B_5&=C_3\left[\left(1-\frac{y}{b}\right)^2\alpha^2V^{(0)}(x,z,t)+\left(\frac{y}{b}\right)^2\alpha^2V^{(b)}(x,z,t)\right]-\\
&\quad\frac{2C_4}{b^2}[V^{(0)}(x,z,t)+V^{(b)}(x,z,t)]+\\
&\quad\zeta^2\left[\left(1-\frac{y}{b}\right)^2V^{(0)}(x,z,t)+\left(\frac{y}{b}\right)^2V^{(b)}(x,z,t)\right],\\
B_6&=\frac{C_1b}{3}\left[\left(1-\frac{y}{b}\right)^3\alpha^2V^{(0)}(x,z,t)-\left(\frac{y}{b}\right)^3\alpha^2V^{(b)}(x,z,t)\right]-\\
&\quad\frac{2C_5}{b}\left[\left(1-\frac{y}{b}\right)V^{(0)}(x,z,t)-\left(\frac{y}{b}\right)V^{(b)}(x,z,t)\right].
\end{aligned}\right.\tag{8}$$

此时式(5)变成

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}C_2\alpha & C_3\beta & -C_1\\ C_3\alpha & C_4\beta & -C_5\\ C_6\beta & C_6\alpha & 0\end{bmatrix}\begin{Bmatrix}\bar U\\ \bar V\\ Z\end{Bmatrix}+$$

$$\begin{Bmatrix}\dfrac{C_2b}{3}\left[\left(1-\dfrac{y}{b}\right)^3\alpha^2V^{(0)}(x,z,t)-\left(\dfrac{y}{b}\right)^3\alpha^2V^{(b)}(x,z,t)\right]-\\ \qquad\dfrac{2C_3}{b}\left[\left(1-\dfrac{y}{b}\right)V^{(0)}(x,z,t)-\left(\dfrac{y}{b}\right)V^{(b)}(x,z,t)\right]\\ \dfrac{C_3b}{3}\left[\left(1-\dfrac{y}{b}\right)^3\alpha^2V^{(0)}(x,z,t)-\left(\dfrac{y}{b}\right)^3\alpha^2V^{(b)}(x,z,t)\right]-\\ \qquad\dfrac{2C_4}{b}\left[\left(1-\dfrac{y}{b}\right)V^{(0)}(x,z,t)-\left(\dfrac{y}{b}\right)V^{(b)}(x,z,t)\right]\\ 0\end{Bmatrix}.\tag{9}$$

现在把式(6)至式(9)中各相关量按下列级数展开:

$$
\left\{
\begin{aligned}
\bar{U} &= \sum_m \sum_n \bar{U}_{mn}(z) \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}\omega_{mn}t}, \\
\bar{V} &= \sum_m \sum_n \bar{V}_{mn}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}\omega_{mn}t}, \\
Z &= \sum_m \sum_n Z_{mn}(z) \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}\omega_{mn}t}, \\
X &= \sum_m \sum_n X_{mn}(z) \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}\omega_{mn}t}, \\
Y &= \sum_m \sum_n Y_{mn}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}\omega_{mn}t}, \\
W &= \sum_m \sum_n W_{mn}(z) \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \mathrm{e}^{\mathrm{i}\omega_{mn}t};
\end{aligned}
\right. \tag{10}
$$

$$
\left\{
\begin{aligned}
V^{(0)}(x,z,t) &= \sum_{m=1}^{\infty} V_m^{(0)}(z) \sin\frac{m\pi x}{a} \mathrm{e}^{\mathrm{i}\omega_{mn}t}, \\
V^{(b)}(x,z,t) &= \sum_{m=1}^{\infty} V_m^{(b)}(z) \sin\frac{m\pi x}{a} \mathrm{e}^{\mathrm{i}\omega_{mn}t};
\end{aligned}
\right. \tag{11}
$$

$$
1 = 1 + \sum_{n=1}^{\infty} 0 \cos\frac{n\pi y}{b}; \tag{12}
$$

$$
\left\{
\begin{aligned}
\frac{y}{b} &= \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{n^2} \cos\frac{n\pi y}{b}, \\
\left(\frac{y}{b}\right)^3 &= \frac{1}{4} + 6\sum_{n=1}^{\infty} \frac{(n^2\pi^2 - 2)\cos n\pi + 2}{n^4\pi^4} \cos\frac{n\pi y}{b}, \\
\left(1 - \frac{y}{b}\right)^3 &= \frac{1}{4} + 6\sum_{n=1}^{\infty} \frac{n^2\pi^2 + 2(\cos n\pi - 1)}{n^4\pi^4} \cos\frac{n\pi y}{b};
\end{aligned}
\right. \tag{13}
$$

$$
1 = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n} \sin\frac{n\pi y}{b}; \tag{14}
$$

$$
\left(\frac{y}{b}\right)^2 = -\frac{2}{\pi^3}\sum_{n=1}^{\infty} \frac{(n^2\pi^2 - 2)\cos n\pi + 2}{n^3} \sin\frac{n\pi y}{b}; \tag{15}
$$

$$
\left(1 - \frac{y}{b}\right)^2 = 2\sum_{n=1}^{\infty} \frac{n^2\pi^2 + 2(\cos n\pi - 1)}{n^3\pi^3} \sin\frac{n\pi y}{b}. \tag{16}
$$

把式(10)至式(16)代入式(7),对每对 m-n, 得到

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}\left[\bar{U}_{mn}(z) \quad \bar{V}_{mn}(z) \quad Z_{mn}(z) \quad X_{mn}(z) \quad Y_{mn}(z) \quad W_{mn}(z)\right]^{\mathrm{T}} = \\
&\qquad \boldsymbol{D}\left[\bar{U}_{mn}(z) \quad \bar{V}_{mn}(z) \quad Z_{mn}(z) \quad X_{mn}(z) \quad Y_{mn}(z) \quad W_{mn}(z)\right]^{\mathrm{T}} + \boldsymbol{B}_{mn}(z),
\end{aligned} \tag{17}
$$

