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用试验数据修正振动系统的双对称阻尼矩阵与刚度矩阵*

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摘要： 讨论用试验数据修正振动系统的双对称阻尼矩阵与刚度矩阵问题.依据特征方程、阻尼矩阵与刚度矩阵的双对称性,利用代数二次特征值反问题的理论和方法,研究了该问题解的存在性与唯一性,提出了修正阻尼矩阵与刚度矩阵的一个新方法.利用双对称矩阵的性质研究了方程的双对称解.给出了二次特征值反问题双对称解的一般表达式,讨论了对任意给定矩阵的最佳逼近问题,并给出了问题的最佳逼近解.用该方法修正的阻尼矩阵与刚度矩阵不仅满足二次特征方程,而且是唯一的双对称矩阵.

关　键　词： 结构模型； 反问题； 修正； 阻尼矩阵； 刚度矩阵； 双对称矩阵

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引　　言

在工程技术,特别是结构动力模型修正技术领域经常遇到的与二次特征值相反的问题,被称为二次特征值反问题(或二次逆特征值问题).2001 年 Tisseur 和 Meerbergen[1] 概述了二次特征值问题的各种应用、数学理论和数值方法.对阻尼结构进行动力分析时,应用有限元方法可得到 n 自由度振动系统的质量矩阵 $\tilde{\boldsymbol{M}}$,阻尼矩阵 $\tilde{\boldsymbol{C}}$ 和刚度矩阵 $\tilde{\boldsymbol{K}}$,从而可求得二次特征值问题的特征值(频率)λ_j 和特征向量(振型)$\boldsymbol{x}_j$, $j=1,2,3,\cdots,2n$.但是有限元模型毕竟是实际结构系统的离散化,并且在离散化过程中还必须对结构部件之间的连接条件、边界条件做力学上的简化.因此,用有限元模型作相应分析时往往存在误差.另一方面,运用测试技术可测得结构的低阶频率和相应的振型.一般来说,有限元方法的计算结果与实测结果之间存在差异.结构动力模型修正技术利用实测模型数据对有限元方法所得的质量矩阵 $\tilde{\boldsymbol{M}}$,阻尼矩阵 $\tilde{\boldsymbol{C}}$ 和刚度矩阵 $\tilde{\boldsymbol{K}}$ 进行修正,使修正的质量矩阵 $\boldsymbol{M}$,阻尼矩阵 $\boldsymbol{C}$ 和刚度矩阵 $\boldsymbol{K}$ 满足理论上的谱约束条件即 $(\lambda_j^2\boldsymbol{M}+\lambda_j\boldsymbol{C}+\boldsymbol{K})\boldsymbol{x}_j=0$, $j=1,2,\cdots,p$,并且矩阵$[\boldsymbol{M},\boldsymbol{C},\boldsymbol{K}]$ 最佳接近矩阵$[\tilde{\boldsymbol{M}},\tilde{\boldsymbol{C}},\tilde{\boldsymbol{K}}]$.文献[2]利用矩阵的奇异值分解研究了二次特征值反问题的中心对称解及其最佳逼近;文献[3]利用矩阵的向量化和 Kroneker 乘积研究了二次特征值反问题的对称次反对称解及其最佳逼近;而文献[4-7]应用不同方法对二次特征值反问题的对称解分别进行了研究.双对称矩阵是对称的

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中心对称矩阵,当系统结构对称时,系统的质量矩阵、阻尼矩阵和刚度矩阵为双对称矩阵,在科学和工程技术中具有广泛的应用[8-10].本文对双对称矩阵的二次特征值反问题及其最佳逼近进行了讨论,即用试验数据最优修正系统的阻尼矩阵和刚度矩阵.

本文中系统的质量矩阵 $\boldsymbol{M}$ 是已知的对称正定矩阵,对 $\boldsymbol{M}$ 进行 Cholesky 分解: $\boldsymbol{M}=\boldsymbol{L}\boldsymbol{L}^{\mathrm{T}}$, 则

$$\boldsymbol{Q}(\lambda)\boldsymbol{x}=\boldsymbol{0}\Leftrightarrow\tilde{\boldsymbol{Q}}(\lambda)(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{x})=\boldsymbol{0},$$

式中 $\boldsymbol{Q}(\lambda)=\lambda^2\boldsymbol{M}+\lambda\boldsymbol{C}+\boldsymbol{K},\tilde{\boldsymbol{Q}}(\lambda)=\lambda^2\boldsymbol{I}_n+\lambda\boldsymbol{L}^{-1}\boldsymbol{C}\boldsymbol{L}^{-\mathrm{T}}+\boldsymbol{L}^{-1}\boldsymbol{K}\boldsymbol{L}^{-\mathrm{T}}$,因此本文假定此反问题中 $\boldsymbol{M}$ 为单位阵,相应问题称为首 1 的二次特征值反问题(即二次束的首项系数矩阵为单位阵).

$R^{m\times n}$ 表示 $m\times n$ 阶实矩阵的集合;$C^{m\times n}$ 表示 $m\times n$ 阶复矩阵的集合;$R_{\mathrm{O}}^{n\times n}$ 为 n 阶正交矩阵的全体;$\boldsymbol{A}^{\mathrm{T}}$,rank($\boldsymbol{A}$) 分别表示矩阵 $\boldsymbol{A}$ 的转置、秩;$R_{\mathrm{S}}^{n\times n}$ 为 n 阶对称矩阵的全体;$\boldsymbol{I}_n$ 表示 n 阶单位矩阵;$\boldsymbol{S}_n$ 表示 n 阶反序单位阵;对 $\boldsymbol{A},\boldsymbol{B}\in C^{m\times n}$,定义内积 $(\boldsymbol{A},\boldsymbol{B})=\mathrm{tr}(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{B})$,则由此内积导出的范数 $\|\boldsymbol{A}\|=\sqrt{(\boldsymbol{A},\boldsymbol{A})}$ 为 Frobenius 范数.

定义[11] 设 $\boldsymbol{A}=(a_{ij})\in R^{n\times n}$,如果 $a_{ij}=a_{ji}=a_{n+1-j,n+1-i},i,j=1,2,\cdots,n$,即 $\boldsymbol{A}^{\mathrm{T}}=\boldsymbol{A},\boldsymbol{S}_n\boldsymbol{A}\boldsymbol{S}_n=\boldsymbol{A}$,则称 $\boldsymbol{A}$ 为 n 阶双对称矩阵.所有 n 阶双对称矩阵的全体记为 $R_{\mathrm{BS}}^{n\times n}$.

问题Ⅰ 给定系统的频率和模态 $\lambda_j,\boldsymbol{x}_j,j=1,2,\cdots,p$,其中 $\lambda_{2j-1}=\bar{\lambda}_{2j}=\alpha_j+\mathrm{i}\beta_j\in\mathbf{C},\alpha_j=\mathrm{Re}(\lambda_j),0<\beta_j=\mathrm{Im}(\lambda_j),j=1,2,\cdots,l;\boldsymbol{x}_{2j-1}=\bar{\boldsymbol{x}}_{2j}=\boldsymbol{y}_j+\mathrm{i}\boldsymbol{z}_j\in C^n,\boldsymbol{y}_j=\mathrm{Re}(\boldsymbol{x}_j),\boldsymbol{z}_j=\mathrm{Im}(\boldsymbol{x}_j),j=1,2,\cdots,l;\lambda_j\in R,\boldsymbol{x}_j\in R^n,j=2l+1,2l+2,\cdots,p$(模态向量为对称向量或反对称向量,即 $\boldsymbol{x}_j=\boldsymbol{S}\boldsymbol{x}_j$ 或 $\boldsymbol{x}_j=-\boldsymbol{S}\boldsymbol{x}_j$).求双对称的阻尼矩阵和刚度矩阵 $\boldsymbol{C},\boldsymbol{K}\in R_{\mathrm{BS}}^{n\times n}$, 使得

$$\lambda_j^2\boldsymbol{x}_j+\lambda_j\boldsymbol{C}\boldsymbol{x}_j+\boldsymbol{K}\boldsymbol{x}_j=\boldsymbol{0},\qquad j=1,2,\cdots,p.\tag{1}$$

令

$$\boldsymbol{\Lambda}=\mathrm{diag}\left\{\begin{bmatrix}\alpha_1&\beta_1\\-\beta_1&\alpha_1\end{bmatrix},\begin{bmatrix}\alpha_3&\beta_3\\-\beta_3&\alpha_3\end{bmatrix},\cdots,\begin{bmatrix}\alpha_{2l-1}&\beta_{2l-1}\\-\beta_{2l-1}&\alpha_{2l-1}\end{bmatrix},\lambda_{2l+1},\lambda_{2l+2},\cdots,\lambda_p\right\}\in R^{p\times p},\tag{2}$$

$$\boldsymbol{X}=[\boldsymbol{y}_1,\boldsymbol{z}_1,\boldsymbol{y}_3,\boldsymbol{z}_3,\cdots,\boldsymbol{y}_{2l-1},\boldsymbol{z}_{2l-1},\boldsymbol{x}_{2l+1},\boldsymbol{x}_{2l+2},\cdots,\boldsymbol{x}_p]\in R^{n\times p},\tag{3}$$

