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# 求解带有间断系数二维扩散方程的修正有限体积方法*

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**摘要**： 针对具有间断系数的二维扩散方程, 通过改进通量函数和调和平均系数的求解方法,提出了一种修正的有限体积方法.新方法得到的是无条件稳定的隐格式.数值实验结果表明该方法在处理间断系数问题时较经典的有限体积方法更为有效.

**关 键 词**： 二维间断系数扩散方程； 修正有限体积方法； 界面问题
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## 引 言

在自然科学中, 很多现象可以用扩散方程来描述.例如在分子的扩散现象中, 由于分子间无规则地碰撞, 分子从浓度高的地方向浓度低的地方传播, 温度场满足扩散方程;再比如河流、湖泊中的悬移质泥沙和污染物的输移过程也满足扩散方程.扩散方程的数值计算方法一直是计算数学和计算流体力学工作者研究的热门课题.国内外针对常系数扩散方程已经有了大量的研究并取得了较好的成果, 如高精度紧致差分格式[1-4]、交替分组显式方法[5-7]以及有限体积方法[8-9]等.而对于带间断扩散系数的扩散方程的研究还较少, 国内学者在这方面做过研究的诸如宋淑红和王双虎[10-11]发展的自适应有限体积方法, 这种方法对每个网格节点(1D)或者边(2D)用最小二乘“孪生逼近”方法构造多项式分布, 保证了温度连续和能流连续.对于二维大变形网格情况, 该格式能自适应地选取一个虚网格, 和虚网格相交的单元参与最小二乘“孪生逼近”, 并且根据模板点和边的距离确定权重的大小.自适应方法的优点是能较好地应对网格扭曲情形, 并且适用于间断扩散系数问题.赵强、袁光伟和董志伟[12]在经典的九点格式基础上, 基于对扩散通量连续假设的情况下构造出节点未知量的一种新的有限体积格式.在扭曲网格上, 该格式对具有连续或间断扩散系数的问题能够保持较高的精度.本文在经典的有限体积方法基础上, 通过改进对通量函数的求解方法构造出求解带间断系数扩散方程的新的差分格式.

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# 1　间断系数扩散方程的修正有限体积离散

## 1.1　问题描述

考虑二维间断系数扩散方程的初边值问题：

$$\begin{cases} \dfrac{\partial u}{\partial t} = k(x,y)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + f(x,y,t), \\ u(0,y,t) = \alpha_0(y,t),\ u(1,y,t) = \alpha_1(y,t), \\ u(x,0,t) = \beta_0(x,t),\ u(x,1,t) = \beta_1(x,t), \\ u(x,y,0) = \varphi(x,y), \end{cases} \tag{1}$$

其中 $\Omega = \{(x,y) \mid 0 < x,y < 1\}, t > 0, u$ 表示未知量，$k(x,y)$ 是扩散系数且恒大于某个正实数，$f(x,y,t)$ 是给定的源项.

首先给出区间 $\Omega$ 的一个标准网格划分，为方便作图和描述，取空间步长为 $h_x = h_y = h, h = 1/N, N$ 为 $x$ 或 $y$ 方向上的网格划分个数.网格划分如下：

$$x_0 = 0,\ x_1 = x_0 + h/2,\ x_i = x_{i-1} + h \quad (i = 2,3,\cdots,N),\ x_{N+1} = x_N + h/2 = 1,$$

$$y_0 = 0,\ y_1 = y_0 + h/2,\ y_j = y_{j-1} + h \quad (j = 2,3,\cdots,N),\ y_{N+1} = y_N + h/2 = 1,$$

其中网格点 $(x_i, y_j), i \in \{1,N\}, j \in \{1,2,\cdots,N\}$ 和 $(x_i, y_j), i \in \{1,2,\cdots,N\}, j \in \{1,N\}$ 区别于其它网格点的是它们与边界点相邻且与边界点相距 $h/2$.所有的网格点都可以看做是这样的一个空间控制体 $V_{i,j}$ 的中心：

$$V_{i,j} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}],$$

其中

$$x_{i\pm1/2} = x_i \pm h/2,\ y_{j\pm1/2} = y_j \pm h/2.$$

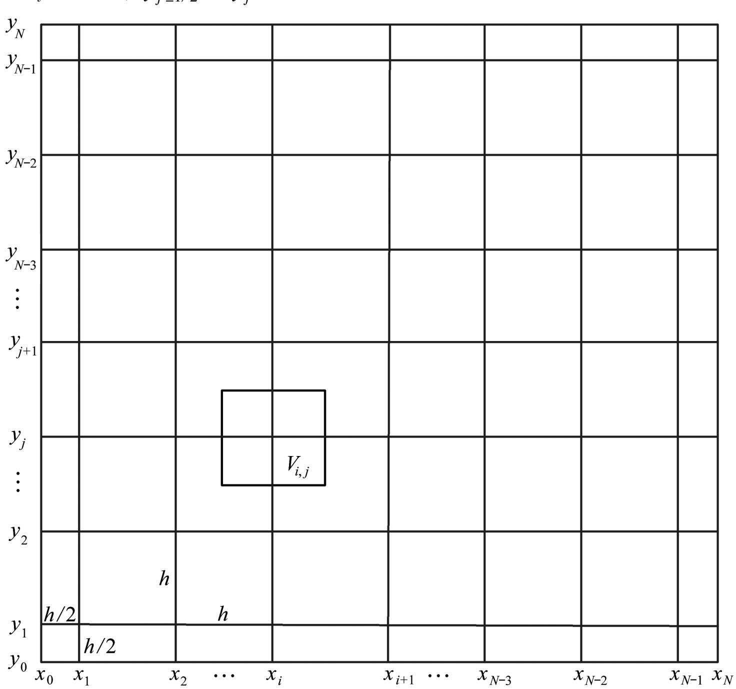


**图 1**　网格及控制体

Fig.1　Mesh and control body

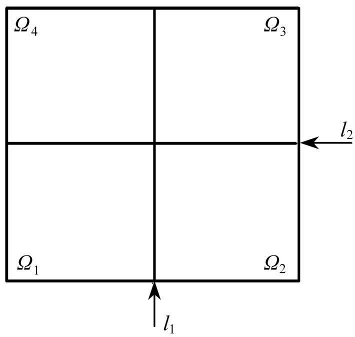


**图 2** 界面对区域的划分

Fig.2 Regional division

时间方向采用均匀网格剖分，步长用 $\tau$ 表示.

以图 1 表示网格的划分及控制体 $V_{i,j}$.

扩散系数 $k(x,y)$ 在分别位于区间 $\Omega$ 的 $x$ 方向和 $y$ 方向上的两条直线 $l_1$ 和 $l_2$ 处发生间断,区间 $\Omega=[0,1]\times[0,1]$ 也被这两条直线分割为 4 个部分 $\Omega_1,\Omega_2,\Omega_3$ 和 $\Omega_4$,如图 2 所示.这两条直线 $l_1$ 和 $l_2$ 平行于网格线且互相垂直，称之为界面.假设扩散系数 $k(x,y)$ 在每个子区间 $\Omega_i$ 内是连续的，在界面 $l_1$ 和 $l_2$ 处是间断的.

设

$$F(x,y,t)=k(x,y)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)$$

为通量.因为扩散系数 $k(x,y)$ 是间断的，那么通过界面的解 $u$ 以及通量 $F$ 要想满足条件

$$u(l_1-0,y,t)=u(l_1+0,y,t),$$
$$u(x,l_2-0,t)=u(x,l_2+0,t);$$
$$F(l_1-0,y,t)=F(l_1+0,y,t),$$
$$F(x,l_2-0,t)=F(x,l_2+0,t)$$

的充分条件[13-14]是

1）存在正常数 $M$ 和 $N$,使得扩散系数 $k(x,y)$ 和源项 $f(x,y,t)$ 满足

$$k(x,y)\leqslant M,\qquad \forall(x,y)\in\Omega,$$
$$f(x,y,t)\leqslant N,\qquad \forall(x,y,t)\in\Omega\times(0,\infty);$$

2）初值条件 $\varphi(x,y)$ 满足

$$k(l_1-0,y)\left(\frac{\partial\varphi}{\partial x}(l_1-0,y)+\frac{\partial\varphi}{\partial y}(l_1-0,y)\right)=$$
$$k(l_1+0,y)\left(\frac{\partial\varphi}{\partial x}(l_1+0,y)+\frac{\partial\varphi}{\partial y}(l_1+0,y)\right),$$
$$k(x,l_2-0)\left(\frac{\partial\varphi}{\partial x}(x,l_2-0)+\frac{\partial\varphi}{\partial y}(x,l_2-0)\right)=$$
$$k(x,l_2+0)\left(\frac{\partial\varphi}{\partial x}(x,l_2+0)+\frac{\partial\varphi}{\partial y}(x,l_2+0)\right);$$

3）边值条件 $\alpha_0(y,t),\alpha_1(y,t),\beta_0(x,t)$ 和 $\beta_1(x,t)$ 在$[0,T]$上是光滑连续函数,并且满足相容性条件:

$$u(0,y,0)=\alpha(0,y,0),\ u(1,y,0)=\alpha(1,y,0),$$
$$u(x,0,0)=\beta(x,0,0),\ u(x,1,0)=\beta(x,1,0).$$