式中

$$\boldsymbol{D}=\begin{bmatrix} 0 & 0 & 0 & C_8 & 0 & -\zeta \\ 0 & 0 & 0 & 0 & C_9 & \eta \\ 0 & 0 & 0 & \zeta & -\eta & -\rho\omega^2 \\ C_2\zeta^2+C_6\eta^2-\rho\omega^2 & -(C_3+C_6)\zeta\eta & C_1\zeta & 0 & 0 & 0 \\ -(C_3+C_6)\zeta\eta & C_6\zeta^2+C_4\eta^2-\rho\omega^2 & -C_5\eta & 0 & 0 & 0 \\ -C_1\zeta & C_5\eta & C_7 & 0 & 0 & 0 \end{bmatrix}, \tag{18}$$

式中　$\zeta=\dfrac{m\pi}{a},\ \eta=\dfrac{n\pi}{b},\ \omega=\omega_{mn};$

$$\boldsymbol{B}_{mn}(z)=[\,B_{mn}^{(1)}(z)\quad B_{mn}^{(2)}(z)\quad 0\quad B_{mn}^{(4)}(z)\quad B_{mn}^{(5)}(z)\quad B_{mn}^{(6)}(z)\,]^{\mathrm{T}},$$

其中

$$\begin{cases} B_{mn}^{(1)}(z)=-\dfrac{b\zeta}{12}\dfrac{\mathrm{d}}{\mathrm{d}z}\left(V_m^{(0)}(z)-V_m^{(b)}(z)\right), \\ B_{mn}^{(2)}(z)=B_{mn}^{(5)}(z)=0, \\ B_{mn}^{(4)}(z)=\dfrac{b\zeta(C_2\zeta^2-\rho\omega_{mn}^2)}{12}\left(V_m^{(0)}(z)-V_m^{(b)}(z)\right)+ \\ \qquad \dfrac{C_3\zeta}{b}\left(V_m^{(0)}(z)-V_m^{(b)}(z)\right), \\ B_{mn}^{(6)}(z)=-\dfrac{C_1b\zeta^2}{12}\left(V_m^{(0)}(z)-V_m^{(b)}(z)\right)-\dfrac{C_5}{b}\left(V_m^{(0)}(z)-V_m^{(b)}(z)\right); \end{cases} \qquad (n=0) \tag{19}$$

$$\begin{cases} B_{mn}^{(1)}(z)=-2b\zeta\,\dfrac{n^2\pi^2+2(\cos n\pi-1)}{n^4\pi^4}\dfrac{\mathrm{d}}{\mathrm{d}z}V_m^{(0)}(z)+ \\ \qquad 2b\zeta\,\dfrac{(n^2\pi^2-2)\cos n\pi+2}{n^4\pi^4}\dfrac{\mathrm{d}}{\mathrm{d}z}V_m^{(b)}(z), \\ B_{mn}^{(2)}(z)=-2\,\dfrac{n^2\pi^2+2(\cos n\pi-1)}{n^3\pi^3}\dfrac{\mathrm{d}}{\mathrm{d}z}V_m^{(0)}(z)+ \\ \qquad 2\,\dfrac{(n^2\pi^2-2)\cos n\pi+2}{n^3\pi^3}\dfrac{\mathrm{d}}{\mathrm{d}z}V_m^{(b)}(z), \\ B_{mn}^{(4)}(z)=2b\zeta(C_2\zeta^2-\rho\omega_{mn}^2)\left[\dfrac{n^2\pi^2+2(\cos n\pi-1)}{n^4\pi^4}V_m^{(0)}(z)-\right. \\ \qquad \left.\dfrac{(n^2\pi^2-2)\cos n\pi+2}{n^4\pi^4}V_m^{(b)}(z)\right]-\dfrac{4C_3\zeta}{b}\dfrac{\cos n\pi-1}{n^2\pi^2}\left[V_m^{(0)}(z)+V_m^{(b)}(z)\right], \\ B_{mn}^{(5)}(z)=-2(C_3\zeta^2+\rho\omega_{mn}^2)\left[\dfrac{n^2\pi^2+2(\cos n\pi-1)}{n^3\pi^3}V_m^{(0)}(z)-\right. \\ \qquad \left.\dfrac{(n^2\pi^2-2)\cos n\pi+2}{n^3\pi^3}V_m^{(b)}(z)\right]-\dfrac{4C_4}{b^2}\dfrac{1-\cos n\pi}{n\pi}\left[V_m^{(0)}(z)+V_m^{(b)}(z)\right], \\ B_{mn}^{(6)}(z)=-2C_1b\zeta^2\left[\dfrac{n^2\pi^2+2(\cos n\pi-1)}{n^4\pi^4}V_m^{(0)}(z)-\right. \\ \qquad \left.\dfrac{(n^2\pi^2-2)\cos n\pi+2}{n^4\pi^4}V_m^{(b)}(z)\right]+\dfrac{4C_5}{b}\dfrac{\cos n\pi-1}{n^2\pi^2}\left[V_m^{(0)}(z)+V_m^{(b)}(z)\right], \end{cases}$$

$$(n\neq 0). \tag{20}$$

方程(17)即本问题的状态方程.

现在考察边界条件.由式(6)、(9)和式(10)可以看到,在简支边($x=0,a$)处,$\sigma_x=W=V=0$已满足.在自由边($y=0,b$)处,$\tau_{yz}=0$和$\tau_{xy}=0$也已满足,尚待满足的边界条件是$\sigma_y=0$.由此定解$V^{(0)}(x,z,t)$和$V^{(b)}(x,z,t)$,再通过式(11)至(16)可以看到,实际上是定解$V_m^{(0)}(z)$和$V_m^{(b)}(z)$.