则式(1)等价于

$$\boldsymbol{X}\boldsymbol{\Lambda}^2+\boldsymbol{C}\boldsymbol{X}\boldsymbol{\Lambda}+\boldsymbol{K}\boldsymbol{X}=\boldsymbol{0}.\tag{4}$$

问题Ⅱ 给定 $\tilde{\boldsymbol{C}},\tilde{\boldsymbol{K}}\in R^{n\times n}$,求 $[\hat{\boldsymbol{C}},\hat{\boldsymbol{K}}]\in S_{\boldsymbol{CK}}$ 使得

$$\|\tilde{\boldsymbol{C}}-\hat{\boldsymbol{C}}\|_{\boldsymbol{W}}^2+\|\tilde{\boldsymbol{K}}-\hat{\boldsymbol{K}}\|_{\boldsymbol{W}}^2=\inf_{\forall[\boldsymbol{C},\boldsymbol{K}]\in S_{\boldsymbol{CK}}}(\|\tilde{\boldsymbol{C}}-\boldsymbol{C}\|_{\boldsymbol{W}}^2+\|\tilde{\boldsymbol{K}}-\boldsymbol{K}\|_{\boldsymbol{W}}^2),\tag{5}$$

其中 $S_{\boldsymbol{CK}}=\{[\boldsymbol{C},\boldsymbol{K}]\mid\boldsymbol{X}\boldsymbol{\Lambda}^2+\boldsymbol{C}\boldsymbol{X}\boldsymbol{\Lambda}+\boldsymbol{K}\boldsymbol{X}=\boldsymbol{0},\boldsymbol{C},\boldsymbol{K}\in R_{\mathrm{BS}}^{n\times n}\}$ 是问题Ⅰ的解集合, $\|\cdot\|_{\boldsymbol{W}}$ 是加权 Frobenius 范数,即 $\|\boldsymbol{A}\|_{\boldsymbol{W}}=\|\boldsymbol{W}\boldsymbol{A}\boldsymbol{W}\|$,$\boldsymbol{W}$ 为对称正定矩阵.

1 问题Ⅰ的解

引理 1[9] $\boldsymbol{A}\in R_{\mathrm{BS}}^{n\times n}$ 的充分必要条件是 $\boldsymbol{A}$ 的一般表达式可表示为

$$\boldsymbol{A}=\boldsymbol{D}\begin{bmatrix}\boldsymbol{A}_1&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{A}_2\end{bmatrix}\boldsymbol{D}^{\mathrm{T}},\qquad\forall\boldsymbol{A}_1\in R_{\mathrm{S}}^{(n-k)\times(n-k)},\ \forall\boldsymbol{A}_2\in R_{\mathrm{S}}^{k\times k},\tag{6}$$

其中, $n=2k$ 时,

$$\boldsymbol{D}=\frac{1}{\sqrt{2}}\begin{bmatrix}\boldsymbol{I}_k&\boldsymbol{I}_k\\\boldsymbol{S}_k&-\boldsymbol{S}_k\end{bmatrix}\in R^{n\times n},$$

$n = 2k + 1$ 时,

$$D = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_k & \boldsymbol{0} & \boldsymbol{I}_k \\ \boldsymbol{0} & \sqrt{2} & \boldsymbol{0} \\ \boldsymbol{S}_k & \boldsymbol{0} & -\boldsymbol{S}_k \end{bmatrix}.$$

引理 2 已知 $\boldsymbol{G} \in R_S^{p\times p}$,$\boldsymbol{\Lambda} \in R^{p\times p}$,存在对称矩阵 $\boldsymbol{A} \in R_S^{p\times p}$,使 $\boldsymbol{G}\boldsymbol{\Lambda}^2 + \boldsymbol{A}\boldsymbol{\Lambda}$ 对称当且仅当存在对称矩阵 $\boldsymbol{H}$,$\boldsymbol{H}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T\boldsymbol{H}$,使 $\boldsymbol{A} = -\boldsymbol{\Lambda}^T\boldsymbol{G} - \boldsymbol{G}\boldsymbol{\Lambda} + \boldsymbol{H}$.

证明 充分性显然.

必要性:取 $\boldsymbol{A}_0 = -\boldsymbol{\Lambda}^T\boldsymbol{G} - \boldsymbol{G}\boldsymbol{\Lambda}$,显然 $\boldsymbol{A}_0^T = \boldsymbol{A}_0$,可证 $\boldsymbol{G}\boldsymbol{\Lambda}^2 + \boldsymbol{A}_0\boldsymbol{\Lambda}$ 为对称矩阵.设对称矩阵 $\boldsymbol{A} \in R_S^{p\times p}$,使 $\boldsymbol{G}\boldsymbol{\Lambda}^2 + \boldsymbol{A}\boldsymbol{\Lambda}$ 对称,取 $\boldsymbol{H} = \boldsymbol{A} - \boldsymbol{A}_0$,则 $\boldsymbol{H} = \boldsymbol{H}^T$,$\boldsymbol{H}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T\boldsymbol{H}$.

引理 3 已知 $\boldsymbol{\Omega} = \mathrm{diag}\{\mu_1\boldsymbol{I}_{n_1}, \mu_2\boldsymbol{I}_{n_2}, \cdots, \mu_s\boldsymbol{I}_{n_s}\} \in R^{p\times p}$,$\mu_i \neq 0$ 互不相同,$i = 1, 2, \cdots, s$,$\sum_{i=1}^{s} n_i = p$,$\boldsymbol{Y} \in R_S^{p\times p}$,若 $\boldsymbol{Y}\boldsymbol{\Omega} = \boldsymbol{\Omega}\boldsymbol{Y}$,则 $\boldsymbol{Y} = \mathrm{diag}\{\boldsymbol{Y}_{11}, \boldsymbol{Y}_{22}, \cdots, \boldsymbol{Y}_{ss}\}$,$\boldsymbol{Y}_{ii} \in R_S^{n_i\times n_i}$.

证明 设 $\boldsymbol{Y} = (\boldsymbol{Y}_{ij}) \in R_S^{p\times p}$, $\boldsymbol{Y}_{ij} \in R^{n_i\times n_j}$, 由 $\boldsymbol{Y}\boldsymbol{\Omega} = \boldsymbol{\Omega}\boldsymbol{Y}$, 得 $\boldsymbol{Y}_{ij}\mu_j = \mu_i\boldsymbol{Y}_{ji}^T = \mu_i\boldsymbol{Y}_{ij}$,当 $i \neq j$ 时,$\boldsymbol{Y}_{ij} = \boldsymbol{0}$.故结论成立.

引理 4 已知 $\boldsymbol{\Lambda} \in R^{p\times p}$ 如公式(2),则存在对称矩阵 $\boldsymbol{H}$,使 $\boldsymbol{H}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T\boldsymbol{H}$,且 $\boldsymbol{H}$ 可表示为

$$\boldsymbol{H} = \mathrm{diag}\left\{\begin{bmatrix} \varepsilon_1 & \delta_1 \\ \delta_1 & -\varepsilon_1 \end{bmatrix}, \begin{bmatrix} \varepsilon_3 & \delta_3 \\ \delta_3 & -\varepsilon_3 \end{bmatrix}, \cdots, \begin{bmatrix} \varepsilon_{2l-1} & \delta_{2l-1} \\ \delta_{2l-1} & -\varepsilon_{2l-1} \end{bmatrix}, \xi_{2l+1}, \xi_{2l+2}, \cdots, \xi_p\right\}, \tag{7}$$

其中 ε_{2j-1},$\delta_{2j-1}(j = 1,2,\cdots,l)$,$\xi_j(j = 2l+1, 2l+2, \cdots, p)$ 为任意实数.

证明 将式(2)中矩阵 $\boldsymbol{\Lambda} \in R^{p\times p}$ 变形为

$$\boldsymbol{\Lambda} = \boldsymbol{\Omega}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Omega}, \tag{8}$$

其中

$$\boldsymbol{\Omega} = \mathrm{diag}\{|\lambda_1|\boldsymbol{I}_2, |\lambda_3|\boldsymbol{I}_2, \cdots, |\lambda_{2l-1}|\boldsymbol{I}_2, \lambda_{2l+1}, \lambda_{2l+2}, \cdots, \lambda_p\} \in R^{p\times p},$$

$$\boldsymbol{V} = \mathrm{diag}\left\{\frac{1}{|\lambda_1|}\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \frac{1}{|\lambda_3|}\begin{bmatrix} \alpha_3 & \beta_3 \\ -\beta_3 & \alpha_3 \end{bmatrix}, \cdots, \frac{1}{|\lambda_{2l-1}|}\begin{bmatrix} \alpha_{2l-1} & \beta_{2l-1} \\ -\beta_{2l-1} & \alpha_{2l-1} \end{bmatrix}, 1, \cdots, 1\right\} \in R_0^{p\times p}.$$

由 $\boldsymbol{H}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T\boldsymbol{H}$,可得 $\boldsymbol{H}\boldsymbol{V}\boldsymbol{\Omega} = \boldsymbol{\Omega}\boldsymbol{V}^T\boldsymbol{H}$.由引理 3,可得

$$\boldsymbol{H}\boldsymbol{V} = \mathrm{diag}\{\boldsymbol{X}_1, \boldsymbol{X}_3, \cdots, \boldsymbol{X}_{2l-1}, \xi_{2l+1}, \xi_{2l+2}, \cdots, \xi_p\},$$

其中 $\boldsymbol{X}_i \in R_S^{2\times 2}$,$i = 1,3,\cdots,2l-1$,$\xi_j(j = 2l+1, 2l+2, \cdots, p)$ 为任意实数.