## 1.2 有限体积离散

将网格中的每一个网格点 $p_{i,j}$ 放在对应的控制体 $V_{i,j}$ 中进行离散，方程(1)中第 1 项的连续形式则转变成对每个网格点 $p_{i,j}$ 在区间$[x_{i-1/2},x_{i+1/2}]\times[y_{j-1/2},y_{j+1/2}]\times[t_n,t_{n+1}]$内的离散形式.下面给出网格点 $p_{i,j}$ 相对应的控制体 $V_{i,j}$ 在时间区间$[t_n,t_{n+1}]$内的平衡方程:

$$\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t=$$
$$\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}t+\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}f(x,y,t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t.\qquad(2)$$

利用中矩形公式对式(2)等号右侧第 1 项进行积分运算,得

$$\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}t \approx$$
$$h\int_{t_n}^{t_{n+1}}(F(x_{i+1/2},y_j,t)-F(x_{i-1/2},y_j,t))\mathrm{d}t+$$
$$h\int_{t_n}^{t_{n+1}}(F(x_i,y_{j+1/2},t)-F(x_i,y_{j-1/2},t))\mathrm{d}t. \tag{3}$$

利用梯形公式继续对式(3)进行积分运算

$$\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}t \approx$$
$$\frac{\tau h}{2}((F(x_{i+1/2},y_j,t_{n+1})-F(x_{i-1/2},y_j,t_{n+1}))+$$
$$(F(x_{i+1/2},y_j,t_n)-F(x_{i-1/2},y_j,t_n)))+$$
$$\frac{\tau h}{2}((F(x_i,y_{j+1/2},t_{n+1})-F(x_i,y_{j-1/2},t_{n+1}))+$$
$$(F(x_i,y_{j+1/2},t_n)-F(x_i,y_{j-1/2},t_n))). \tag{4}$$

考虑通量方程:

$$F(x,y,t)=k(x,y)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right),$$

当 $y$ 值给定为 $y_j$ 时通量方程重写为

$$F(x,y_j,t)=k(x,y_j)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y_j}\right).$$

显然 $\partial u/\partial y_j=0$, 那么整理通量方程可得

$$\frac{F(x,y_j,t)}{k(x,y_j)}=\frac{\partial u(x,y_j,t)}{\partial x}. \tag{5}$$

对式(5)在区间 $(x_i,x_{i+1})$ 内进行积分运算得

$$\int_{x_i}^{x_{i+1}}\frac{F(x,y_j,t)}{k(x,y_j)}\mathrm{d}x=\int_{x_i}^{x_{i+1}}\frac{\partial u(x,y_j,t)}{\partial x}\mathrm{d}x. \tag{6}$$

若假定通量 $F(x,y_j,t)$ 在界面处是二次连续可微的, 则 $F(x,y_j,t)$ 可在点 $x_{i+1/2}$ 处展开:

$$F(x,y_j,t)\approx F(x_{i+1/2},y_j,t)+(x-x_{i+1/2})\frac{\partial F(x_{i+1/2},y_j,t)}{\partial x}+$$
$$\frac{(x-x_{i+1/2})^2}{2}\frac{\partial^2 F(\delta,y_j,t)}{\partial x^2},\qquad \delta\in[x_i,x_{i+1}]. \tag{7}$$

将式(7)右侧的一阶偏导项 $\partial F(x_{i+1/2},y_j,t)/\partial x$ 用向后差分进行离散,可以得到式(7)的一个逼近形式:

$$F(x,y_j,t)\approx F(x_{i+1/2},y_j,t)+$$
$$(x-x_{i+1/2})\frac{F(x_{i+1/2},y_j,t)-F(x_{i-1/2},y_j,t)}{h}+O(h^2). \tag{8}$$

将式(8)代入式(6)得

$$\int_{x_i}^{x_{i+1}}\frac{\partial u(x,y_j,t)}{\partial x}\mathrm{d}x=F(x_{i+1/2},y_j,t)\int_{x_i}^{x_{i+1}}\frac{1}{k(x,y_j)}\mathrm{d}x+$$

$$\frac{F(x_{i+1/2},y_j,t)-F(x_{i-1/2},y_j,t)}{h}\int_{x_i}^{x_{i+1}}\frac{x-x_{i+1/2}}{k(x,y_j)}\mathrm{d}x+O(h^3). \tag{9}$$

令

$$\kappa_{i+1/2}=\left(\frac{1}{h}\int_{x_i}^{x_{i+1}}\frac{1}{k(x,y_j)}\mathrm{d}x\right)^{-1},\ \phi_{i+1/2}=\frac{1}{h^2}\kappa_{i+1/2}\int_{x_i}^{x_{i+1}}\frac{x-x_{i+1/2}}{k(x,y_j)}\mathrm{d}x,$$

则式(9)可简写为

$$\kappa_{i+1/2}\frac{u(x_{i+1},y_j,t)-u(x_i,y_j,t)}{h}\approx$$
$$F(x_{i+1/2},y_j,t)+\phi_{i+1/2}(F(x_{i+1/2},y_j,t)-F(x_{i-1/2},y_j,t))+O(h^2),$$

忽略等式右边的高阶项得

$$\kappa_{i+1/2}\frac{u(x_{i+1},y_j,t)-u(x_i,y_j,t)}{h}=$$
$$F(x_{i+1/2},y_j,t)+\phi_{i+1/2}(F(x_{i+1/2},y_j,t)-F(x_{i-1/2},y_j,t)). \tag{10}$$

式(10)中 $\kappa_{i+1/2}$ 是扩散系数 $k(x,y)$ 在区间 $(x_i,x_{i+1})$ 内的调和平均系数，它在求解界面处数值解的过程中起到关键性作用，$\phi_{i+1/2}$ 是一阶偏导项的系数.经典的有限体积方法中只截取了通量 $F(x,y_j,t)$ 在点 $x_{i+1/2}$ 处 Taylor(泰勒)展式(7)右侧的第 1 项，本文为了提高对通量 $F(x,y_j,t)$ 的计算精度，截取式(7)右侧的前两项.

同样地，可以将通量 $F(x,y_j,t)$ 在点 $x_{i-1/2}$ 处展开，并进行类似的运算后可得

$$\kappa_{i-1/2}\frac{u(x_i,y_j,t)-u(x_{i-1},y_j,t)}{h}=$$
$$F(x_{i-1/2},y_j,t)+\phi_{i-1/2}(F(x_{i+1/2},y_j,t)-F(x_{i-1/2},y_j,t)). \tag{11}$$

当 $x$ 的值给定为 $x_i$ 时,同上面的方法一样,可以将通量 $F(x_i,y,t)$ 在点 $y_{j+1/2}$ 和点 $y_{j-1/2}$ 处展开,积分并化简后可得

$$\kappa_{j+1/2}\frac{u(x_i,y_{j+1},t)-u(x_i,y_j,t)}{h}=$$
$$F(x_i,y_{j+1/2},t)+\phi_{j+1/2}(F(x_i,y_{j+1/2},t)-F(x_i,y_{j-1/2},t)), \tag{12}$$

$$\kappa_{j-1/2}\frac{u(x_i,y_j,t)-u(x_i,y_{j-1},t)}{h}=$$
$$F(x_i,y_{j-1/2},t)+\phi_{j-1/2}(F(x_i,y_{j+1/2},t)-F(x_i,y_{j-1/2},t)). \tag{13}$$

由式(10)和式(11)可得

$$(1+\phi_{i+1/2}-\phi_{i-1/2})(F(x_{i+1/2},y_j,t)-F(x_{i-1/2},y_j,t))=$$
$$\kappa_{i+1/2}\frac{u(x_{i+1},y_j,t)-u(x_i,y_j,t)}{h}-\kappa_{i-1/2}\frac{u(x_i,y_j,t)-u(x_{i-1},y_j,t)}{h}.$$

令 $\rho_i=(1+\phi_{i+1/2}-\phi_{i-1/2})^{-1}$，则上式可改写为

$$F(x_{i+1/2},y_j,t)-F(x_{i-1/2},y_j,t)=$$
$$\rho_i\kappa_{i+1/2}\frac{u(x_{i+1},y_j,t)-u(x_i,y_j,t)}{h}-\rho_i\kappa_{i-1/2}\frac{u(x_i,y_j,t)-u(x_{i-1},y_j,t)}{h}. \tag{14}$$

同样地,令 $\rho_j=(1+\phi_{j+1/2}-\phi_{j-1/2})^{-1}$ 可得

$$F(x_i,y_{j+1/2},t)-F(x_i,y_{j-1/2},t)=$$
$$\rho_j\kappa_{j+1/2}\frac{u(x_i,y_{j+1},t)-u(x_i,y_j,t)}{h}-\rho_j\kappa_{j-1/2}\frac{u(x_i,y_j,t)-u(x_i,y_{j-1},t)}{h}. \tag{15}$$