2 叠层板的状态方程及其解

叠层板由p层正交异性材料组成,坐标轴平行弹性主方向,$x=0,a$两边简支,$y=0,b$两边自由.上节中论述的$V_m^{(0)}(z)$和$V_m^{(b)}(z)$是自由边位移函数沿厚度z方向的分布规律,需要通过边界条件的满足来确定.然而,不论是怎样分布规律,只要沿厚度方向把板分成足够多的薄层,在每个薄层内,这些函数总可以认为是线性分布的.为此,把任一层j($j=1,2,\cdots,p$)等分成k_j个薄层,叠层板共分成薄层数$k=k_1+k_2+\cdots+k_p$.为了简化程序计算,把每个薄层的厚度都取为d,k的数目或薄层厚度d根据精度要求通过试算确定.现将第1层中的第1个薄层示于图2.根据式(17),其状态方程是

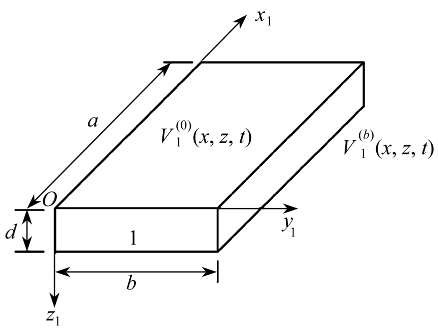


图2 第一层内第一个薄层示意图

Fig. 2 First thin layer in the first layer

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\bar{U}_{mn}(z)\quad \bar{V}_{mn}(z)\quad Z_{mn}(z)\quad X_{mn}(z)\quad Y_{mn}(z)\quad W_{mn}(z)\right]_1^{\mathrm{T}}=$$
$$\boldsymbol{D}_1\left[\bar{U}_{mn}(z)\quad \bar{V}_{mn}(z)\quad Z_{mn}(z)\quad X_{mn}(z)\quad Y_{mn}(z)\quad W_{mn}(z)\right]_1^{\mathrm{T}}+\{\boldsymbol{B}_{mn}(z)\}_1,\tag{21}$$

式中,$\boldsymbol{D}_1$的表达式见式(18),其中弹性模量应根据第1个薄层的材料计算.列阵$\{\boldsymbol{B}_{mn}(z)\}_1$只要将式(19)和(20)中的$V_m^{(0)}(z)$和$V_m^{(b)}(z)$换成$V_{m1}^{(0)}(z)$和$V_{m1}^{(b)}(z)$即可.因为$d$足够小,故可认为$V_{m1}^{(0)}(z)$和$V_{m1}^{(b)}(z)$沿厚度方向是线性分布的,即有

$$\begin{cases}V_{m1}^{(0)}(z)=A_{m1}-\dfrac{A_{m1}-A_{m2}}{d}z,\\ V_{m1}^{(b)}(z)=B_{m1}-\dfrac{B_{m1}-B_{m2}}{d}z,\end{cases}\qquad z\in[0,d],\tag{22}$$

$A_{m1},A_{m2},\cdots,A_{m,k+p}$;$B_{m1},B_{m2},\cdots,B_{m,k+p}$等是线性函数在薄层端点的函数值,它们应由边界条件的满足来确定.像这样的待定常数有$2(k+p)$个.需要说明的是,考虑不同材料交界面处y方向位移的连续性,即j层下表面和$j+1$层上表面y方向的位移相等,前述的待定常数应减去$2(p-1)$,所以实际上独立待定常数共有$2(k+1)$个.

方程(21)的解是

$$\boldsymbol{R}_1(z)=\boldsymbol{G}_1(z)\boldsymbol{R}_1(0)+\boldsymbol{C}_1(z),\qquad z\in[0,d],\tag{23}$$

其中

$$\boldsymbol{R}_1(z)=\left[\bar{U}_{mn}(z)\quad \bar{V}_{mn}(z)\quad Z_{mn}(z)\quad X_{mn}(z)\quad Y_{mn}(z)\quad W_{mn}(z)\right]_1^{\mathrm{T}},\tag{24}$$

$$\boldsymbol{R}_1(0)=[\bar{U}_{mn}(0)\quad \bar{V}_{mn}(0)\quad Z_{mn}(0)\quad X_{mn}(0)\quad Y_{mn}(0)\quad W_{mn}(0)]_1^{\mathrm{T}},\tag{25}$$

$$\boldsymbol{G}_1(z)=\mathrm{e}^{\boldsymbol{D}_1\cdot z},\tag{26}$$

$$\boldsymbol{C}_1(z)=\int_0^z \mathrm{e}^{\boldsymbol{D}_1\cdot(z-\tau)}\{\boldsymbol{B}_{mn}(\tau)\}_1\mathrm{d}\tau.\tag{27}$$

对每个薄层进行类似的推导,再利用薄层间应力和位移的连续条件,最后可把第 k 个薄层下表面和第 1 个薄层上表面的力学量用下式联系起来:

$$\boldsymbol{R}_k(d)=\boldsymbol{\Pi}\boldsymbol{R}_1(0)+\bar{\boldsymbol{\Pi}},\tag{28}$$

式中

$$\boldsymbol{\Pi}=\prod_{i=k}^{1}\boldsymbol{G}_i(d),\tag{29}$$

$$\bar{\boldsymbol{\Pi}}=\prod_{i=k}^{2}\boldsymbol{G}_i(d)\boldsymbol{C}_1(d)+\prod_{i=k}^{3}\boldsymbol{G}_i(d)\boldsymbol{C}_2(d)+\cdots+\boldsymbol{G}_1(d)\boldsymbol{C}_{k-1}(d)+\boldsymbol{C}_k(d).\tag{30}$$