上式等号两端右乘 $\boldsymbol{V}^T$, 得

$$\boldsymbol{H} = \mathrm{diag}\{\boldsymbol{X}_1, \boldsymbol{X}_3, \cdots, \boldsymbol{X}_{2l-1}, \xi_{2l+1}, \xi_{2l+2}, \cdots, \xi_p\}\boldsymbol{V}^T =$$

$$\mathrm{diag}\left\{\frac{\boldsymbol{X}_1}{|\lambda_1|}\begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix}, \frac{\boldsymbol{X}_3}{|\lambda_3|}\begin{bmatrix} \alpha_3 & -\beta_3 \\ \beta_3 & \alpha_3 \end{bmatrix}, \cdots, \frac{\boldsymbol{X}_{2l-1}}{|\lambda_{2l-1}|}\begin{bmatrix} \alpha_{2l-1} & -\beta_{2l-1} \\ \beta_{2l-1} & \alpha_{2l-1} \end{bmatrix}, \xi_{2l+1}, \xi_{2l+2}, \cdots, \xi_p\right\}.$$

记 $\boldsymbol{H}_i = \dfrac{\boldsymbol{X}_i}{|\lambda_i|}\begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}$,$i = 1,3,\cdots,2l-1$, 进而

$$\boldsymbol{H}_i = \frac{1}{|\lambda_i|}\begin{bmatrix} x_{i,i} & x_{i,i+1} \\ x_{i,i+1} & x_{i+1,i+1}\end{bmatrix}\begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix} =$$

$$\frac{1}{|\lambda_i|}\begin{bmatrix} x_{i,i}\alpha_i + x_{i,i+1}\beta_i & -x_{i,i}\beta_i + x_{i,i+1}\alpha_i \\ x_{i,i+1}\alpha_i + x_{i+1,i+1}\beta_i & -x_{i,i+1}\beta_i + x_{i+1,i+1}\alpha_i \end{bmatrix}.$$

由 $\boldsymbol{H} = \boldsymbol{H}^{\mathrm{T}}$，有 $\boldsymbol{H}_i = \boldsymbol{H}_i^{\mathrm{T}}, i = 1,3,\cdots,2l-1$，因为 $\beta_i > 0$，可得 $x_{i,i} = -x_{i+1,i+1}$，令

$$\varepsilon_i = \frac{1}{|\lambda_i|}(x_{i,i}\alpha_i + x_{i,i+1}\beta_i),\ \delta_i = \frac{1}{|\lambda_i|}(-x_{i,i}\beta_i + x_{i,i+1}\alpha_i),$$

得

$$\boldsymbol{H}_i = \begin{bmatrix} \varepsilon_i & \delta_i \\ \delta_i & -\varepsilon_i \end{bmatrix},$$

故 $\boldsymbol{H}$ 可表示为式(7).

对模态矩阵 $\boldsymbol{X}$ 进行奇异值分解

$$\boldsymbol{X} = \boldsymbol{U}\begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{U}_1\boldsymbol{\Sigma}\boldsymbol{Q}_1^{\mathrm{T}}, \tag{9}$$

其中

$$\boldsymbol{U} = (\boldsymbol{U}_1, \boldsymbol{U}_2) \in R_0^{n\times n},\ \boldsymbol{U}_1 \in R^{n\times r},\ \boldsymbol{U}_2 \in R^{n\times(n-r)},\ \boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1,\sigma_2,\cdots,\sigma_r) > 0,$$
$$r = \mathrm{rank}(\boldsymbol{X}),\ \boldsymbol{Q} = (\boldsymbol{Q}_1, \boldsymbol{Q}_2) \in R_0^{p\times p},$$
$$\boldsymbol{Q}_1 = (\boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_r) \in R^{p\times r},\ \boldsymbol{Q}_2 = (\boldsymbol{q}_{r+1}, \boldsymbol{q}_{r+2}, \cdots, \boldsymbol{q}_p) \in R^{p\times(p-r)}.$$

令 $\boldsymbol{q}_j = (q_{1j}, q_{2j}, \cdots, q_{pj})^{\mathrm{T}} \in R^p$（下文中 $q_{i,j}$ 简写为 q_{ij}），$\boldsymbol{G} = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}$，$\boldsymbol{g}_j = \boldsymbol{G}\boldsymbol{\Lambda}\boldsymbol{q}_j$，$j = r+1,\ r+2,\ \cdots, p$，

$$\boldsymbol{g} = \begin{bmatrix} \boldsymbol{g}_{r+1} \\ \boldsymbol{g}_{r+2} \\ \vdots \\ \boldsymbol{g}_p \end{bmatrix} \in R^{p(p-r)}, \tag{10}$$

$$\boldsymbol{N}_j = \mathrm{diag}\left\{\begin{bmatrix} q_{1j} & q_{2j} \\ -q_{2j} & q_{1j}\end{bmatrix}, \begin{bmatrix} q_{3j} & q_{4j} \\ -q_{4j} & q_{3j}\end{bmatrix}, \cdots, \begin{bmatrix} q_{2l-1,j} & q_{2l,j} \\ -q_{2l,j} & q_{2l-1,j}\end{bmatrix}, q_{2l+1,j}, q_{2l+2,j}, \cdots, q_{p,j}\right\},$$
$$j = r+1, r+2, \cdots, p,$$

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_{r+1} \\ \boldsymbol{N}_{r+2} \\ \vdots \\ \boldsymbol{N}_p \end{bmatrix} \in R^{p(p-r)\times p}, \tag{11}$$

$$\boldsymbol{h} = (\varepsilon_1, \delta_1, \varepsilon_3, \delta_3, \cdots, \varepsilon_{2l-1}, \delta_{2l-1}, \xi_{2l+1}, \xi_{2l+2}, \cdots, \xi_p)^{\mathrm{T}} \in R^p.$$

定理 1 给定频率矩阵 $\boldsymbol{\Lambda} \in R^{p\times p}$，模态矩阵 $\boldsymbol{X} \in R^{n\times p}$ 如式(2)、(3)，$\boldsymbol{X}$ 的奇异值分解为式(9)，$\boldsymbol{g}, \boldsymbol{N}$ 如式(10)、(11)，则存在对称的阻尼矩阵和刚度矩阵 $\boldsymbol{C}, \boldsymbol{K} \in R_{\mathrm{S}}^{n\times n}$，使得 $\boldsymbol{X}\boldsymbol{\Lambda}^2 + \boldsymbol{C}\boldsymbol{X}\boldsymbol{\Lambda} + \boldsymbol{K}\boldsymbol{X} = \boldsymbol{0}$ 成立的充分必要条件为

$$\boldsymbol{N}\boldsymbol{N}^{+}\boldsymbol{g} = \boldsymbol{g}, \tag{12}$$

且 $\boldsymbol{C}, \boldsymbol{K}$ 的形式为

$$C = \begin{bmatrix} -(X\Lambda X^+)^T - X\Lambda X^+ + (X^+)^T H X^+ & U_1 C_{12} U_2^T \\ U_2 C_{12}^T U_1^T & U_2 C_{22} U_2^T \end{bmatrix}, \tag{13}$$

$$K = \begin{bmatrix} (X\Lambda X^+)^T X\Lambda X^+ - (X^+)^T H \Lambda X^+ & -(X\Lambda X^+)^T U_1 C_{12} U_2^T \\ -U_2 C_{12}^T U_1^T X\Lambda X^+ & U_2 K_{22} U_2^T \end{bmatrix}, \tag{14}$$

其中 $C_{12} \in R^{r\times(n-r)}$ 为任意矩阵，$C_{22}, K_{22} \in R_S^{(n-r)\times(n-r)}$ 为任意对称矩阵，H 如式(7)，H 中的元素由 $h = N^+ g + (I_p - N^+ N)y$，$\forall y \in R^p$ 给出。