将式(14)和式(15)代入式(4)得

$$\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}t\approx$$
$$\frac{\tau h}{2}\left(\rho_i\kappa_{i+1/2}\frac{u(x_{i+1},y_j,t_{n+1})-u(x_i,y_j,t_{n+1})}{h}-\rho_i\kappa_{i-1/2}\frac{u(x_i,y_j,t_{n+1})-u(x_{i-1},y_j,t_{n+1})}{h}\right)+$$
$$\frac{\tau h}{2}\left(\rho_i\kappa_{i+1/2}\frac{u(x_{i+1},y_j,t_n)-u(x_i,y_j,t_n)}{h}-\rho_i\kappa_{i-1/2}\frac{u(x_i,y_j,t_n)-u(x_{i-1},y_j,t_n)}{h}\right)+$$
$$\frac{\tau h}{2}\left(\rho_j\kappa_{j+1/2}\frac{u(x_i,y_{j+1},t_{n+1})-u(x_i,y_j,t_{n+1})}{h}-\rho_j\kappa_{j-1/2}\frac{u(x_i,y_j,t_{n+1})-u(x_i,y_{j-1},t_{n+1})}{h}\right)+$$
$$\frac{\tau h}{2}\left(\rho_j\kappa_{j+1/2}\frac{u(x_i,y_{j+1},t_n)-u(x_i,y_j,t_n)}{h}-\rho_j\kappa_{j-1/2}\frac{u(x_i,y_j,t_n)-u(x_i,y_{j-1},t_n)}{h}\right).\tag{16}$$

考虑式(2)等式左侧的积分项:

$$\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}\frac{\partial u}{\partial t}\mathrm{d}x\mathrm{d}y\mathrm{d}t\approx\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}(u(x,y,t_{n+1})-u(x,y,t_n))\mathrm{d}x\mathrm{d}y.$$

利用中矩形积分公式进行积分运算得

$$\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}\frac{\partial u}{\partial t}\mathrm{d}x\mathrm{d}y\mathrm{d}t\approx h^2(u(x_i,y_j,t_{n+1})-u(x_i,y_j,t_n)).\tag{17}$$

再考虑式(2)右侧的源项积分项:

$$\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}f(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

首先对时间层利用梯形公式进行积分运算得

$$\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}f(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t\approx\frac{\tau}{2}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}(f(x,y,t_{n+1})+f(x,y,t_n))\mathrm{d}x\mathrm{d}y.$$

对空间层利用中矩形公式进行积分运算得

$$\frac{\tau}{2}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}}(f(x,y,t_{n+1})+f(x,y,t_n))\mathrm{d}x\mathrm{d}y\approx$$
$$\frac{\tau}{2}h^2(f(x_i,y_j,t_{n+1})+f(x_i,y_j,t_n)).\tag{18}$$

结合式(16)、(17)和(18)得出本文对间断系数扩散方程的离散格式:

$$-\frac{\tau}{2}\frac{\rho_i\kappa_{i-1/2}}{h^2}u_{i-1,j}^{n+1}-\frac{\tau}{2}\frac{\rho_j\kappa_{j-1/2}}{h^2}u_{i,j-1}^{n+1}+$$
$$\left(1+\frac{\tau}{2}\left(\frac{\rho_i}{h^2}(\kappa_{i+1/2}+\kappa_{i-1/2})+\frac{\rho_j}{h^2}(\kappa_{j+1/2}+\kappa_{j-1/2})\right)\right)u_{i,j}^{n+1}-$$

$$\frac{\tau}{2}\frac{\rho_i\kappa_{i+1/2}}{h^2}u_{i+1,j}^{n+1}-\frac{\tau}{2}\frac{\rho_j\kappa_{j+1/2}}{h^2}u_{i,j+1}^{n+1}=$$
$$\frac{\tau}{2}\frac{\rho_i\kappa_{i-1/2}}{h^2}u_{i-1,j}^{n}+\frac{\tau}{2}\frac{\rho_j\kappa_{j-1/2}}{h^2}u_{i,j-1}^{n}+$$
$$\left(1-\frac{\tau}{2}\left(\frac{\rho_i}{h^2}(\kappa_{i+1/2}+\kappa_{i-1/2})+\frac{\rho_j}{h^2}(\kappa_{j+1/2}+\kappa_{j-1/2})\right)\right)u_{i,j}^n+$$
$$\frac{\tau}{2}\frac{\rho_i\kappa_{i+1/2}}{h^2}u_{i+1,j}^{n}+\frac{\tau}{2}\frac{\rho_j\kappa_{j+1/2}}{h^2}u_{i,j+1}^{n}+\frac{\tau}{2}(f_{i,j}^{n+1}+f_{i,j}^n),\tag{19}$$

其中 $u_{i,j}^n=u(x_i,y_j,t_n)$，$f_{i,j}^n=f(x_i,y_j,t_n)$，　$i=1,2,\cdots,N$.

### 1.3 误差分析

为书写简洁,式(19)中的系数重新定义为

$$a_1=-\frac{\tau}{2}\frac{\rho_i\kappa_{i-1/2}}{h^2},\ a_2=-\frac{\tau}{2}\frac{\rho_i\kappa_{i+1/2}}{h^2},\ b_1=-\frac{\tau}{2}\frac{\rho_j\kappa_{j-1/2}}{h^2},\ b_2=-\frac{\tau}{2}\frac{\rho_j\kappa_{j+1/2}}{h^2}.$$

下面考虑格式(19)的截断误差，首先对重写后的格式(19)进行移项变形得

$$a_1(u_{i-1,j}^{n+1}-u_{i,j}^{n+1}+u_{i-1,j}^n-u_{i,j}^n)+a_2(u_{i+1,j}^{n+1}-u_{i,j}^{n+1}+u_{i+1,j}^n-u_{i,j}^n)+$$
$$b_1(u_{i,j-1}^{n+1}-u_{i,j}^{n+1}+u_{i,j-1}^n-u_{i,j}^n)+b_2(u_{i,j+1}^{n+1}-u_{i,j}^{n+1}+u_{i,j+1}^n-u_{i,j}^n)+$$
$$(u_{i,j}^{n+1}-u_{i,j}^n)=\frac{\tau}{2}(f_{i,j}^{n+1}+f_{i,j}^n).\tag{20}$$

用微分方程的解 $u(x_i,y_j,t_n)$ 代替式(20)中的全部近似解 $u_{i,j}^n$,这样得到方程两边的差就是截断误差.将式(20)中等式左侧的第1项 $a_1(u_{i-1,j}^{n+1}-u_{i,j}^{n+1}+u_{i-1,j}^n-u_{i,j}^n)$ 做 Taylor 展式，$u_{i-1,j}^{n+1},u_{i,j}^{n+1},u_{i-1,j}^n$ 全部在点$(x_i,y_j,t_n)$处展开.

由于格式(20)为二维隐式格式，所以首先对$(u_{i-1,j}^{n+1}-u_{i,j}^{n+1}+u_{i-1,j}^n-u_{i,j}^n)$在时间层上进行 Taylor 展开：

$$(u_{i-1,j}^{n+1}-u_{i,j}^{n+1}+u_{i-1,j}^n-u_{i,j}^n)\approx$$
$$\left(u_{i-1,j}^n+\tau\left(\frac{\partial u}{\partial t}\right)_{i-1,j}^n+\frac{\tau^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_{i-1,j}^n+\frac{\tau^3}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)_{i-1,j}^n+\frac{\tau^4}{24}\left(\frac{\partial^4 u}{\partial t^4}\right)_{i-1,j}^n+O(\tau^5)\right)-$$
$$\left(u_{i,j}^n+\tau\left(\frac{\partial u}{\partial t}\right)_{i,j}^n+\frac{\tau^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j}^n+\frac{\tau^3}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)_{i,j}^n+\frac{\tau^4}{24}\left(\frac{\partial^4 u}{\partial t^4}\right)_{i,j}^n+O(\tau^5)\right)+$$
$$\left(u_{i,j}^n-h\left(\frac{\partial u}{\partial x}\right)_{i,j}^n+\frac{h^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}^n-\frac{h^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}^n+\frac{h^4}{24}\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j}^n+O(h^5)\right)-u_{i,j}^n.$$

得到如上的 Taylor 展式后再在空间层上 Taylor 展开,得

$$(u_{i-1,j}^{n+1}-u_{i,j}^{n+1}+u_{i-1,j}^n-u_{i,j}^n)\approx$$
$$-2h\left(\frac{\partial u}{\partial x}\right)_{i,j}^n+h^2\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}^n-\frac{h^3}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}^n+\frac{h^4}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j}^n+O(h^5)-$$
$$\tau h\left(\frac{\partial^2 u}{\partial t\partial x}\right)_{i,j}^n+\frac{\tau h^2}{2}\left(\frac{\partial^3 u}{\partial t\partial x^2}\right)_{i,j}^n-\frac{\tau h^3}{6}\left(\frac{\partial^4 u}{\partial t\partial x^3}\right)_{i,j}^n+\frac{\tau h^4}{24}\left(\frac{\partial^5 u}{\partial t\partial x^4}\right)_{i,j}^n+O(\tau h^5)-$$
$$\frac{\tau^2 h}{2}\left(\frac{\partial^3 u}{\partial t^2\partial x}\right)_{i,j}^n+\frac{\tau^2 h^2}{4}\left(\frac{\partial^4 u}{\partial t^2\partial x^2}\right)_{i,j}^n-\frac{\tau^2 h^3}{12}\left(\frac{\partial^5 u}{\partial t^2\partial x^3}\right)_{i,j}^n+\frac{\tau^2 h^4}{60}\left(\frac{\partial^6 u}{\partial t^2\partial x^4}\right)_{i,j}^n+O(\tau^2h^5)-$$
$$\frac{\tau^3 h}{6}\left(\frac{\partial^4 u}{\partial t^2\partial x^2}\right)_{i,j}^n+\frac{\tau^3 h^2}{24}\left(\frac{\partial^5 u}{\partial t^3\partial x^2}\right)_{i,j}^n-\frac{\tau^3 h^3}{120}\left(\frac{\partial^6 u}{\partial t^3\partial x^3}\right)_{i,j}^n+O(\tau^3h^5).$$