式(28)中 $\boldsymbol{R}_1(0)$ 为初始值,$\boldsymbol{\Pi}$ 是 6 阶方阵,$\bar{\boldsymbol{\Pi}}$ 为 6 阶列阵,含待定常数 A_{mi},B_{mi}($i=1,2,\cdots,k+p$). 在通常情况下作用在板上、下表面的外力是预先给定的,所以,式(28)实际上是板上、下表面 6 个位移分量以及上述待定常数的矩阵方程. 展开式(28),取出其中第 3 ~5 行,经简单运算后,有

$$\begin{Bmatrix}\bar{U}_{mn}(0)\\ \bar{V}_{mn}(0)\\ W_{mn}(0)\end{Bmatrix}=$$

$$\begin{bmatrix}\Pi_{31}&\Pi_{32}&\Pi_{36}\\ \Pi_{41}&\Pi_{42}&\Pi_{46}\\ \Pi_{51}&\Pi_{52}&\Pi_{56}\end{bmatrix}^{-1}\left(\begin{Bmatrix}Z_{mn}(d)\\ X_{mn}(d)\\ Y_{mn}(d)\end{Bmatrix}-\begin{bmatrix}\Pi_{33}&\Pi_{34}&\Pi_{35}\\ \Pi_{43}&\Pi_{44}&\Pi_{45}\\ \Pi_{53}&\Pi_{54}&\Pi_{55}\end{bmatrix}\begin{Bmatrix}Z_{mn}(0)\\ X_{mn}(0)\\ Y_{mn}(0)\end{Bmatrix}-\begin{Bmatrix}\bar{\Pi}_3\\ \bar{\Pi}_4\\ \bar{\Pi}_5\end{Bmatrix}\right).\tag{31}$$

当板上表面受均布压力 q 作用时,把 q 按式(10) 中的 Z- 级数形式展开:

$$q=\sum_m\sum_n q_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}.$$

当 $n=0$ 时,有

$$q_{m0}=\frac{4q}{m\pi}\qquad(m=1,3,5,\cdots);$$

当 $n\neq 0$ 时,$q_{mn}=0$,此时有 $Z_{mn}(0)=-q_{m0}=-4q/(m\pi)$,$X_{mn}(0)=Y_{mn}(0)=0$,$X_{mn}(d)=Y_{mn}(d)=Z_{mn}(d)=0$, 式(31)简化为

$$\begin{Bmatrix}\bar{U}_{mn}(0)\\ \bar{V}_{mn}(0)\\ W_{mn}(0)\end{Bmatrix}=\begin{bmatrix}\Pi_{31}&\Pi_{32}&\Pi_{36}\\ \Pi_{41}&\Pi_{42}&\Pi_{46}\\ \Pi_{51}&\Pi_{52}&\Pi_{56}\end{bmatrix}^{-1}\left(\frac{4q}{m\pi}\begin{Bmatrix}\Pi_{33}\\ \Pi_{43}\\ \Pi_{53}\end{Bmatrix}-\begin{Bmatrix}\bar{\Pi}_3\\ \bar{\Pi}_4\\ \bar{\Pi}_5\end{Bmatrix}\right).\tag{32}$$

通过边界条件的满足求出待定常数,即求出 $[\bar{\Pi}_3\quad \bar{\Pi}_4\quad \bar{\Pi}_5]^{\mathrm{T}}$ 后,则$[\bar{U}_{mn}(0)\quad \bar{V}_{mn}(0)\quad W_{mn}(0)]^{\mathrm{T}}$

可由上式求出，于是初始值 $\boldsymbol{R}_1(0)$ 为已知，初始值求出后，一切问题均可求得。

剩下的问题是通过边界条件 $\sigma_y = 0$ 的满足求式(22) 中的待定常数。为此，由式(9) 第2式求出 σ_y 表达式，在考虑式(11) 后，对于每个 m，在 $y = 0$ 处和 $y = b$ 处分别得到

$$\begin{cases} \sum_n \left[-C_3\zeta\bar{U}_{mn}(z) + C_4\beta\bar{V}_{mn}(z) - C_5 Z_{mn}(z) \right] - \\ \qquad \dfrac{C_3 b}{3}\zeta^2 V_m^{(0)}(z) - \dfrac{2C_4}{b} V_m^{(0)}(z) = 0, \\ \sum_n (-1)^n \left[-C_3\zeta\bar{U}_{mn}(z) + C_4\beta\bar{V}_{mn}(z) - C_5 Z_{mn}(z) \right] + \dfrac{C_3 b}{3}\zeta^2 V_m^{(b)}(z) + \\ \qquad \dfrac{2C_4}{b} V_m^{(b)}(z) = 0. \end{cases} \tag{33}$$

利用对称性知 $V_m^{(0)}(z) = -V_m^{(b)}(z)$，故知当 n 为偶数时式(33) 中二式相同，可用其中第1式作为定解方程。当 $V_m^{(0)}(z) = -V_m^{(b)}(z)$ 时，式(19) 和(20) 可简化成十分简单的形式。本问题的定解方程是

$$\sum_n \left[-C_3\zeta\bar{U}_{mn}(z) + C_4\beta\bar{V}_{mn}(z) - C_5 Z_{mn}(z) \right] - \frac{C_3 b}{3}\zeta^2 V_m^{(0)}(z) - \frac{2C_4}{b} V_m^{(0)}(z) = 0 \qquad (m = 1,3,5,\cdots; n = 0,2,4,\cdots). \tag{34}$$

为了写出式(34) 中的 $\bar{U}_{mn}(z)$，$\bar{V}_{mn}(z)$ 和 $Z_{mn}(z)$ 的表达式，需将板内任一薄层的力学量用初始值表示。仿照式(28) 和(29) 的推导，第 i 层内的力学量和初始值的关系为

$$\boldsymbol{R}_i(z) = \boldsymbol{\Pi}_i(z)\boldsymbol{R}_1(0) + \bar{\boldsymbol{\Pi}}_i(z), \tag{35}$$

其中

$$\boldsymbol{\Pi}_i(z) = \boldsymbol{G}_i(z)\left[\boldsymbol{G}_{i-1}(d)\cdots\boldsymbol{G}_2(d)\boldsymbol{G}_1(d)\right], \tag{36}$$

$$\bar{\boldsymbol{\Pi}}_i(z) = \boldsymbol{G}_i(z)\{[\boldsymbol{G}_{i-1}(d)\cdots\boldsymbol{G}_3(d)\boldsymbol{G}_2(d)]\boldsymbol{C}_1(d) + [\boldsymbol{G}_{i-1}(d)\cdots\boldsymbol{G}_4(d)\boldsymbol{G}_3(d)]\boldsymbol{C}_2(d) + \cdots + \boldsymbol{C}_{i-1}(d)\} + \boldsymbol{C}_i(z). \tag{37}$$