$X\Lambda^2 + CX\Lambda + KX = 0$ 有唯一对称解 $C, K \in R_S^{n\times n}$ 的充分必要条件为

$$NN^+ g = g,\ \mathrm{rank}(N) = p, \tag{15}$$

解的形式如式(13)、(14)，H 中的元素由 $h = N^+ g$ 给出。

证明 将模态矩阵 X 的奇异值分解式(9)代入式(4)，得

$$\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} Q^T \Lambda^2 + U^T C U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} Q^T \Lambda + U^T K U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} Q^T = 0. \tag{16}$$

记

$$U^T C U = \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{pmatrix},\ U^T K U = \begin{pmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{pmatrix}, \tag{17}$$

其中 $C_{11}, K_{11} \in R_S^{r\times r}$，$C_{22}, K_{22} \in R_S^{(n-r)\times(n-r)}$，将其代入式(16)，有

$$\Sigma Q_1^T \Lambda^2 + C_{11} \Sigma Q_1^T \Lambda + K_{11} \Sigma Q_1^T = 0, \tag{18}$$

$$C_{12}^T \Sigma Q_1^T \Lambda + K_{12}^T \Sigma Q_1^T = 0. \tag{19}$$

记

$$G = Q_1 \Sigma^2 Q_1^T,\ A = Q_1 \Sigma C_{11} \Sigma Q_1^T,\ B = Q_1 \Sigma K_{11} \Sigma Q_1^T, \tag{20}$$

式(18)等价于

$$G\Lambda^2 + A\Lambda + B = 0. \tag{21}$$

由于矩阵 B 为对称矩阵，式(21)等价于存在对称矩阵 A，使得

$$G\Lambda^2 + A\Lambda = \Lambda^T A + (\Lambda^T)^2 G. \tag{22}$$

由引理 2，可知式(22)的解为

$$A = -\Lambda^T G - G\Lambda + H, \tag{23}$$

其中 H 由式(7)给出，将上式代入式(21)，得

$$B = \Lambda^T G \Lambda - H\Lambda. \tag{24}$$

利用式(20)，由式(23)、(24)，可得

$$Q_1 \Sigma C_{11} \Sigma Q_1^T = -\Lambda^T G - G\Lambda + H, \tag{25}$$

$$Q_1 \Sigma K_{11} \Sigma Q_1^T = \Lambda^T G \Lambda - H\Lambda. \tag{26}$$

式(25)与(26)有对称解 $C_{11}, K_{11} \in R_S^{r\times r}$ 当且仅当存在对称矩阵 H，使得

$$HQ_2 = G\Lambda Q_2, \tag{27}$$

$$H\Lambda Q_2 = \Lambda^T G \Lambda Q_2. \tag{28}$$

若式(27)成立，则式(28)必然成立。故由式(27)可得

$$Hq_j = G\Lambda q_j,\quad j = r+1, r+2, \cdots, p, \tag{29}$$

记

$$q_j = (q_{1j}, q_{2j}, \cdots, q_{pj})^T \in R^p,\ j = r+1, r+2, \cdots, p,$$

式(29)等价于

$$\begin{bmatrix} \begin{bmatrix} q_{1j} & q_{2j} \\ -q_{2j} & q_{1j} \end{bmatrix} & & & & & & \\ & \begin{bmatrix} q_{3j} & q_{4j} \\ -q_{4j} & q_{3j} \end{bmatrix} & & & \mathbf{0} & & \\ & & \ddots & & & & \\ & & & \begin{bmatrix} q_{2l-1,j} & q_{2l,j} \\ -q_{2l,j} & q_{2l-1,j} \end{bmatrix} & & & \\ & \mathbf{0} & & & q_{2l+1,j} & & \\ & & & & & q_{2l+2,j} & \\ & & & & & & \ddots \\ & & & & & & & q_{p,j} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \delta_1 \\ \varepsilon_3 \\ \delta_3 \\ \vdots \\ \varepsilon_{2l-1} \\ \delta_{2l-1} \\ \xi_{2l+1} \\ \xi_{2l+2} \\ \vdots \\ \xi_p \end{bmatrix} = \boldsymbol{G\Lambda q}_j,$$

$$j = r+1, r+2, \cdots, p. \tag{30}$$

$\boldsymbol{h} = (\varepsilon_1, \delta_1, \varepsilon_3, \delta_3, \cdots, \varepsilon_{2l-1}, \delta_{2l-1}, \xi_{2l+1}, \xi_{2l+2}, \cdots, \xi_p)^{\mathrm{T}} \in R^p$, $\boldsymbol{g}$, $\boldsymbol{N}$ 如式(10)、(11).式(30)等价于 $\boldsymbol{Nh} = \boldsymbol{g}$, 由此可得问题有解的充分必要条件及解的表达形式.由式(25)、(26),可得

$$\boldsymbol{C}_{11} = -\boldsymbol{\Sigma}^{-1}\boldsymbol{Q}_1^{\mathrm{T}}(\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G} + \boldsymbol{G\Lambda})\boldsymbol{Q}_1\boldsymbol{\Sigma}^{-1} + \boldsymbol{\Sigma}^{-1}\boldsymbol{Q}_1^{\mathrm{T}}\boldsymbol{HQ}_1\boldsymbol{\Sigma}^{-1}, \tag{31}$$

$$\boldsymbol{K}_{11} = \boldsymbol{\Sigma}^{-1}\boldsymbol{Q}_1^{\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G\Lambda Q}_1\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{Q}_1^{\mathrm{T}}\boldsymbol{H\Lambda Q}_1\boldsymbol{\Sigma}^{-1}. \tag{32}$$

由式(19),可得

$$\boldsymbol{K}_{12} = -\boldsymbol{\Sigma}^{-1}\boldsymbol{Q}_1^{\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{Q}_1\boldsymbol{\Sigma C}_{12}. \tag{33}$$

将 $\boldsymbol{C}_{11}, \boldsymbol{K}_{11}, \boldsymbol{K}_{12}$ 代入式(17),再利用 $\boldsymbol{X} = \boldsymbol{U}_1\boldsymbol{\Sigma Q}_1^{\mathrm{T}}$, $\boldsymbol{X}^{+} = \boldsymbol{Q}_1\boldsymbol{\Sigma}^{-1}\boldsymbol{U}_1^{\mathrm{T}}$, $\boldsymbol{G} = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}$, 可得式(13)、(14).定理证毕.

当模态矩阵 $\boldsymbol{X} \in R^{n\times p}$ 列满秩时,即 $\mathrm{rank}(\boldsymbol{X}) = p$, $\boldsymbol{X}$ 的奇异值分解为

$$\boldsymbol{X} = \boldsymbol{U}\begin{bmatrix} \boldsymbol{\Sigma} \\ \mathbf{0} \end{bmatrix}\boldsymbol{Q}^{\mathrm{T}}, \tag{34}$$

其中

$$\boldsymbol{U} = (\boldsymbol{U}_1, \boldsymbol{U}_2) \in R_0^{n\times n},\ \boldsymbol{U}_1 \in R^{n\times p},\ \boldsymbol{U}_2 \in R^{n\times(n-p)},$$

$$\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_p) > 0,\ \boldsymbol{Q} \in R_0^{p\times p}.$$

此时由定理 1 可得文献[4]的定理 2.1.

推论 1 给定频率矩阵 $\boldsymbol{\Lambda} \in R^{p\times p}$,模态矩阵 $\boldsymbol{X} \in R^{n\times p}$ 如式(2)、(3), $\mathrm{rank}(\boldsymbol{X}) = p$, $\boldsymbol{X}$ 的奇异值分解为式(34),则 $\boldsymbol{X\Lambda}^2 + \boldsymbol{CX\Lambda} + \boldsymbol{KX} = \mathbf{0}$ 有解(对称的阻尼矩阵和刚度矩阵) $\boldsymbol{C}, \boldsymbol{K} \in R_{\mathrm{S}}^{n\times n}$,且 $\boldsymbol{C}, \boldsymbol{K}$ 的形式为

$$\boldsymbol{C} = \boldsymbol{U}\begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{C}_{12}^{\mathrm{T}} & \boldsymbol{C}_{22} \end{bmatrix}\boldsymbol{U}^{\mathrm{T}},\ \boldsymbol{K} = \boldsymbol{U}\begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{12}^{\mathrm{T}} & \boldsymbol{K}_{22} \end{bmatrix}\boldsymbol{U}^{\mathrm{T}},$$

其中

$$\boldsymbol{C}_{11} = -\boldsymbol{\Sigma}^{-1}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G} + \boldsymbol{G\Lambda})\boldsymbol{Q\Sigma}^{-1} + \boldsymbol{\Sigma}^{-1}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{HQ\Sigma}^{-1} \in R^{p\times p},\ \boldsymbol{G} = \boldsymbol{Q\Sigma}^2\boldsymbol{Q}^{\mathrm{T}} \in R^{p\times p},$$

$$\boldsymbol{K}_{11} = \boldsymbol{\Sigma}^{-1}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G\Lambda Q\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{H\Lambda Q\Sigma}^{-1} \in R^{p\times p},$$

$$\boldsymbol{K}_{12}^{\mathrm{T}} = -\boldsymbol{C}_{12}^{\mathrm{T}}\boldsymbol{\Sigma Q}^{\mathrm{T}}\boldsymbol{\Lambda Q\Sigma}^{-1} \in R^{(n-p)\times p},$$

$\boldsymbol{C}_{12} \in R^{p\times(n-p)}$ 为任意矩阵, $\boldsymbol{C}_{22}, \boldsymbol{K}_{22} \in R_{\mathrm{S}}^{(n-p)\times(n-p)}$ 为任意对称矩阵, $\boldsymbol{H}$ 如式(7).