用同样的方法对式(20)左侧的第2,3和4项进行Taylor展开.再将各项的展式代入式(20)得

$$
\begin{aligned}
&2h\left(\frac{\partial u}{\partial x}\right)_{i,j}^{n}(a_2-a_1)+h^2\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}^{n}(a_2+a_1)+\frac{h^3}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}^{n}(a_2-a_1)+\\
&\quad\frac{h^4}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j}^{n}(a_2+a_1)+\tau h\left(\frac{\partial^2 u}{\partial t\partial x}\right)_{i,j}^{n}(a_2-a_1)+\frac{\tau h^2}{2}\left(\frac{\partial^3 u}{\partial t\partial x^2}\right)_{i,j}^{n}(a_2+a_1)+\\
&\quad\frac{\tau h^3}{6}\left(\frac{\partial^4 u}{\partial t\partial x^3}\right)_{i,j}^{n}(a_2-a_1)+\frac{\tau h^4}{24}\left(\frac{\partial^5 u}{\partial t\partial x^4}\right)_{i,j}^{n}(a_2+a_1)+\frac{\tau^2 h}{2}\left(\frac{\partial^3 u}{\partial t^2\partial x}\right)_{i,j}^{n}(a_2-a_1)+\\
&\quad\frac{\tau^2 h^2}{4}\left(\frac{\partial^4 u}{\partial t^2\partial x^2}\right)_{i,j}^{n}(a_2+a_1)+\frac{\tau^2 h^3}{12}\left(\frac{\partial^5 u}{\partial t^2\partial x^3}\right)_{i,j}^{n}(a_2-a_1)+\\
&\quad\frac{\tau^2 h^4}{60}\left(\frac{\partial^6 u}{\partial t^2\partial x^4}\right)_{i,j}^{n}(a_2+a_1)+\frac{\tau^3 h}{6}\left(\frac{\partial^4 u}{\partial t^2\partial x^2}\right)_{i,j}^{n}(a_2-a_1)+\\
&\quad\frac{\tau^3 h^2}{24}\left(\frac{\partial^5 u}{\partial t^3\partial x^2}\right)_{i,j}^{n}(a_2+a_1)+\frac{\tau^3 h^3}{120}\left(\frac{\partial^6 u}{\partial t^3\partial x^3}\right)_{i,j}^{n}(a_2-a_1)+\\
&\quad 2h\left(\frac{\partial u}{\partial y}\right)_{i,j}^{n}(b_2-b_1)+h^2\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j}^{n}(b_2+b_1)+\frac{h^3}{3}\left(\frac{\partial^3 u}{\partial y^3}\right)_{i,j}^{n}(b_2-b_1)+\\
&\quad\frac{h^4}{12}\left(\frac{\partial^4 u}{\partial y^4}\right)_{i,j}^{n}(b_2+b_1)+\tau h\left(\frac{\partial^2 u}{\partial t\partial y}\right)_{i,j}^{n}(b_2-b_1)+\frac{\tau h^2}{2}\left(\frac{\partial^3 u}{\partial t\partial y^2}\right)_{i,j}^{n}(b_2+b_1)+\\
&\quad\frac{\tau h^3}{6}\left(\frac{\partial^4 u}{\partial t\partial y^3}\right)_{i,j}^{n}(b_2-b_1)+\frac{\tau h^4}{24}\left(\frac{\partial^5 u}{\partial t\partial y^4}\right)_{i,j}^{n}(b_2+b_1)+\frac{\tau^2 h}{2}\left(\frac{\partial^3 u}{\partial t^2\partial y}\right)_{i,j}^{n}(b_2-b_1)+\\
&\quad\frac{\tau^2 h^2}{4}\left(\frac{\partial^4 u}{\partial t^2\partial y^2}\right)_{i,j}^{n}(b_2+b_1)+\frac{\tau^2 h^3}{12}\left(\frac{\partial^5 u}{\partial t^2\partial y^3}\right)_{i,j}^{n}(b_2-b_1)+\\
&\quad\frac{\tau^2 h^4}{60}\left(\frac{\partial^6 u}{\partial t^2\partial y^4}\right)_{i,j}^{n}(b_2+b_1)+\frac{\tau^3 h}{6}\left(\frac{\partial^4 u}{\partial t^2\partial y^2}\right)_{i,j}^{n}(b_2-b_1)+\\
&\quad\frac{\tau^3 h^2}{24}\left(\frac{\partial^5 u}{\partial t^3\partial y^2}\right)_{i,j}^{n}(b_2+b_1)+\frac{\tau^3 h^3}{120}\left(\frac{\partial^6 u}{\partial t^3\partial y^3}\right)_{i,j}^{n}(b_2-b_1)+\\
&\quad\tau\left(\frac{\partial u}{\partial t}\right)_{i,j}^{n}+\frac{\tau^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j}^{n}+O(\tau h^4)\approx\frac{\tau}{2}(f_{i,j}^{n+1}+f_{i,j}^{n}).
\end{aligned}
\tag{21}
$$

从上一小节可知,当网格划分均匀且扩散系数不变时有$a_1=a_2,b_1=b_2$,又因为$a_1,a_2,b_1,b_2$中含有时间步长$\tau$和空间步长$h$,所以令

$$
a_1=\frac{\tau}{h^2}a_1',\ a_2=\frac{\tau}{h^2}a_2',\ b_1=\frac{\tau}{h^2}b_1',\ b_2=\frac{\tau}{h^2}b_2'.
$$

将$a_1',a_2',b_1',b_2'$代入式(21),整理后得出格式(19)的截断误差为

$$
\begin{aligned}
T(x,y,t)=&\frac{h^2}{12}(a_1'+a_2')\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j}^{n}+\frac{\tau}{2}(a_1'+a_2')\left(\frac{\partial^3 u}{\partial t\partial x^2}\right)_{i,j}^{n}+\\
&\frac{\tau h^2}{24}(a_1'+a_2')\left(\frac{\partial^5 u}{\partial t\partial x^4}\right)_{i,j}^{n}+\frac{\tau}{4}(a_1'+a_2')\left(\frac{\partial^4 u}{\partial t^2\partial x^2}\right)_{i,j}^{n}+\frac{\tau^2 h^2}{60}(a_1'+a_2')\left(\frac{\partial^6 u}{\partial t^2\partial x^4}\right)_{i,j}^{n}+\\
&\frac{\tau^3}{24}(a_1'+a_2')\left(\frac{\partial^5 u}{\partial t^3\partial x^2}\right)_{i,j}^{n}+\frac{h^2}{12}(b_1'+b_2')\left(\frac{\partial^4 u}{\partial y^4}\right)_{i,j}^{n}+\frac{\tau}{2}(b_1'+b_2')\left(\frac{\partial^3 u}{\partial t\partial y^2}\right)_{i,j}^{n}+
\end{aligned}
$$

$$\frac{\tau h^2}{24}(b_1' + b_2')\left(\frac{\partial^5 u}{\partial t \partial y^4}\right)_{i,j}^n + \frac{\tau}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j}^n + \frac{\tau}{4}(b_1' + b_2')\left(\frac{\partial^4 u}{\partial t^2 \partial y^2}\right)_{i,j}^n +$$
$$\frac{\tau^2 h^2}{60}\left(\frac{\partial^6 u}{\partial t^2 \partial y^4}\right)_{i,j}^n + \frac{\tau^3}{24}(b_1' + b_2')\left(\frac{\partial^5 u}{\partial t^3 \partial y^2}\right)_{i,j}^n + O(h^4) + O(\tau^2).$$

由上式可知格式(19)的截断误差为 $O(\tau + h^2)$.

### 1.4 界面处理

假设两个界面 $l_1$ 和 $l_2$ 分别位于区间 $[x_i, x_{i+1}]$ 和 $[y_j, y_{j+1}]$ 内, 则可以表示为 $l_1 = x_i + \theta_1 h$, $l_2 = y_j + \theta_2 h$,其中 $0 \leqslant \theta_1, \theta_2 \leqslant 1$.若 $\theta_1 = 0$ 或 $\theta_1 = 1$,那么界面 $l_1$ 便与网格线重合,这种情况下经过离散后得出的差分格式与对扩散方程直接进行有限差分离散所得到差分格式一致,对这种情形我们不做考虑,而总是取 $0 < \theta_1 < 1$,同样地取 $0 < \theta_2 < 1$.