展开式(35)，取出其中前 3 行，得

$$\begin{Bmatrix} \bar{U}_{mn}(z) \\ \bar{V}_{mn}(z) \\ Z_{mn}(z) \end{Bmatrix}_i = \begin{bmatrix} \Pi_{11}(z) & \Pi_{12}(z) & \Pi_{16}(z) \\ \Pi_{21}(z) & \Pi_{22}(z) & \Pi_{26}(z) \\ \Pi_{31}(z) & \Pi_{32}(z) & \Pi_{36}(z) \end{bmatrix}_i \begin{Bmatrix} \bar{U}_{mn}(0) \\ \bar{V}_{mn}(0) \\ W_{mn}(0) \end{Bmatrix} + \begin{bmatrix} \Pi_{13}(z) & \Pi_{14}(z) & \Pi_{15}(z) \\ \Pi_{23}(z) & \Pi_{24}(z) & \Pi_{25}(z) \\ \Pi_{33}(z) & \Pi_{34}(z) & \Pi_{35}(z) \end{bmatrix}_i \begin{Bmatrix} Z_{mn}(0) \\ X_{mn}(0) \\ Y_{mn}(0) \end{Bmatrix} + \begin{Bmatrix} \bar{\Pi}_1(z) \\ \bar{\Pi}_2(z) \\ \bar{\Pi}_3(z) \end{Bmatrix}_i. \tag{38}$$

把初始值的表达式(31)代入上式，得

$$\begin{Bmatrix} \bar{U}_{mn}(z) \\ \bar{V}_{mn}(z) \\ Z_{mn}(z) \end{Bmatrix}_i = \begin{bmatrix} \Pi_{11}(z) & \Pi_{12}(z) & \Pi_{16}(z) \\ \Pi_{21}(z) & \Pi_{22}(z) & \Pi_{26}(z) \\ \Pi_{31}(z) & \Pi_{32}(z) & \Pi_{36}(z) \end{bmatrix}_i \begin{bmatrix} \Pi_{31} & \Pi_{32} & \Pi_{36} \\ \Pi_{41} & \Pi_{42} & \Pi_{46} \\ \Pi_{51} & \Pi_{52} & \Pi_{56} \end{bmatrix}^{-1} \left(\begin{Bmatrix} Z_{mn}(d) \\ X_{mn}(d) \\ Y_{mn}(d) \end{Bmatrix} - \right.$$

$$
\left.\begin{bmatrix} \Pi_{33} & \Pi_{34} & \Pi_{35} \\ \Pi_{43} & \Pi_{44} & \Pi_{45} \\ \Pi_{53} & \Pi_{54} & \Pi_{55} \end{bmatrix} \begin{Bmatrix} Z_{mn}(0) \\ X_{mn}(0) \\ Y_{mn}(0) \end{Bmatrix} - \begin{Bmatrix} \bar{\Pi}_3 \\ \bar{\Pi}_4 \\ \bar{\Pi}_5 \end{Bmatrix} \right) +
$$

$$
\begin{bmatrix} \Pi_{13}(z) & \Pi_{14}(z) & \Pi_{15}(z) \\ \Pi_{23}(z) & \Pi_{24}(z) & \Pi_{25}(z) \\ \Pi_{33}(z) & \Pi_{34}(z) & \Pi_{35}(z) \end{bmatrix}_i \begin{Bmatrix} Z_{mn}(0) \\ X_{mn}(0) \\ Y_{mn}(0) \end{Bmatrix} + \begin{Bmatrix} \bar{\Pi}_1(z) \\ \bar{\Pi}_2(z) \\ \bar{\Pi}_3(z) \end{Bmatrix}_i
$$

$$
(m,n = 0,1,2,\cdots; i = 1,2,\cdots,k). \tag{39}
$$

若板上表面受均布压力 q 作用,上式简化为

$$
\begin{Bmatrix} \bar{U}_{mn}(z) \\ \bar{V}_{mn}(z) \\ Z_{mn}(z) \end{Bmatrix}_i =
$$

$$
\begin{bmatrix} \Pi_{11}(z) & \Pi_{12}(z) & \Pi_{16}(z) \\ \Pi_{21}(z) & \Pi_{22}(z) & \Pi_{26}(z) \\ \Pi_{31}(z) & \Pi_{32}(z) & \Pi_{36}(z) \end{bmatrix}_i \begin{bmatrix} \Pi_{31} & \Pi_{32} & \Pi_{36} \\ \Pi_{41} & \Pi_{42} & \Pi_{46} \\ \Pi_{51} & \Pi_{52} & \Pi_{56} \end{bmatrix}^{-1} \left(\frac{4q}{m\pi} \begin{Bmatrix} \Pi_{33} \\ \Pi_{43} \\ \Pi_{53} \end{Bmatrix} - \begin{Bmatrix} \bar{\Pi}_3 \\ \bar{\Pi}_4 \\ \bar{\Pi}_5 \end{Bmatrix} \right) +
$$

$$
\begin{bmatrix} \Pi_{13}(z) & \Pi_{14}(z) & \Pi_{15}(z) \\ \Pi_{23}(z) & \Pi_{24}(z) & \Pi_{25}(z) \\ \Pi_{33}(z) & \Pi_{34}(z) & \Pi_{35}(z) \end{bmatrix}_i \begin{Bmatrix} Z_{mn}(0) \\ X_{mn}(0) \\ Y_{mn}(0) \end{Bmatrix} + \begin{Bmatrix} \bar{\Pi}_1(z) \\ \bar{\Pi}_2(z) \\ \bar{\Pi}_3(z) \end{Bmatrix}_i
$$

$$
(m = 1,3,5,\cdots;\ n = 0,2,4,\cdots;\ j = 1,2,\cdots,k). \tag{40}
$$

对于每对 m-n,在式(39)中,令 z 分别等于各个薄层端点的 z 坐标值,并将它代入式(34)中,便得到关于 A_{mi} 的一个方程,令 $i = 1,2,\cdots,k$,总共得到 $k+1$ 个方程,用来定解相同数目的待定常数。当这些常数求出后,通过式(31)可求出初始值,于是可以求解出所有力学量。