当模态矩阵 $\boldsymbol{X} \in R^{n\times p}$ 行满秩时,即 $\mathrm{rank}(\boldsymbol{X}) = n$ 时,$\boldsymbol{X}$ 的奇异值分解为

$$\boldsymbol{X} = \boldsymbol{U}(\boldsymbol{\Sigma},\boldsymbol{0})\boldsymbol{Q}^{\mathrm{T}}, \tag{35}$$

其中

$$\boldsymbol{U} \in R_0^{n\times n},\ \boldsymbol{Q} = (\boldsymbol{Q}_1,\boldsymbol{Q}_2) \in R_0^{p\times p},\ \boldsymbol{Q}_1 = (\boldsymbol{q}_1,\boldsymbol{q}_2,\cdots,\boldsymbol{q}_n) \in R^{p\times n},$$

$$\boldsymbol{Q}_2 = (\boldsymbol{q}_{n+1},\boldsymbol{q}_{n+2},\cdots,\boldsymbol{q}_p) \in R^{p\times(p-n)},\ \boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1,\sigma_2,\cdots,\sigma_n) > 0.$$

令 $\boldsymbol{q}_j = (q_{1j},q_{2j},\cdots,q_{pj})^{\mathrm{T}} \in R^p$, $\boldsymbol{G} = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}$, $\boldsymbol{g}_j = \boldsymbol{G\Lambda q}_j$, $j = n+1,n+2,\cdots,p$,

$$\tilde{\boldsymbol{g}} = \begin{bmatrix} \boldsymbol{g}_{n+1} \\ \boldsymbol{g}_{n+2} \\ \vdots \\ \boldsymbol{g}_p \end{bmatrix} \in R^{p(p-n)}, \tag{36}$$

$$\boldsymbol{N}_j = \mathrm{diag}\left\{\begin{bmatrix} q_{1j} & q_{2j} \\ -q_{2j} & q_{1j} \end{bmatrix},\begin{bmatrix} q_{3j} & q_{4j} \\ -q_{4j} & q_{3j} \end{bmatrix},\cdots,\begin{bmatrix} q_{2l-1,j} & q_{2l,j} \\ -q_{2l,j} & q_{2l-1,j} \end{bmatrix},q_{2l+1,j},q_{2l+2,j},\cdots,q_{p,j}\right\},$$
$$j = n+1,n+2,\cdots,p,$$

$$\tilde{\boldsymbol{N}} = \begin{bmatrix} \boldsymbol{N}_{n+1} \\ \boldsymbol{N}_{n+2} \\ \vdots \\ \boldsymbol{N}_p \end{bmatrix} \in R^{p(p-n)\times p}, \tag{37}$$

$$\boldsymbol{h} = (\varepsilon_1,\delta_1,\varepsilon_3,\delta_3,\cdots,\varepsilon_{2l-1},\delta_{2l-1},\xi_{2l+1},\xi_{2l+2},\cdots,\xi_p)^{\mathrm{T}} \in R^p.$$

下面的推论 2 是对文献[5]中定理的推广,当 $p = n+1$ 时,即为文献[5]中定理 2.1 的结果.

推论 2 给定频率矩阵 $\boldsymbol{\Lambda} \in R^{p\times p}$,模态矩阵 $\boldsymbol{X} \in R^{n\times p}$ 如式(2)、(3),$\mathrm{rank}(\boldsymbol{X})=n$,$\boldsymbol{X}$ 的奇异值分解为式(35),$\tilde{\boldsymbol{g}}$,$\tilde{\boldsymbol{N}}$ 如式(36)、(37),则存在对称的阻尼矩阵和刚度矩阵 $\boldsymbol{C},\boldsymbol{K} \in R_{\mathrm{S}}^{n\times n}$,使得 $\boldsymbol{X\Lambda}^2 + \boldsymbol{CX\Lambda} + \boldsymbol{KX} = \boldsymbol{0}$ 成立的充分必要条件为

$$\tilde{\boldsymbol{N}}\tilde{\boldsymbol{N}}^{+}\tilde{\boldsymbol{g}} = \tilde{\boldsymbol{g}}, \tag{38}$$

且 $\boldsymbol{C},\boldsymbol{K}$ 的形式为

$$\boldsymbol{C} = -(\boldsymbol{X\Lambda X}^{+})^{\mathrm{T}} - \boldsymbol{X\Lambda X}^{+} + (\boldsymbol{X}^{+})^{\mathrm{T}}\boldsymbol{HX}^{+}, \tag{39}$$

$$\boldsymbol{K} = (\boldsymbol{X\Lambda X}^{+})^{\mathrm{T}}\boldsymbol{X\Lambda X}^{+} - (\boldsymbol{X}^{+})^{\mathrm{T}}\boldsymbol{H\Lambda X}^{+}, \tag{40}$$

其中 $\boldsymbol{H}$ 如式(7),$\boldsymbol{H}$ 中的元素由 $\boldsymbol{h} = \tilde{\boldsymbol{N}}^{+}\tilde{\boldsymbol{g}} + (\boldsymbol{I}_p - \tilde{\boldsymbol{N}}^{+}\tilde{\boldsymbol{N}})\boldsymbol{y}$, $\forall \boldsymbol{y} \in R^p$ 给出.

$\boldsymbol{X\Lambda}^2 + \boldsymbol{CX\Lambda} + \boldsymbol{KX} = \boldsymbol{0}$ 有唯一对称解 $\boldsymbol{C},\boldsymbol{K} \in R_{\mathrm{S}}^{n\times n}$ 的充分必要条件为

$$\tilde{\boldsymbol{N}}\tilde{\boldsymbol{N}}^{+}\tilde{\boldsymbol{g}} = \tilde{\boldsymbol{g}},\ \mathrm{rank}(\tilde{\boldsymbol{N}}) = p, \tag{41}$$

解的形式如式(39)、(40),$\boldsymbol{H}$ 中的元素由 $\boldsymbol{h} = \tilde{\boldsymbol{N}}^{+}\tilde{\boldsymbol{g}}$ 给出.

令 $\boldsymbol{D}^{\mathrm{T}}\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}$,$\boldsymbol{X}_1 \in R^{(n-k)\times p}$,$\boldsymbol{X}_2 \in R^{k\times p}$;对矩阵 $\boldsymbol{X}_i(i = 1,2)$ 进行奇异值分解,

$$\boldsymbol{X}_i = \boldsymbol{U}^{(i)}\begin{bmatrix} \boldsymbol{\Sigma}_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{Q}^{(i)\mathrm{T}} \quad (i = 1,2), \tag{42}$$

其中

$$\boldsymbol{U}^{(1)} \in R_0^{(n-k)\times(n-k)},\ \boldsymbol{U}^{(2)} \in R_0^{k\times k},\ r_i = \mathrm{rank}(\boldsymbol{X}_i),\ \boldsymbol{\Sigma}_1 = \mathrm{diag}(\mu_1,\mu_2,\cdots,\mu_{r_1}) > 0,$$

$$\boldsymbol{\Sigma}_2 = \mathrm{diag}(\nu_1,\nu_2,\cdots,\nu_{r_2}) > 0,\ \boldsymbol{Q}^{(i)} = (\boldsymbol{Q}_1^{(i)},\boldsymbol{Q}_2^{(i)}) \in R_0^{p\times p},$$

$$\boldsymbol{Q}_1^{(i)} = (\boldsymbol{q}_1^{(i)},\boldsymbol{q}_2^{(i)},\cdots,\boldsymbol{q}_{r_i}^{(i)}) \in R^{p\times r_i},$$

$$Q_2^{(i)}=(q_{r_i+1}^{(i)},q_{r_i+2}^{(i)},\cdots,q_p^{(i)})\in R^{p\times(p-r_i)},\qquad i=1,2.$$