将界面 $l_1$ 和 $l_2$ 确定在区间 $[x_i, x_{i+1}]$ 和 $[y_j, y_{j+1}]$ 内意味着扩散系数 $k(x,y)$ 在这两个区间内发生间断,那么在区间 $[x_i, x_{i+1}]$ 和 $[y_j, y_{j+1}]$ 内求调和平均系数 $\kappa_{i+1/2}$ 时需要利用积分区间可加性.

$$\kappa_{i+1/2} = \left(\frac{1}{h}\int_{x_i}^{x_{i+1}} \frac{1}{k(x,y_j)}\mathrm{d}x\right)^{-1} = \left(\frac{1}{h}\int_{x_i}^{l_1} \frac{1}{k(x,y_j)}\mathrm{d}x + \frac{1}{h}\int_{l_1}^{x_{i+1}} \frac{1}{k(x,y_j)}\mathrm{d}x\right)^{-1}.$$

为了提高整体的计算精度,对于调和平均系数的积分求解本文采用具有三阶精度的 Simpson(辛普森)公式对其进行运算:

$$\kappa_{i+1/2} \approx \left(\frac{\theta_1}{6}\left(\frac{1}{k(x_i,y_j)} + \frac{4}{k\left(\frac{x_i + l_1^-}{2}, y_j\right)} + \frac{1}{k(l_1^-, y_j)}\right) + \frac{1-\theta_1}{6}\left(\frac{1}{k(l_1^+, y_j)} + \frac{4}{k\left(\frac{l_1^+ + x_{i+1}}{2}, y_j\right)} + \frac{1}{k(x_{i+1}, y_j)}\right)\right)^{-1}.$$

其它的调和平均系数 $\kappa_{i-1/2}, \kappa_{j+1/2}$ 和 $\kappa_{j-1/2}$ 以及 $\phi_{i+1/2}, \phi_{i-1/2}, \phi_{j+1/2}$ 和 $\phi_{j-1/2}$ 采用同样的方法进行积分运算.当空间步长 $h$ 充分小时, 可得到如下近似:

$$k(x_i, y_j) \approx k\left(\frac{x_i + l_1^-}{2}, y_j\right) \approx k(l_1^-, y_j),$$
$$k(l_1^+, y_j) \approx k\left(\frac{l_1^+ + x_{i+1}}{2}, y_j\right) \approx k(x_{i+1}, y_j).$$

将上面的近似值代入 $\kappa_{i+1/2}$ 中得

$$\kappa_{i+1/2} = \left(\frac{\theta_1}{k(x_i, y_j)} + \frac{1-\theta_1}{k(x_{i+1}, y_j)}\right)^{-1}. \tag{22}$$

同样地, 有

$$\kappa_{i-1/2} = \left(\frac{\theta_1}{k(x_{i-1}, y_j)} + \frac{1-\theta_1}{k(x_i, y_j)}\right)^{-1},$$
$$\kappa_{j+1/2} = \left(\frac{\theta_2}{k(x_i, y_j)} + \frac{1-\theta_2}{k(x_i, y_{j+1})}\right)^{-1},$$
$$\kappa_{j-1/2} = \left(\frac{\theta_2}{k(x_i, y_{j-1})} + \frac{1-\theta_2}{k(x_i, y_j)}\right)^{-1}.$$

将近似值代入 $\phi_{i+1/2}, \phi_{i-1/2}, \phi_{j+1/2}$ 和 $\phi_{j-1/2}$ 中可得

$$\phi_{i+1/2}=\frac{\theta_1}{6}\frac{3\theta_1-1}{k(x_i,y_j)}+\frac{1-\theta_1}{6}\frac{3\theta_1-2}{k(x_{i+1},y_j)},$$
$$\phi_{i-1/2}=\frac{\theta_1}{6}\frac{3\theta_1-1}{k(x_{i-1},y_j)}+\frac{1-\theta_1}{6}\frac{3\theta_1-2}{k(x_i,y_j)},$$
$$\phi_{j+1/2}=\frac{\theta_2}{6}\frac{3\theta_2-1}{k(x_i,y_j)}+\frac{1-\theta_2}{6}\frac{3\theta_2-2}{k(x_i,y_{j+1})},$$
$$\phi_{j-1/2}=\frac{\theta_2}{6}\frac{3\theta_2-1}{k(x_i,y_{j-1})}+\frac{1-\theta_2}{6}\frac{3\theta_2-2}{k(x_i,y_j)}.$$

### 1.5　稳定性分析

1.2 小节给出的隐格式(19)可以写成如下矩阵形式：

$$(\boldsymbol{I}+\lambda\boldsymbol{G}_1)\boldsymbol{u}^{n+1}=(\boldsymbol{I}-\lambda\boldsymbol{G}_2)\boldsymbol{u}^n+\boldsymbol{F}^n, \tag{23}$$

其中 $\boldsymbol{I}$ 为单位矩阵，$\lambda=\tau/2h^2$，

$$\boldsymbol{u}^n=(u_{1,1}^n,u_{1,2}^n,\cdots,u_{1,N}^n,u_{2,1}^n,u_{2,2}^n\cdots,u_{2,N}^n,\cdots,u_{N,1}^n,u_{N,2}^n,\cdots,u_{N,N}^n)^{\mathrm{T}},$$

$$\boldsymbol{F}^n=\Big(\frac{\tau}{2}(f_{1,1}^{n+1}+f_{1,1}^n),\frac{\tau}{2}(f_{1,2}^{n+1}+f_{1,2}^n),\cdots,\frac{\tau}{2}(f_{1,N}^{n+1}+f_{1,N}^n),$$
$$\frac{\tau}{2}(f_{2,1}^{n+1}+f_{2,1}^n),\frac{\tau}{2}(f_{2,2}^{n+1}+f_{2,2}^n),\cdots,\frac{\tau}{2}(f_{2,N}^{n+1}+f_{2,N}^n),\cdots,\frac{\tau}{2}(f_{N,N}^{n+1}+f_{N,N}^n)\Big)^{\mathrm{T}},$$

$\boldsymbol{u}$ 和 $\boldsymbol{F}$ 是 $N^2\times 1$ 矩阵，$\boldsymbol{G}_1$ 和 $\boldsymbol{G}_2$ 是 $N^2\times N^2$ 矩阵.将 $\boldsymbol{G}_1$ 和 $\boldsymbol{G}_2$ 写出后容易看出 $\boldsymbol{G}_1=\boldsymbol{G}_2$.

为了证明方法的稳定性，给出如下两个引理.

**引理 1**　由式(23)给出的两个矩阵为非负实阵.

**引理 2**　(Kellogg)如果 $\varphi>0$，矩阵 $\boldsymbol{G}$ 为非负实阵，则

$$\|(\varphi\boldsymbol{I}+\boldsymbol{G})^{-1}\|_2\leqslant 1,\ \|(\varphi\boldsymbol{I}-\boldsymbol{G})(\varphi\boldsymbol{I}+\boldsymbol{G})^{-1}\|_2\leqslant 1.$$

**定理 1**　求解间断系数扩散方程(1)的隐格式(19)无条件稳定.

定理 1 的证明还需要考虑到文中所规定的一些约束条件：

1）通量函数 $F(x,y,t)$ 在界面处二次连续可微；

2）扩散系数 $k(x,y)$ 在每个子区间具有连续的二阶导数.

**证明**　容易得出格式(19)的增长矩阵为

$$\boldsymbol{G}=(\boldsymbol{I}+\lambda\boldsymbol{G}_1)^{-1}(\boldsymbol{I}-\lambda\boldsymbol{G}_2).$$

应用引理 1 和引理 2 有

$$\|\boldsymbol{G}^n\|_2=\|(\boldsymbol{I}+\lambda\boldsymbol{G}_1)^{-1}\|_2^n\|(\boldsymbol{I}-\lambda\boldsymbol{G}_2)\|_2^n,$$

又因为 $\boldsymbol{G}_1=\boldsymbol{G}_2$，所以

$$\|\boldsymbol{G}^n\|_2=\|(\boldsymbol{I}+\lambda\boldsymbol{G}_1)^{-1}\|_2^n\|(\boldsymbol{I}-\lambda\boldsymbol{G}_1)\|_2^n\leqslant 1.$$

定理 1 得证.