3 算　　例

分别计算单层板和三层叠层板,$x=0,a$ 两边简支,$y=0,b$ 两边自由,板的上表面受均布力 q 作用。三层板的第1、3两层材料相同,每层都有如下的弹性常数:

$C_{12}/C_{11} = 0.246\ 269$, $C_{13}/C_{11} = 0.083\ 171\ 5$,

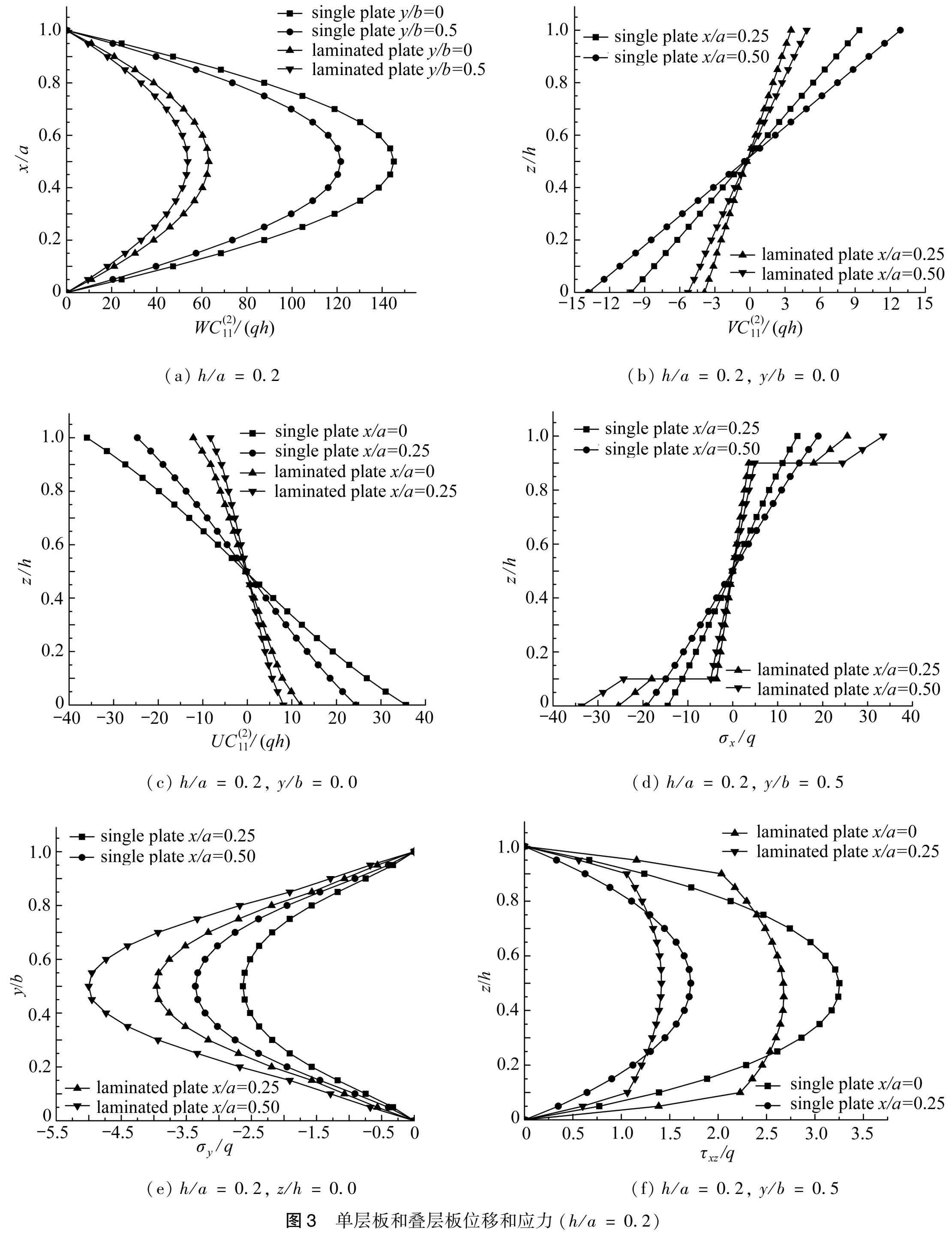
$C_{22}/C_{11} = 0.543\ 103$, $C_{23}/C_{11} = 0.115\ 017$,

$C_{33}/C_{11} = 0.530\ 172$, $C_{44}/C_{11} = 0.266\ 810$,

$C_{55}/C_{11} = 0.159\ 914$, $C_{66}/C_{11} = 0.262\ 931$,

三层板的 $C_{11}^{(1)}/C_{11}^{(2)} = 5$,几何参数是 $a=b$, $h_1 = h_3 = 0.1h$, $h_2 = 0.8\ h$。高跨比 h/a 分别取0.1,0.2和0.4进行计算。对应的级数取值为 $m = 1,3,5,\cdots,9$; $n = 0,2,4,\cdots,16$。

高跨比 h/a 取 0.2 时,单层板和叠层板位移和应力的部分计算结果见图 3(a)~(f)。将计算结果与有限元方法解的结果进行了对比,采用两种方法计算的单层板和叠层板静力计算结果分别见表 1 和表 2,其中有限元方法解是用有限元计算软件 ANSYS 完成的。



(a) $h/a=0.2$　(b) $h/a=0.2$, $y/b=0.0$

(c) $h/a=0.2$, $y/b=0.0$　(d) $h/a=0.2$, $y/b=0.5$

(e) $h/a=0.2$, $z/h=0.0$　(f) $h/a=0.2$, $y/b=0.5$

图 3　单层板和叠层板位移和应力($h/a=0.2$)

Fig. 3　Displacements and stresses of single plates and laminated plates ($h/a=0.2$)

从图 3 可以看出,采用提出的边界位移函数,对状态方程中的变量以级数形式展开,通过

边界条件的满足可以求解任意厚度具有自由边单层板和叠层板的位移和应力,并且满足层间位移和应力连续条件. 图3(e)为 σ_y 的计算结果,可以看出满足自由边界条件 $\sigma_y=0$. 表1和表2对本文解和有限元方法解进行了对比,本文解与有限元解除边界应力有点差别外,内部各力学量的差别较小,而二者 τ_{xz} 的差别则很大. 主要原因为有限元方法一般对边界应力的计算并不准确,有限元方法 τ_{xz} 计算不准确的原因则是由于并不满足层间连续条件造成的.