令 $q_j^{(i)}=(q_{1j}^{(i)},q_{2j}^{(i)},\cdots,q_{pj}^{(i)})^{\mathrm T}\in R^p$, $G_i=X_i^{\mathrm T}X_i$, $g_j^{(i)}=G_i\Lambda q_j^{(i)}$, $j=r_i+1,r_i+2,\cdots,p$,

$$g^{(i)}=\begin{bmatrix}g_{r_i+1}^{(i)}\\ g_{r_i+2}^{(i)}\\ \vdots\\ g_p^{(i)}\end{bmatrix}\in R^{p(p-r_i)},\qquad i=1,2,\tag{43}$$

$$N_j^{(i)}=\mathrm{diag}\left\{\begin{bmatrix}q_{1j}^{(i)} & q_{2j}^{(i)}\\ -q_{2j}^{(i)} & q_{1j}^{(i)}\end{bmatrix},\begin{bmatrix}q_{3j}^{(i)} & q_{4j}^{(i)}\\ -q_{4j}^{(i)} & q_{3j}^{(i)}\end{bmatrix},\cdots,\right.$$
$$\left.\begin{bmatrix}q_{2l-1,j}^{(i)} & q_{2l,j}^{(i)}\\ -q_{2l,j}^{(i)} & q_{2l-1,j}^{(i)}\end{bmatrix},q_{2l+1,j}^{(i)},q_{2l+2,j}^{(i)},\cdots,q_{p,j}^{(i)}\right\},$$

$$N^{(i)}=\begin{bmatrix}N_{r_i+1}^{(i)}\\ N_{r_i+2}^{(i)}\\ \vdots\\ N_p^{(i)}\end{bmatrix}\in R^{p(p-r_i)\times p},\qquad i=1,2,\tag{44}$$

$$h_i=(\varepsilon_1^{(i)},\delta_1^{(i)},\varepsilon_3^{(i)},\delta_3^{(i)},\cdots,\varepsilon_{2l-1}^{(i)},\delta_{2l-1}^{(i)},\xi_{2l+1}^{(i)},\xi_{2l+2}^{(i)},\cdots,\xi_p^{(i)})^{\mathrm T}\in R^p,\qquad i=1,2.$$

定理 2 给定频率矩阵 $\Lambda\in R^{p\times p}$，模态矩阵 $X\in R^{n\times p}$ 如式(2)、(3)，$D^{\mathrm T}X=\begin{bmatrix}X_1\\ X_2\end{bmatrix}$，对 X_i $(i=1,2)$ 进行奇异值分解如式(42)，则问题 I 有解 $C,K\in R_{\mathrm{BS}}^{n\times n}$ 的充分必要条件为

$$N^{(i)}N^{(i)+}g^{(i)}=g^{(i)},\qquad i=1,2,\tag{45}$$

且其通解为

$$C=D\begin{bmatrix}C_1 & 0\\ 0 & C_2\end{bmatrix}D^{\mathrm T},\ K=D\begin{bmatrix}K_1 & 0\\ 0 & K_2\end{bmatrix}D^{\mathrm T},\tag{46}$$

其中

$$C_i=U^{(i)}\begin{bmatrix}-\Sigma_i^{-1}Q_1^{(i)\mathrm T}(\Lambda^{\mathrm T}G_i+G_i\Lambda)Q_1^{(i)}\Sigma_i^{-1}+\Sigma_i^{-1}Q_1^{(i)\mathrm T}H_iQ_1^{(i)}\Sigma_i^{-1} & C_{12}^{(i)}\\ C_{12}^{(i)T} & C_{22}^{(i)}\end{bmatrix}U^{(i)\mathrm T},\tag{47}$$

$$K_i=U^{(i)}\begin{bmatrix}\Sigma_i^{-1}Q_1^{(i)\mathrm T}\Lambda^{\mathrm T}G_i\Lambda Q_1^{(i)}\Sigma_i^{-1}-\Sigma_i^{-1}Q_1^{(i)\mathrm T}H_i\Lambda Q_1^{(i)}\Sigma_i^{-1} & -\Sigma_i^{-1}Q_1^{(i)\mathrm T}\Lambda^{\mathrm T}Q_1^{(i)}\Sigma_iC_{12}^{(i)}\\ -C_{12}^{(i)\mathrm T}\Sigma_iQ_1^{(i)\mathrm T}\Lambda Q_1^{(i)}\Sigma_i^{-1} & K_{22}^{(i)}\end{bmatrix}U^{(i)\mathrm T},\tag{48}$$

式中 $i=1,2$，$G_i=Q_1^{(i)}\Sigma_i^2Q_1^{(i)\mathrm T}$，$C_{12}^{(1)}\in R^{r_1\times(n-k-r_1)}$，$C_{12}^{(2)}\in R^{r_2\times(k-r_2)}$ 为任意矩阵，$C_{22}^{(1)},K_{22}^{(1)}\in R_{\mathrm S}^{(n-k-r_1)\times(n-k-r_1)}$，$C_{22}^{(2)},K_{22}^{(2)}\in R_{\mathrm S}^{(k-r_2)\times(k-r_2)}$ 为任意对称矩阵，

$$U^{(1)}\in R_0^{(n-k)\times(n-k)},\ U^{(2)}\in R_0^{k\times k},$$

$$\Sigma_1=\mathrm{diag}\{\mu_1,\mu_2,\cdots,\mu_{r_1}\}>0,\ \Sigma_2=\mathrm{diag}\{\nu_1,\nu_2,\cdots,\nu_{r_2}\}>0,$$

$$H_i=\mathrm{diag}\left\{\begin{bmatrix}\varepsilon_1^{(i)} & \delta_1^{(i)}\\ \delta_1^{(i)} & -\varepsilon_1^{(i)}\end{bmatrix},\begin{bmatrix}\varepsilon_3^{(i)} & \delta_3^{(i)}\\ \delta_3^{(i)} & -\varepsilon_3^{(i)}\end{bmatrix},\cdots,\right.$$

$$\begin{bmatrix} \varepsilon_{2l-1}^{(i)} & \delta_{2l-1}^{(i)} \\ \delta_{2l-1}^{(i)} & -\varepsilon_{2l-1}^{(i)} \end{bmatrix}, \xi_{2l+1}^{(i)}, \xi_{2l+2}^{(i)}, \cdots, \xi_p^{(i)} \Bigg\}, \tag{49}$$

式中 $\varepsilon_{2j-1}^{(i)}$，$\delta_{2j-1}^{(i)}(j=1,2,\cdots,l)$，$\xi_j^{(i)}(j=2l+1,\ 2l+2,\ \cdots,\ p)$ 由

$$\boldsymbol{h}_i = \boldsymbol{N}^{(i)+}\boldsymbol{g}^{(i)} + (\boldsymbol{I}_p - \boldsymbol{N}^{(i)+}\boldsymbol{N}^{(i)})\boldsymbol{y}_i, \qquad \forall \boldsymbol{y}_i \in R^p$$

给出.

$\boldsymbol{X\Lambda}^2 + \boldsymbol{CX\Lambda} + \boldsymbol{KX} = \boldsymbol{0}$ 有唯一双对称解 $\boldsymbol{C},\boldsymbol{K} \in R_{\mathrm{BS}}^{n\times n}$ 的充分必要条件为

$$\boldsymbol{N}^{(i)}\boldsymbol{N}^{(i)+}\boldsymbol{g}^{(i)} = \boldsymbol{g}^{(i)},\ \mathrm{rank}(\boldsymbol{N}^{(i)}) = p, \qquad i = 1,2,$$

解的形式如式(46)，$\boldsymbol{H}_i$ 中的元素由 $\boldsymbol{h}_i = \boldsymbol{N}^{(i)+}\boldsymbol{g}^{(i)}$，$i=1,2$ 给出.

证明 由引理 1 知，$\boldsymbol{C},\boldsymbol{K}$ 可表示为式(46)，其中 $\boldsymbol{C}_1,\boldsymbol{K}_1 \in R_{\mathrm{S}}^{(n-k)\times(n-k)}$，$\boldsymbol{C}_2,\boldsymbol{K}_2 \in R_{\mathrm{S}}^{k\times k}$.因此式(4)等价于

$$\boldsymbol{X\Lambda}^2 + \boldsymbol{D}\begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_2 \end{bmatrix}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{X\Lambda} + \boldsymbol{D}\begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_2 \end{bmatrix}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{X} = \boldsymbol{0},$$

即

$$\boldsymbol{X}_i\boldsymbol{\Lambda}^2 + \boldsymbol{C}_i\boldsymbol{X}_i\boldsymbol{\Lambda} + \boldsymbol{K}_i\boldsymbol{X}_i = \boldsymbol{0} \qquad (i=1,2). \tag{50}$$

所以问题 Ⅰ 有解的充分必要条件是式(50)有对称解；问题 Ⅰ 有唯一解的充分必要条件是式(50)有唯一对称解.故由定理 1 可得问题 Ⅰ 的解.