## 2　数 值 算 例

**例 1**　考虑方程(1)的定解问题.

扩散系数

$$k(x,y)=\begin{cases}1, & (x,y)\in[0,0.5)\times[0,0.5),\\ 0.5, & (x,y)\in[0.5,1]\times[0,0.5),\\ 0.5, & (x,y)\in[0,0.5)\times[0.5,1],\\ 1, & (x,y)\in[0.5,1]\times[0.5,1].\end{cases}$$

初始条件

$$u(x,y,0)=\begin{cases}\sin\pi x\sin\pi y, & (x,y)\in[0,0.5)\times[0,0.5),\\ 2\sin\pi x\sin\pi y, & (x,y)\in[0.5,1]\times[0,0.5),\\ 2\sin\pi x\sin\pi y, & (x,y)\in[0,0.5)\times[0.5,1],\\ \sin\pi x\sin\pi y, & (x,y)\in[0.5,1]\times[0.5,1].\end{cases}$$

源项

$$f(x,y,t)=\begin{cases}0, & (x,y)\in[0,0.5)\times[0,0.5),\\ -2\pi^2 e^{-2\pi^2 t}\sin\pi x\sin\pi y, & (x,y)\in[0.5,1]\times[0,0.5),\\ -2\pi^2 e^{-2\pi^2 t}\sin\pi x\sin\pi y, & (x,y)\in[0,0.5)\times[0.5,1],\\ 0, & (x,y)\in[0.5,1]\times[0.5,1].\end{cases}$$

边界条件

$$u(0,y,t)=0,\ u(1,y,t)=0,\ u(x,0,t)=0,\ u(x,1,t)=0.$$

精确解为

$$u(x,y,t)=\begin{cases}e^{-2\pi^2 t}\sin\pi x\sin\pi y, & (x,y)\in[0,0.5)\times[0,0.5),\\ 2e^{-2\pi^2 t}\sin\pi x\sin\pi y, & (x,y)\in[0.5,1]\times[0,0.5),\\ 2e^{-2\pi^2 t}\sin\pi x\sin\pi y, & (x,y)\in[0,0.5)\times[0.5,1],\\ e^{-2\pi^2 t}\sin\pi x\sin\pi y, & (x,y)\in[0.5,1]\times[0.5,1].\end{cases}$$

用经典的有限体积方法与本文方法进行比较，同时给出精确解相对应，结果如表 1~表 8 所示。

**表 1** 当 $t=0.002\,5$, $h=0.025$, $\lambda=0.1$ 时，两种方法的结果和精确值比较(节点(20, 21)、(21, 20))

Table 1 Results of the two methods in comparison with the exact solution(nodes(20, 21), (21, 20)), $t=0.002\,5$, $h=0.025$, $\lambda=0.1$

| method | $t=0.002\,5$ $h=0.025$ $\lambda=0.1$ | | |
|---|---|---|---|
| | nodes(20, 21), (21, 20) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 1.994 455 254 766 29 | – | – |
| classical solution | 1.994 344 454 406 76 | 5.555 419 669E−5 | 1.108 003 595E−4 |
| present solution | 1.994 441 960 195 78 | 6.665 765 240E−6 | 1.329 457 051E−5 |

**表 2** 当 $t=0.002\,5$, $h=0.025$, $\lambda=0.1$ 时，两种方法的结果和精确值比较(节点(20, 20)、(21, 21))

Table 2 Results of the two methods in comparison with the exact solution(nodes(20, 20), (21, 21)), $t=0.002\,5$, $h=0.025$, $\lambda=0.1$

| method | $t=0.002\,5$ $h=0.025$ $\lambda=0.1$ | | |
|---|---|---|---|
| | nodes(20, 20), (21, 21) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 0.997 227 627 383 14 | – | – |
| classical solution | 0.998 458 798 533 32 | 0.001 234 593 904 5 | 0.001 231 171 150 18 |
| present solution | 0.998 458 666 988 11 | 0.001 234 461 993 5 | 0.001 231 039 604 97 |

**表 3** 当 $t=0.002\,5$, $h=0.025$, $\lambda=0.5$ 时，两种方法的结果和精确值比较(节点(20, 21)、(21, 20))

Table 3 Results of the two methods in comparison with the exact solution(nodes(20, 21), (21, 20)), $t=0.002\,5$, $h=0.025$, $\lambda=0.5$

| method | $t=0.002\,5$ $h=0.025$ $\lambda=0.5$ | | |
|---|---|---|---|
| | nodes(20, 21), (21, 20) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 1.984 637 257 447 77 | – | – |
| classical solution | 1.984 102 831 285 29 | 2.692 815 326E−4 | 5.344 261 624E−4 |
| present solution | 1.984 586 243 108 69 | 2.570 461 624E−5 | 5.101 433 908E−5 |

**表 4** 当 $t = 0.002\,5$, $h = 0.025$, $\lambda = 0.5$ 时，两种方法的结果和精确值比较(节点(20, 20)、(21, 21))

Table 4 Results of the two methods in comparison with the exact solution(nodes(20, 20), (21, 21)), $t = 0.002\,5$, $h = 0.025$, $\lambda = 0.5$

| method | $t = 0.002\,5$ $h = 0.025$ $\lambda = 0.5$ | | |
|---|---|---|---|
| | nodes(20, 20), (21, 21) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 0.992 318 628 723 89 | - | - |
| classical solution | 0.998 459 320 814 51 | 0.006 188 226 153 243 6 | 0.006 140 692 090 62 |
| present solution | 0.998 458 667 470 36 | 0.006 187 567 751 666 6 | 0.006 140 038 746 47 |

**表 5** 当 $t = 0.002\,5$, $h = 0.025$, $\lambda = 1$ 时，两种方法的结果和精确值比较(节点(20, 21)、(21, 20))

Table 5 Results of the two methods in comparison with the exact solution(nodes(20, 21), (21, 20)), $t = 0.002\,5$, $h = 0.025$, $\lambda = 1$

| method | $t = 0.002\,5$ $h = 0.025$ $\lambda = 1$ | | |
|---|---|---|---|
| | nodes(20, 21), (21, 20) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 1.972 579 840 136 85 | - | - |
| classical solution | 1.971 795 083 179 64 | 3.978 327 980 6E-4 | 7.847 569 572 1E-4 |
| present solution | 1.972 508 745 128 65 | 3.604 163 783 6E-5 | 7.109 500 820 1E-5 |

**表 6** 当 $t = 0.002\,5$, $h = 0.025$, $\lambda = 1$ 时，两种方法的结果和精确值比较(节点(20, 20)、(21, 21))

Table 6 Results of the two methods in comparison with the exact solution(nodes(20, 20), (21, 21)), $t = 0.002\,5$, $h = 0.025$, $\lambda = 1$

| method | $t = 0.002\,5$ $h = 0.025$ $\lambda = 1$ | | |
|---|---|---|---|
| | nodes(20, 20), (21, 21) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 0.988 975 429 674 13 | - | - |
| classical solution | 0.997 253 881 263 76 | 0.008 370 735 350 18 | 0.008 278 451 589 63 |
| present solution | 0.997 253 442 053 65 | 0.008 370 291 243 99 | 0.008 278 012 379 52 |

**表 7** 当 $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$ 时，两种方法的结果和精确值比较(节点(20, 21)、(21, 20))

Table 7 Results of the two methods in comparison with the exact solution(nodes(20, 21), (21, 20)), $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$

| method | $t = 0.002\,5$ $h = 0.025$ $\lambda = 10$ | | |
|---|---|---|---|
| | nodes(20, 21), (21, 20) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 1.963 824 796 044 53 | - | - |
| classical solution | 1.963 000 156 232 02 | 4.199 151 646 1E-4 | 8.246 398 125 1E-4 |
| present solution | 1.963 743 701 036 33 | 4.129 442 115 4E-5 | 8.109 500 820 1E-5 |

**表 8** 当 $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$ 时，两种方法的结果和精确值比较(节点(20, 20)、(21, 21))

Table 8 Results of the two methods in comparison with the exact solution(nodes(20, 20), (21, 21)), $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$

| method | $t = 0.002\,5$ $h = 0.025$ $\lambda = 10$ | | |
|---|---|---|---|
| | nodes(20, 20), (21, 21) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 0.987 425 796 471 82 | - | - |
| classical solution | 0.997 001 210 155 94 | 0.009 697 350 138 44 | 0.009 575 413 684 12 |
| present solution | 0.997 001 197 109 77 | 0.009 697 336 926 14 | 0.009 575 400 637 95 |

表 1～表 8 中考虑了 4 个网格点的解的情况，这 4 个网格点正好位于两个界面相交的 4 个顶点处，其中点(20,20)与点(21,21)关于界面的交点相对称，点(20,21)与点(21,20)关于界面的交点相对称，所以它们的计算结果相同.8 个表格中分别给出了当 $\lambda = 0.1$, $\lambda = 0.5$, $\lambda =$

1 和 $\lambda=10$ 时 4 个网格点的计算结果及相对误差和绝对误差.对比可以看出本文方法能更好地与精确解相吻合，准确程度比经典的有限体积方法更高.

**表 9** 当 $t=0.001$，$\lambda=0.1$ 时，两种方法与精确解的误差以及绝对误差的收敛阶

Table 9 Absolute errors of the two methods compared with the exact solution and the orders of absolute errors, $t=0.001$, $\lambda=0.1$

| $N$ | classical method | | present method | |
|---|---|---|---|---|
| | error $\Delta u$ | order $n$ | error $\Delta u$ | order $n$ |
| 20 | 1.401 2E−3 | – | 1.201 4E−3 | – |
| 40 | 4.201 6E−4 | 1.736 4 | 3.678 7E−4 | 1.705 8 |
| 80 | 1.457 8E−4 | 1.527 1 | 1.246 7E−4 | 1.561 1 |
| 160 | 5.012 4E−5 | 1.540 2 | 4.127 3E−5 | 1.594 8 |

表 9 显示的是当 $t=0.001$，$\lambda=0.1$ 时通过网格加密得出的本文格式在 $L_2$- 范数下的收敛阶，从表中可以看出，新方法与经典方法都达不到理论上的二阶精度，但是新方法较经典方法的收敛速度更快一些.