与文献[12]相比,方便处理了自由边的边界条件以及叠层板的层间连续条件问题,而且采用较少的级数项可以取得收敛解,算例中 m 和 n 分别取9和16,各力学量基本收敛. 文献[12]类似的算例在计算过程中 m 和 n 分别取29和99.

表1 $x=0$, a 两边简支, $y=0$, b 两边自由的单层板的位移和应力

Table 1 Displacements and stresses of single plates

		$h/a=0.1$		$h/a=0.2$		$h/a=0.4$	
		present study	FEM	present study	FEM	present study	FEM
		$WC_{11}/(qh)$					
$x=a/2$, $y=b/2$	$z=0$	1 674.709 1	1 644.060 4	121.786 9	118.591 4	12.395 9	13.343 7
$x=a/2$, $y=0$	$z=0$	1 969.062 6	1 973.612 1	144.149 1	145.262 3	14.432 4	15.788 5
		σ_x/q					
$(x=a/2, y=b/2)$	$z=0$	−74.312 8	−70.838 0	−19.166 5	−18.010 2	−5.439 8	−4.759 5
	$z=0.1h$	−59.019 9	−58.852 0	−14.892 8	−14.807 4	−3.897 8	−3.768 3
	$z=0.9h$	58.929 7	58.801 4	14.810 6	14.766 1	3.790 4	3.738 4
	$z=h$	74.184 7	70.787 3	19.006 8	17.968 4	5.769 2	4.728 3
$(x=a/2, y=0)$	$z=0$	−76.464 3	−74.965 5	−18.951 8	−18.543 4	−4.890 9	−4.669 9
	$z=0.1h$	−60.321 0	−61.529 0	−14.472 3	−15.050 2	−3.334 3	−3.640 9
	$z=0.9h$	60.480 6	61.634 9	14.637 0	15.134 7	3.476 4	3.698 4
	$z=h$	76.594 0	75.071 3	19.057 4	18.627 9	4.556 4	4.727 8
		σ_y/q					
$(x=a/2, y=b/2)$	$z=0$	−13.728 7	−11.182 8	−3.360 6	−2.426 2	−0.844 0	−0.431 4
	$z=0.1h$	−10.894 6	−10.484 6	−2.592 4	−2.306 2	−0.569 7	−0.426 1
	$z=0.9h$	10.866 7	10.470 4	2.567 7	2.296 5	0.544 7	0.420 1
	$z=h$	13.692 1	11.168 7	3.317 2	2.416 6	0.997 2	0.425 6
		$VC_{11}/(qh)$					
$(x=a/2, y=0)$	$z=0$	−93.009 2	−93.711 1	−13.822 9	−14.563 7	−2.438 1	−2.591 1
	$z=0.1h$	−74.549 2	−75.633 2	−11.120 5	−11.869 0	−1.969 6	−2.149 7
	$z=0.9h$	72.650 8	73.748 8	10.171 3	10.916 3	1.479 6	1.669 3
	$z=h$	91.105 3	91.826 6	12.867 1	13.610 8	1.900 8	2.109 7
		$UC_{11}/(qh)$					
$(x=a/4, y=0)$	$z=0$	197.319 6	199.390 3	24.370 8	24.913 0	3.119 2	3.163 9
	$z=0.1h$	155.580 8	157.823 8	18.589 6	19.385 2	2.114 6	2.339 3
	$z=0.9h$	−156.217 8	−157.819 9	−18.901 2	−19.369 0	−2.228 8	−2.315 2
	$z=h$	−197.941 7	−199.386 4	−24.667 1	−24.896 9	−3.174 7	−3.140 2
		τ_{xz}/q					
$(x=0, y=b/2)$	$z=0.1h$	2.579 0	1.002 4	1.389 2	0.201 1	0.897 5	0.059 4
	$z=0.9h$	2.535 6	0.810 0	1.239 1	0.160 7	0.590 3	0.046 9

注 本表结果采用三层板程序计算,即令 $C_{11}^{(1)}=C_{11}^{(2)}=C_{11}$.

Note The results are calculated with the program for three-layer plates, i.e. $C_{11}^{(1)}=C_{11}^{(2)}=C_{11}$.