定理 3 给定频率矩阵 $\boldsymbol{\Lambda} \in R^{p\times p}$，振型矩阵 $\boldsymbol{X} \in R^{n\times p}$ 如式(2)、(3)，$\boldsymbol{D}^{\mathrm{T}}\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}$，设 $\mathrm{rank}(\boldsymbol{X}_i) = p$，对 $\boldsymbol{X}_i(i=1,2)$ 进行奇异值分解 $\boldsymbol{X}_i = \boldsymbol{U}^{(i)}\begin{pmatrix} \boldsymbol{\Sigma}_i \\ \boldsymbol{0} \end{pmatrix}\boldsymbol{Q}^{(i)\mathrm{T}}$，其中 $\boldsymbol{U}^{(1)} \in R_0^{(n-k)\times(n-k)}$，$\boldsymbol{U}^{(2)} \in R_0^{k\times k}$，$\boldsymbol{\Sigma}_1 = \mathrm{diag}(\mu_1,\mu_2,\cdots,\mu_p) > 0$，$\boldsymbol{\Sigma}_2 = \mathrm{diag}(\nu_1,\nu_2,\cdots,\nu_p) > 0$，$\boldsymbol{Q}^{(i)} \in R_0^{p\times p}$，则问题 Ⅰ 必有解如式(46)，其中

$$\boldsymbol{C}_i = \boldsymbol{U}^{(i)}\begin{bmatrix} -\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}(\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}_i + \boldsymbol{G}_i\boldsymbol{\Lambda})\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1} + \boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{H}_i\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1} & \boldsymbol{C}_{12}^{(i)} \\ \boldsymbol{C}_{12}^{(i)\mathrm{T}} & \boldsymbol{C}_{22}^{(i)} \end{bmatrix}\boldsymbol{U}^{(i)\mathrm{T}},$$

$$\boldsymbol{K}_i = \boldsymbol{U}^{(i)}\begin{bmatrix} \boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}_i\boldsymbol{\Lambda}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{H}_i\boldsymbol{\Lambda}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1} & -\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i\boldsymbol{C}_{12}^{(i)} \\ -\boldsymbol{C}_{12}^{(i)\mathrm{T}}\boldsymbol{\Sigma}_i\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1} & \boldsymbol{K}_{22}^{(i)} \end{bmatrix}\boldsymbol{U}^{(i)\mathrm{T}},$$

式中 $i=1,2$，$\boldsymbol{G}_i = \boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^2\boldsymbol{Q}^{(i)\mathrm{T}}$，$\boldsymbol{C}_{12}^{(1)} \in R^{p\times(n-k-p)}$，$\boldsymbol{C}_{12}^{(2)} \in R^{p\times(k-p)}$ 为任意矩阵，$\boldsymbol{C}_{22}^{(1)},\boldsymbol{K}_{22}^{(1)} \in R_{\mathrm{S}}^{(n-k-p)\times(n-k-p)}$，$\boldsymbol{C}_{22}^{(2)},\boldsymbol{K}_{22}^{(2)} \in R_{\mathrm{S}}^{(k-p)\times(k-p)}$ 为任意对称矩阵，

$$\boldsymbol{H}_i = \mathrm{diag}\Bigg\{\begin{bmatrix} \varepsilon_1^{(i)} & \delta_1^{(i)} \\ \delta_1^{(i)} & -\varepsilon_1^{(i)} \end{bmatrix}, \begin{bmatrix} \varepsilon_3^{(i)} & \delta_3^{(i)} \\ \delta_3^{(i)} & -\varepsilon_3^{(i)} \end{bmatrix}, \cdots,$$

$$\begin{bmatrix} \varepsilon_{2l-1}^{(i)} & \delta_{2l-1}^{(i)} \\ \delta_{2l-1}^{(i)} & -\varepsilon_{2l-1}^{(i)} \end{bmatrix}, \xi_{2l+1}^{(i)}, \xi_{2l+2}^{(i)}, \cdots, \xi_p^{(i)} \Bigg\},$$

式中 $\varepsilon_{2j-1}^{(i)},\delta_{2j-1}^{(i)}(j=1,2,\cdots,l)$，$\xi_j^{(i)}(j=2l+1,2l+2,\cdots,p)$ 为任意实数.

2 问题 Ⅱ 的解

应用多元函数的微分法可得如下的结论：

引理 5 给定 $\boldsymbol{G} \in R^{n\times n}$，则问题

$$\|S - G\| = \min,$$

在 $R_S^{n\times n}$ 内存在唯一解 $\hat{S} = (G + G^T)/2$.

引理 6 给定 $D_1, D_2, F_1, F_2 \in R^{m\times n}$, $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \in R^{m\times m}$, 则问题

$$\|G - D_1\|^2 + \|G - D_2\|^2 + \|\Lambda G - F_1\|^2 + \|\Lambda G - F_2\|^2 = \min,$$

有唯一解 $\hat{G} = \frac{1}{2}\Phi(D_1 + D_2 + \Lambda F_1 + \Lambda F_2) \in R^{m\times n}$, 其中

$$\Phi = \operatorname{diag}\left\{\frac{1}{1+\lambda_1^2}, \frac{1}{1+\lambda_2^2}, \cdots, \frac{1}{1+\lambda_m^2}\right\} \in R^{m\times m}.$$

当 $\operatorname{rank}(X_i) = p\,(i = 1,2)$,即矩阵 X_i 为列满秩阵时,问题Ⅰ是恒有解的,易知 S_{CK} 是一个闭凸集,因此给定 $\tilde{C}, \tilde{K} \in R^{n\times n}$,存在唯一解 $[\hat{C}, \hat{K}] \in S_{CK}$ 使得等式(4)成立.

设 $\tilde{C}, \tilde{K} \in R^{n\times n}$,对任意的矩阵 $[C, K] \in S_{CK}$, 有

$$\|\tilde{C} - C\|_W^2 + \|\tilde{K} - K\|_W^2 = \left\|\tilde{C} - D\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} D^T\right\|_W^2 + \left\|\tilde{K} - D\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} D^T\right\|_W^2 = \left\|W\tilde{C}W - WD\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} D^T W\right\|^2 + \left\|W\tilde{K}W - WD\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} D^T W\right\|^2. \tag{51}$$

令 $\Gamma_1 = \operatorname{diag}\{\Sigma_1, I_{n-k-p}\}$, $\Gamma_2 = \operatorname{diag}\{\Sigma_2, I_{k-p}\}$, 取

$$W = D\begin{bmatrix} U^{(1)} & 0 \\ 0 & U^{(2)} \end{bmatrix}\begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}\begin{bmatrix} U^{(1)T} & 0 \\ 0 & U^{(2)T} \end{bmatrix} D^T,$$

此时,记

$$\begin{bmatrix} U^{(1)T} & 0 \\ 0 & U^{(2)T} \end{bmatrix} D^T W\tilde{C}WD \begin{bmatrix} U^{(1)} & 0 \\ 0 & U^{(2)} \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{bmatrix},$$

$$\begin{bmatrix} U^{(1)T} & 0 \\ 0 & U^{(2)T} \end{bmatrix} D^T W\tilde{K}WD \begin{bmatrix} U^{(1)} & 0 \\ 0 & U^{(2)} \end{bmatrix} = \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{21} & \tilde{K}_{22} \end{bmatrix},$$

其中 $\tilde{C}_{11}, \tilde{K}_{11} \in R^{(n-k)\times(n-k)}$, $\tilde{C}_{22}, \tilde{K}_{22} \in R^{k\times k}$.

式(51)中 $\|\tilde{C} - C\|_W^2 + \|\tilde{K} - K\|_W^2 = \min$ 等价于

$$\left\|\begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{bmatrix} - \begin{bmatrix} \Gamma_1 U^{(1)T} C_1 U^{(1)} \Gamma_1 & 0 \\ 0 & \Gamma_2 U^{(2)T} C_2 U^{(2)} \Gamma_2 \end{bmatrix}\right\|^2 + \left\|\begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{21} & \tilde{K}_{22} \end{bmatrix} - \begin{bmatrix} \Gamma_1 U^{(1)T} K_1 U^{(1)} \Gamma_1 & 0 \\ 0 & \Gamma_2 U^{(2)T} K_2 U^{(2)} \Gamma_2 \end{bmatrix}\right\|^2 = \min, \tag{52}$$

上式中

$$\Gamma_i U^{(i)T} C_i U^{(i)} \Gamma_i = \begin{bmatrix} -Q^{(i)T}(\Lambda^T G_i + G_i \Lambda) Q^{(i)} + Q^{(i)T} H_i Q^{(i)} & \Sigma_i C_{12}^{(i)} \\ C_{12}^{(i)T} \Sigma_i & C_{22}^{(i)} \end{bmatrix}, \tag{53}$$

$$\Gamma_i U^{(i)T} K_i U^{(i)} \Gamma_i = \begin{bmatrix} Q^{(i)T} \Lambda^T G_i \Lambda Q^{(i)} - Q^{(i)T} H_i \Lambda Q^{(i)} & -Q^{(i)T} \Lambda^T Q^{(i)} \Sigma_i C_{12}^{(i)} \\ -C_{12}^{(i)T} \Sigma_i Q^{(i)T} \Lambda Q^{(i)} & K_{22}^{(i)} \end{bmatrix}. \tag{54}$$

令

$$\tilde{\boldsymbol{C}}_{ii}=\begin{bmatrix}\tilde{\boldsymbol{C}}_{11}^{(i)} & \tilde{\boldsymbol{C}}_{12}^{(i)}\\ \tilde{\boldsymbol{C}}_{21}^{(i)} & \tilde{\boldsymbol{C}}_{22}^{(i)}\end{bmatrix},\ \tilde{\boldsymbol{K}}_{ii}=\begin{bmatrix}\tilde{\boldsymbol{K}}_{11}^{(i)} & \tilde{\boldsymbol{K}}_{12}^{(i)}\\ \tilde{\boldsymbol{K}}_{21}^{(i)} & \tilde{\boldsymbol{K}}_{22}^{(i)}\end{bmatrix},\qquad i=1,2, \tag{55}$$

其中 $\tilde{\boldsymbol{C}}_{11}^{(i)},\tilde{\boldsymbol{K}}_{11}^{(i)}\in R^{p\times p},\ i=1,2,\ \tilde{\boldsymbol{C}}_{22}^{(1)},\tilde{\boldsymbol{K}}_{22}^{(1)}\in R^{(n-k-p)\times(n-k-p)},\tilde{\boldsymbol{C}}_{22}^{(2)},\tilde{\boldsymbol{K}}_{22}^{(2)}\in R^{(k-p)\times(k-p)}$.