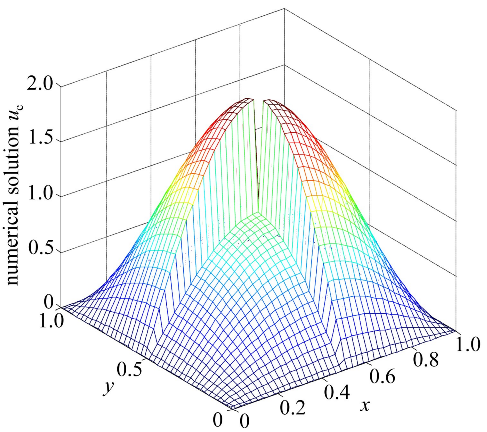


**图 3** 本文方法（$t=0.002\ 5$，$h=1/40$，$\lambda=0.1$）

Fig.3 Present method ($t=0.002\ 5$, $h=1/40$, $\lambda=0.1$)

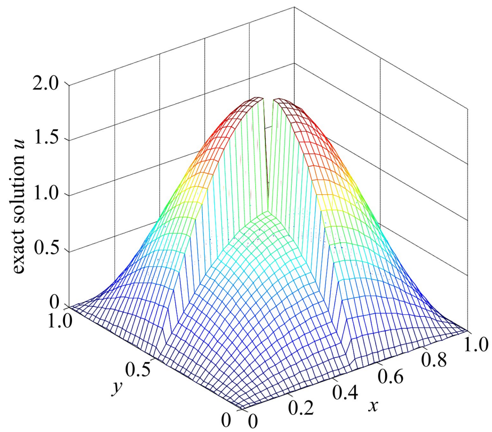


**图 4** 精确解（$t=0.002\ 5$，$h=1/40$，$\lambda=0.1$）

Fig.4 Classical method ($t=0.002\ 5$, $h=1/40$, $\lambda=0.1$)

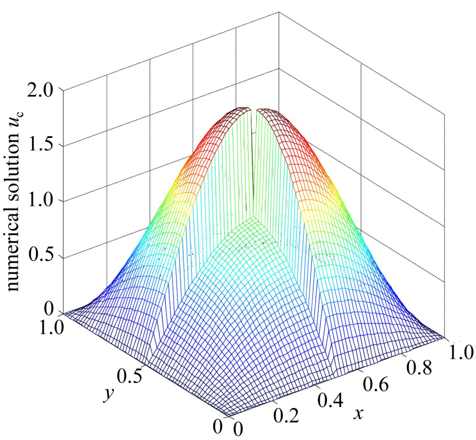


**图 5** 本文方法（$t=0.001$，$h=1/60$，$\lambda=1$）

Fig.5 Present method ($t=0.001$, $h=1/60$, $\lambda=1$)

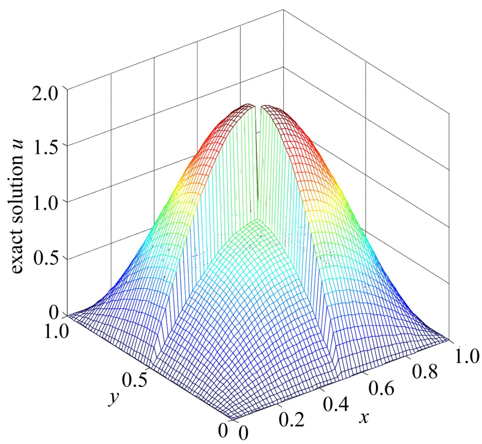


**图 6** 精确解（$t=0.001$，$h=1/60$，$\lambda=1$）

Fig.6 Classical method ($t=0.001$, $h=1/60$, $\lambda=1$)

图 3 和图 4 给出的是当 $t = 0.0025, h = 1/40, \lambda = 0.1$ 时本文方法的数值图和精确解的图. 图 5 和图 6 给出的是当 $t = 0.001, h = 1/60, \lambda = 1$ 时本文方法的数值图和精确解的图. 两组图对比不难看出本文方法能很好地逼近精确解.

例 2　考虑在例 1 的基础上对间断系数的跳跃增加一个数量级的情况.

扩散系数

$$k(x,y) = \begin{cases} 1, & (x,y) \in [0,0.5) \times [0,0.5), \\ 5 \times 10^{-2}, & (x,y) \in [0.5,1] \times [0,0.5), \\ 5 \times 10^{-2}, & (x,y) \in [0,0.5) \times [0.5,1], \\ 1, & (x,y) \in [0.5,1] \times [0.5,1]. \end{cases}$$

初始条件

$$u(x,y,0) = \begin{cases} \sin \pi x \sin \pi y, & (x,y) \in [0,0.5) \times [0,0.5), \\ 10\sin \pi x \sin \pi y, & (x,y) \in [0.5,1] \times [0,0.5), \\ 10\sin \pi x \sin \pi y, & (x,y) \in [0,0.5) \times [0.5,1], \\ \sin \pi x \sin \pi y, & (x,y) \in [0.5,1] \times [0.5,1]. \end{cases}$$

源项

$$f(x,y,t) = \begin{cases} 0, & (x,y) \in [0,0.5) \times [0,0.5), \\ -19\pi^2 e^{-2\pi^2 t} \sin \pi x \sin \pi y, & (x,y) \in [0.5,1] \times [0,0.5), \\ -19\pi^2 e^{-2\pi^2 t} \sin \pi x \sin \pi y, & (x,y) \in [0,0.5) \times [0.5,1], \\ 0, & (x,y) \in [0.5,1] \times [0.5,1]. \end{cases}$$

边界条件

$$u(0,y,t) = 0,\ u(1,y,t) = 0,\ u(x,0,t) = 0,\ u(x,1,t) = 0.$$

精确解为

$$u(x,y,t) = \begin{cases} e^{-2\pi^2 t} \sin \pi x \sin \pi y, & (x,y) \in [0,0.5) \times [0,0.5), \\ 10e^{-2\pi^2 t} \sin \pi x \sin \pi y, & (x,y) \in [0.5,1] \times [0,0.5), \\ 10e^{-2\pi^2 t} \sin \pi x \sin \pi y, & (x,y) \in [0,0.5) \times [0.5,1], \\ e^{-2\pi^2 t} \sin \pi x \sin \pi y, & (x,y) \in [0.5,1] \times [0.5,1]. \end{cases}$$

同样地,用经典的有限体积方法与本文的方法进行比较, 同时给出精确解相对应, 结果如表 10~14 所示.

表 10~14 是当间断系数的跳跃较例 1 增加一个数量级后所得结果. 对照表 1~14 能够看出,当间断系数跳跃增大时经典方法和本文的方法依然可以较好地逼近精确解,误差和收敛阶变化不大. 并且,本文方法依然比经典方法更加精确.

**表 10**　当 $t = 0.0025$, $h = 0.025$, $\lambda = 0.1$ 时, 两种方法的结果和精确值比较(节点(20, 21)、(21, 20))

Table 10　Results of the two methods in comparison with the exact solution(nodes(20, 21), (21, 20)), $t = 0.0025$, $h = 0.025$, $\lambda = 0.1$

| method | $t = 0.0025$　$h = 0.025$　$\lambda = 0.1$ | | |
|---|---|---|---|
| | nodes(20, 21), (21, 20) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 9.898 731 988 164 36 | - | - |
| classical solution | 9.898 607 398 550 96 | 1.247 478 937E-5 | 1.245 896 134E-4 |
| present solution | 9.898 717 962 795 64 | 1.404 318 673E-6 | 1.402 536 872E-5 |

**表 11** 当 $t = 0.002\,5$, $h = 0.025$, $\lambda = 0.1$ 时, 两种方法的结果和精确值比较(节点(20, 20)、(21, 21))

Table 11 Results of the two methods in comparison with the exact solution(nodes(20, 20), (21, 21)), $t = 0.002\,5$, $h = 0.025$, $\lambda = 0.1$

| method | $t = 0.002\,5$ $h = 0.025$ $\lambda = 0.1$ | | |
|---|---|---|---|
| | nodes(20, 20), (21, 21) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 0.989 873 198 816 44 | – | – |
| classical solution | 0.987 219 535 670 72 | 0.001 343 066 357 49 | 0.001 327 684 514 22 |
| present solution | 0.987 221 241 553 41 | 0.001 341 340 711 35 | 0.001 325 978 631 53 |