表2 $x=0$, a 两边简支,$y=0$, b 两边自由的三层板的位移和应力

Table 2 Displacements and stresses of laminated plates

		$h/a=0.1$		$h/a=0.2$		$h/a=0.4$	
		present study	FEM	present study	FEM	present study	FEM
				$WC_{11}^{(2)}/(qh)$			
$x=a/2,\ y=b/2$	$z=0$	616.585 7	595.018 1	53.810 8	52.371 3	7.569 2	9.091 8
$x=a/2,\ y=0$	$z=0$	726.803 8	722.466 5	62.665 9	62.941 5	8.335 3	9.933 3
				σ_x/q			
$\begin{pmatrix}x=a/2,\\ y=b/2\end{pmatrix}$	$z=0$	-126.838 5	-120.197 9	-33.603 7	-30.720 7	-10.165 9	-8.243 8
	$z=0.1h+$	-98.892 4	-99.330 8	-24.269 5	-24.763 2	-5.528 1	-6.038 8
	$z=0.1h-$	-19.778 5	-19.866 2	-4.853 9	-4.952 6	-1.105 6	-1.207 8
	$z=0.9h-$	19.801 0	19.835 8	4.888 5	4.928 0	1.095 6	1.183 7
	$z=0.9h+$	99.005 1	99.178 9	24.442 5	24.640 0	5.478 2	5.918 5
	$z=h$	126.769 5	120.359 4	33.448 5	30.910 6	10.087 9	8.430 2
$\begin{pmatrix}x=a/2,\\ y=0\end{pmatrix}$	$z=0$	-127.436 4	-124.131 4	-31.503 8	-30.333 7	-8.523 6	-7.627 9
	$z=0.1h+$	-97.717 5	-102.576 8	-21.730 5	-24.469 9	-3.944 9	-5.558 6
	$z=0.1h-$	-19.543 5	-20.515 4	-4.346 1	-4.894 0	-0.789 0	-1.111 7
	$z=0.9h-$	19.665 2	20.553 0	4.474 3	4.924 3	0.875 4	1.131 1
	$z=0.9h+$	98.325 9	102.764 9	22.371 6	24.621 6	4.376 8	5.655 5
	$z=h$	127.878 6	124.586 2	31.869 8	30.726 4	8.473 7	7.944 2
				σ_y/q			
$\begin{pmatrix}x=a/2,\\ y=b/2\end{pmatrix}$	$z=0$	-22.301 5	-17.036 5	-4.998 9	-2.983 4	-1.132 8	-0.199 7
	$z=0.1h+$	-17.195 3	-16.460 0	-3.334 2	-3.269 3	-0.223 1	-0.604 0
	$z=0.1h-$	-3.439 1	-3.292 0	-0.666 8	-0.653 9	-0.044 6	-0.120 8
	$z=0.9h-$	3.502 6	3.277 0	0.733 2	0.640 8	0.104 0	0.105 0
	$z=0.9h+$	17.512 8	16.384 9	3.665 8	3.204 0	0.520 1	0.525 0
	$z=h$	22.587 3	17.395 4	5.272 4	3.352 2	1.456 7	0.557 6
				$VC_{11}^{(2)}/(qh)$			
$\begin{pmatrix}x=a/2,\\ y=0\end{pmatrix}$	$z=0$	-34.172 1	-35.406 2	-5.288 4	-5.624 8	-0.918 2	-0.931 4
	$z=0.1h$	-27.381 0	-28.873 5	-4.251 2	-4.705 2	-0.743 8	-0.843 2
	$z=0.9h$	26.420 5	27.844 5	3.787 1	4.200 3	0.518 0	0.602 6
	$z=h$	33.223 9	34.432 2	4.848 2	5.179 1	0.717 3	0.749 7
				$UC_{11}^{(2)}/(qh)$			
$\begin{pmatrix}x=a/4,\\ y=0\end{pmatrix}$	$z=0$	65.761 3	67.098 2	8.107 8	8.421 2	1.102 2	1.132 8
	$z=0.1h$	50.366 1	52.161 8	5.578 1	6.208 1	0.492 5	0.695 1
	$z=0.9h$	-50.718 7	-52.103 1	-5.745 6	-6.145 3	-0.548 3	-0.634 0
	$z=h$	-66.099 2	-67.039 4	-8.264 1	-8.359 4	-1.135 1	-1.073 0
				τ_{xz}/q			
$\begin{pmatrix}x=0,\\ y=b/2\end{pmatrix}$	$z=0.1h$	4.220 2	0.898 0	2.228 8	0.352 4	1.333 0	0.098 1
	$z=0.9h$	4.171 2	0.935 6	2.034 9	0.382 7	0.983 7	0.123 1

注 “+”和“-”分别表示分界面上的外层和内层.

Note “+” and “-” locked on the coordinates denote the outer and interior layers, respectively.

4 结论与讨论

通过设定简单合适自由边界位移函数,采用状态空间方法,可精确求解任意厚度具有自由边叠层板的解析解,而且大大简化了计算过程. 算例结果表明,本文方法可满足层间位移和应力连续条件以及所有边界条件,并且可以得到准确的计算结果. 本方法的计算结果可以作为其它数值算法和半解析算法的参考解,同时可以进一步应用于其它边界条件叠层板的求解以及振动与稳定问题求解,具有实际的科学和工程意义.

本文取 y 的多项式作为自由边界位移函数,但是,函数形式并不是唯一的,例如可取 y 的指数函数形式. 取何种形式应根据计算量小和级数收敛速度快的原则来确定.

自由边对三维问题来说,应该是 1 个正应力和两个剪应力全部为 0. 根据 St. Venant 原理可用弯矩为 0 代替正应力为 0,扭矩和剪力为 0 分别代替水平和垂直剪应力为 0,但是已经不是真正的自由边,因为弯矩、剪力和扭矩实际上在边界上是不存在的,特别是基于各种人为假设的板、壳理论,可能会导致弹性力学基本方程不能全部满足,从而导致自由边上的边界条件也不能全部满足,形成自由边上的边界效应. 本文满足全部弹性力学基本方程、层间位移和应力的连续条件以及自由边上的边界条件,不存在通常意义上的自由边的边界效应.

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Exact Analytic Solution for Laminated Plates With Free-Edges and Arbitrary Thickness

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Abstract: The problem of free-edges in three-dimensional elasticity is always a difficult one. The conditions that both normal stress and shear stress on the free edges equal zero are satisfied very difficultly. Based on the three-dimensional fundamental equations of elasticity and the state space method, the state equation for orthotropic plates was established through introduction boundary displacement function and consideration of all elastic constants of the orthotropic materials. Series expansion was carried out on the variables of the state equation. An exact solution was presented for laminated plates with arbitrary thickness by satisfaction of boundary conditions, which could also satisfy the continuous conditions of stresses and displacements between plies of the laminates. The results of two examples show that the calculation process is simplified and the convergent solution can be achieved with less terms of series when the displacement function of free boundary is adopted. The numerical results with high accuracy can be obtained through comparison with the results of finite element. The results can be used as reference to numerical methods and semi-analytical methods.

Key words: laminated plate; free-edges; state equation; exact analytic solution