**表 12** 当 $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$ 时, 两种方法的结果和精确值比较(节点(20, 21)、(21, 20))

Table 12 Results of the two methods in comparison with the exact solution(nodes(20, 21), (21, 20)), $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$

| method | $t = 0.002\,5$ $h = 0.025$ $\lambda = 10$ | | |
|---|---|---|---|
| | nodes(20, 21), (21, 20) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 9.984 586 668 665 64 | – | – |
| classical solution | 9.983 550 242 951 36 | 1.038 025 657 6E−4 | 1.036 425 714 3E−3 |
| present solution | 9.984 471 454 298 12 | 1.153 922 253 8E−5 | 1.152 143 675 2E−4 |

**表 13** 当 $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$ 时, 两种方法的结果和精确值比较(节点(20, 20)、(21, 21))

Table 13 Results of the two methods in comparison with the exact solution(nodes(20, 20), (21, 21)), $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$

| method | $t = 0.002\,5$ $h = 0.025$ $\lambda = 10$ | | |
|---|---|---|---|
| | nodes(20, 20), (21, 21) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 0.998 458 666 866 56 | – | – |
| classical solution | 0.988 213 302 145 22 | 0.010 261 180 619 02 | 0.010 245 364 721 34 |
| present solution | 0.988 696 125 472 05 | 0.009 777 611 951 78 | 0.009 762 541 394 51 |

**表 14** 当 $t = 0.001$, $\lambda = 0.1$ 时, 两种方法与精确解的误差以及绝对误差的收敛阶

Table 14 Absolute errors of the two methods compared with the exact solution and the orders of absolute errors, $t = 0.001$, $\lambda = 0.1$

| $N$ | classical method | | present method | |
|---|---|---|---|---|
| | error $\Delta u$ | order $n$ | error $\Delta u$ | order $n$ |
| 20 | 1.486 4E−4 | – | 1.302 4E−3 | – |
| 40 | 5.031 2E−4 | 1.562 8 | 4.136 9E−4 | 1.654 5 |
| 80 | 1.827 8E−4 | 1.460 8 | 1.434 6E−4 | 1.527 9 |
| 160 | 6.387 6E−5 | 1.516 7 | 4.856 1E−5 | 1.562 7 |

**例 3** 考虑将例 1 中的间断系数的跳跃再增加一个数量级的情况.

扩散系数

$$k(x,y) = \begin{cases} 1, & (x,y) \in [0,0.5) \times [0,0.5), \\ 5 \times 10^{-3}, & (x,y) \in [0.5,1] \times [0,0.5), \\ 5 \times 10^{-3}, & (x,y) \in [0,0.5) \times [0.5,1], \\ 1, & (x,y) \in [0.5,1] \times [0.5,1]. \end{cases}$$

初始条件

$$u(x,y,0) = \begin{cases} \sin \pi x \sin \pi y, & (x,y) \in [0,0.5) \times [0,0.5), \\ 10^2 \sin \pi x \sin \pi y, & (x,y) \in [0.5,1] \times [0,0.5), \\ 10^2 \sin \pi x \sin \pi y, & (x,y) \in [0,0.5) \times [0.5,1], \\ \sin \pi x \sin \pi y, & (x,y) \in [0.5,1] \times [0.5,1]. \end{cases}$$

源项

$$f(x,y,t)=\begin{cases}0, & (x,y)\in[0,0.5)\times[0,0.5),\\ -199\pi^2 e^{-2\pi^2 t}\sin \pi x\sin \pi y, & (x,y)\in[0.5,1]\times[0,0.5),\\ -199\pi^2 e^{-2\pi^2 t}\sin \pi x\sin \pi y, & (x,y)\in[0,0.5)\times[0.5,1],\\ 0, & (x,y)\in[0.5,1]\times[0.5,1].\end{cases}$$

边界条件

$$u(0,y,t)=0,\ u(1,y,t)=0,\ u(x,0,t)=0,\ u(x,1,t)=0.$$

精确解为

$$u(x,y,t)=\begin{cases}e^{-2\pi^2 t}\sin \pi x\sin \pi y, & (x,y)\in[0,0.5)\times[0,0.5),\\ 10^2 e^{-2\pi^2 t}\sin \pi x\sin \pi y, & (x,y)\in[0.5,1]\times[0,0.5),\\ 10^2 e^{-2\pi^2 t}\sin \pi x\sin \pi y, & (x,y)\in[0,0.5)\times[0.5,1],\\ e^{-2\pi^2 t}\sin \pi x\sin \pi y, & (x,y)\in[0.5,1]\times[0.5,1].\end{cases}$$

表 15~19 显示的是当间断系数的跳跃较例 2 再增加一个数量级后所得结果.从表中能够看出,间断系数的跳跃级数继续增大对收敛阶影响不是很大,而且两种方法继续保持着对精确解的逼近,说明本文方法对跳跃较大的情况也能较好地处理.

**表 15** 当 $t=0.002\,5$, $h=0.025$, $\lambda=0.1$ 时, 两种方法的结果和精确值比较(节点(20, 21)、(21, 20))

Table 15 Results of the two methods in comparison with the exact solution(nodes(20, 21), (21, 20)), $t=0.002\,5$, $h=0.025$, $\lambda=0.1$

| method | $t=0.002\,5$ $h=0.025$ $\lambda=0.1$ | | |
|---|---|---|---|
| | nodes(20, 21), (21, 20) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 98.987 311 988 164 3 | - | - |
| classical solution | 98.984 993 702 850 2 | 2.342 002 492E-5 | 2.318 285 314E-3 |
| present solution | 98.987 068 375 378 1 | 2.461 050 626E-6 | 2.436 127 862E-4 |

**表 16** 当 $t=0.002\,5$, $h=0.025$, $\lambda=0.1$ 时, 两种方法的结果和精确值比较(节点(20, 20)、(21, 21))

Table 16 Results of the two methods in comparison with the exact solution(nodes(20, 20), (21, 21)), $t=0.002\,5$, $h=0.025$, $\lambda=0.1$

| method | $t=0.002\,5$ $h=0.025$ $\lambda=0.1$ | | |
|---|---|---|---|
| | nodes(20, 20), (21, 21) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 0.989 873 198 816 44 | - | - |
| classical solution | 0.987 223 465 137 86 | 0.002 676 841 520 46 | 0.002 649 733 678 58 |
| present solution | 0.987 258 912 674 51 | 0.002 641 031 341 24 | 0.002 614 286 141 93 |

**表 17** 当 $t=0.002\,5$, $h=0.025$, $\lambda=10$ 时, 两种方法的结果和精确值比较(节点(20, 21)、(21, 20))

Table 17 Results of the two methods in comparison with the exact solution(nodes(20, 21), (21, 20)), $t=0.002\,5$, $h=0.025$, $\lambda=10$

| method | $t=0.002\,5$ $h=0.025$ $\lambda=10$ | | |
|---|---|---|---|
| | nodes(20, 21), (21, 20) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 99.845 866 686 656 4 | - | - |
| classical solution | 99.835 679 853 287 3 | 1.020 255 891 2E-4 | 0.010 186 833 369 09 |
| present solution | 99.844 675 914 836 7 | 1.192 610 029 0E-5 | 0.001 190 771 819 67 |

表 18 当 $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$ 时, 两种方法的结果和精确值比较(节点(20, 20)、(21, 21))

Table 18 Results of the two methods in comparison with the exact solution(nodes(20, 20), (21, 21)), $t = 0.002\,5$, $h = 0.025$, $\lambda = 10$

| method | $t = 0.002\,5$ $h = 0.025$ $\lambda = 10$ | | |
|---|---|---|---|
| | nodes(20, 20), (21, 21) | relative error $\varepsilon$ | absolute error $\Delta u$ |
| exact solution | 0.998 458 666 866 56 | – | – |
| classical solution | 0.988 185 697 321 84 | 0.010 288 828 056 31 | 0.010 272 969 544 72 |
| present solution | 0.988 671 347 825 71 | 0.009 802 427 847 68 | 0.009 787 319 040 85 |

表 19 当 $t = 0.001$, $\lambda = 0.1$ 时, 两种方法与精确解的误差以及绝对误差的收敛阶

Table 19 Absolute errors of the two methods compared with the exact solution and the orders of absolute errors, $t = 0.001$, $\lambda = 0.1$

| $N$ | classical method | | present method | |
|---|---|---|---|---|
| | error $\Delta u$ | order $n$ | error $\Delta u$ | order $n$ |
| 20 | 1.126 3E−2 | – | 1.012 4E−2 | – |
| 40 | 3.841 2E−3 | 1.551 9 | 3.437 3E−3 | 1.558 4 |
| 80 | 1.379 4E−3 | 1.477 5 | 1.212 6E−3 | 1.503 2 |
| 160 | 5.132 6E−4 | 1.426 2 | 4.452 7E−4 | 1.445 4 |

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# A Modified Finite Volume Approximation of 2-Dimensional Diffusion Equation With Discontinuous Coefficients

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**Abstract**: A new modified finite volume method was presented to solve the 2-dimensional diffusion equation. Through improvement of the methods for solving the flux function and harmonic average coefficient, a new difference scheme was obtained for the diffusion equation with discontinuous coefficients. This scheme was an implicit difference scheme and was unconditionally stable. Subsequent numerical tests show that the presented method is more accurate than the classical finite volume method.

**Key words**: diffusion equation with discontinuous coefficient; finite volume method; interface problem

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