将式(53)~(55)代入式(52),利用式(8),可得

$$\begin{aligned}&\|\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{C}}_{11}^{(i)}\boldsymbol{Q}^{(i)\mathrm{T}}+\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}_i+\boldsymbol{G}_i\boldsymbol{\Lambda}-\boldsymbol{H}_i\|^2+\\&\quad\|\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{11}^{(i)}\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{V}^{\mathrm{T}}-\boldsymbol{V}^{\mathrm{T}}\boldsymbol{\Omega}\boldsymbol{G}_i\boldsymbol{\Omega}+\boldsymbol{H}_i\boldsymbol{\Omega}\|^2=\min,\end{aligned} \tag{56}$$

$$\begin{cases}\|\tilde{\boldsymbol{C}}_{22}^{(i)}-\boldsymbol{C}_{22}^{(i)}\|^2=\min,\ \text{s.t.}\ \boldsymbol{C}_{22}^{(i)}=\boldsymbol{C}_{22}^{(i)\mathrm{T}},\\ \|\tilde{\boldsymbol{K}}_{22}^{(i)}-\boldsymbol{K}_{22}^{(i)}\|^2=\min,\ \text{s.t.}\ \boldsymbol{K}_{22}^{(i)}=\boldsymbol{K}_{22}^{(i)\mathrm{T}},\end{cases} \tag{57}$$

$$\begin{aligned}&\|\tilde{\boldsymbol{C}}_{12}^{(i)}-\boldsymbol{\Sigma}_i\boldsymbol{C}_{12}^{(i)}\|^2+\|\tilde{\boldsymbol{C}}_{21}^{(i)\mathrm{T}}-\boldsymbol{\Sigma}_i\boldsymbol{C}_{12}^{(i)}\|^2+\|\boldsymbol{V}\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{12}^{(i)}+\boldsymbol{\Omega}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i\boldsymbol{C}_{12}^{(i)}\|^2+\\&\quad\|\boldsymbol{V}\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{21}^{(i)\mathrm{T}}+\boldsymbol{\Omega}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i\boldsymbol{C}_{12}^{(i)}\|^2=\min,\qquad i=1,2.\end{aligned} \tag{58}$$

对于式(56),令

$$\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{C}}_{11}^{(i)}\boldsymbol{Q}^{(i)\mathrm{T}}+\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}_i+\boldsymbol{G}_i\boldsymbol{\Lambda}=[\gamma_{sj}^{(i)}]_{p\times p},\ \boldsymbol{V}^{\mathrm{T}}\boldsymbol{\Omega}\boldsymbol{G}_i\boldsymbol{\Omega}-\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{11}^{(i)}\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{V}^{\mathrm{T}}=[\mu_{sj}^{(i)}]_{p\times p},$$

则上式等价于

$$\begin{cases}\left\|\begin{bmatrix}\varepsilon_{2s-1}^{(i)} & \delta_{2s-1}^{(i)}\\ \delta_{2s-1}^{(i)} & -\varepsilon_{2s-1}^{(i)}\end{bmatrix}-\begin{bmatrix}\gamma_{2s-1,2s-1}^{(i)} & \gamma_{2s-1,2s}^{(i)}\\ \gamma_{2s,2s-1}^{(i)} & \gamma_{2s,2s}^{(i)}\end{bmatrix}\right\|^2+\\ \qquad\left\||\lambda_{2s-1}|\begin{bmatrix}\varepsilon_{2s-1}^{(i)} & \delta_{2s-1}^{(i)}\\ \delta_{2s-1}^{(i)} & -\varepsilon_{2s-1}^{(i)}\end{bmatrix}-\begin{bmatrix}\mu_{2s-1,2s-1}^{(i)} & \mu_{2s-1,2s}^{(i)}\\ \mu_{2s,2s-1}^{(i)} & \mu_{2s,2s}^{(i)}\end{bmatrix}\right\|^2=\min,\\ \|\xi_j^{(i)}-\gamma_{jj}\|^2+\|\lambda_j\xi_j^{(i)}-\mu_{jj}\|^2=\min\\ \qquad\qquad(s=1,2,\cdots,l,\ j=2l+1,2l+2,\cdots p,\ i=1,2).\end{cases}$$

由引理6,可解得

$$\begin{cases}\varepsilon_{2s-1}^{(i)}=\dfrac{1}{2(1+|\lambda_{2s-1}|^2)}[\gamma_{2s-1,2s-1}^{(i)}-\gamma_{2s,2s}^{(i)}+|\lambda_{2s-1}|(\mu_{2s-1,2s-1}^{(i)}-\mu_{2s,2s}^{(i)})],\\ \delta_{2s-1}^{(i)}=\dfrac{1}{2(1+|\lambda_{2s-1}|^2)}[\gamma_{2s-1,2s}^{(i)}+\gamma_{2s,2s-1}^{(i)}+|\lambda_{2s-1}|(\mu_{2s-1,2s}^{(i)}+\mu_{2s,2s-1}^{(i)})],\\ \xi_j^{(i)}=\dfrac{1}{1+\lambda_j^2}(\gamma_{jj}^{(i)}+\lambda_j\mu_{jj}^{(i)})\\ \qquad\qquad(s=1,2,\cdots,l,\ j=2l+1,2l+2,\cdots,p,\ i=1,2).\end{cases} \tag{59}$$

对式(57),由引理5,得

$$\hat{\boldsymbol{C}}_{22}^{(i)}=\frac{1}{2}(\tilde{\boldsymbol{C}}_{22}^{(i)}+\tilde{\boldsymbol{C}}_{22}^{(i)\mathrm{T}}),\ \hat{\boldsymbol{K}}_{22}^{(i)}=\frac{1}{2}(\tilde{\boldsymbol{K}}_{22}^{(i)}+\tilde{\boldsymbol{K}}_{22}^{(i)\mathrm{T}}),\qquad i=1,2. \tag{60}$$

式(58)等价于

$$\begin{aligned}&\|\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{C}}_{12}^{(i)}-\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i\boldsymbol{C}_{12}^{(i)}\|^2+\|\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{C}}_{21}^{(i)\mathrm{T}}-\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i\boldsymbol{C}_{12}^{(i)}\|^2+\\&\quad\|\boldsymbol{V}\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{12}^{(i)}+\boldsymbol{\Omega}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i\boldsymbol{C}_{12}^{(i)}\|^2+\|\boldsymbol{V}\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{21}^{(i)\mathrm{T}}+\boldsymbol{\Omega}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i\boldsymbol{C}_{12}^{(i)}\|^2=\min.\end{aligned} \tag{61}$$

由引理6,得

$$\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i\hat{\boldsymbol{C}}_{12}^{(i)}=\frac{1}{2}\boldsymbol{\Phi}(\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{C}}_{12}^{(i)}+\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{C}}_{21}^{(i)\mathrm{T}}-\boldsymbol{\Omega}\boldsymbol{V}\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{12}^{(i)}-\boldsymbol{\Omega}\boldsymbol{V}\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{21}^{(i)\mathrm{T}}),\qquad i=1,2, \tag{62}$$

其中

$$\boldsymbol{\Phi}=\mathrm{diag}\left\{\frac{1}{1+|\lambda_1|^2}\boldsymbol{I}_2,\frac{1}{1+|\lambda_3|^2}\boldsymbol{I}_2,\cdots,\frac{1}{1+|\lambda_{2l-1}|^2}\boldsymbol{I}_2,\frac{1}{1+\lambda_{2l+1}^2},\frac{1}{1+\lambda_{2l+2}^2},\cdots,\frac{1}{1+\lambda_p^2}\right\}\in R^{p\times p},$$

进而

$$\hat{\boldsymbol{C}}_{12}^{(i)}=\frac{1}{2}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{\Phi}(\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{C}}_{12}^{(i)}+\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{C}}_{21}^{(i)\mathrm{T}}-\boldsymbol{\Lambda}\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{12}^{(i)}-\boldsymbol{\Lambda}\boldsymbol{Q}^{(i)}\tilde{\boldsymbol{K}}_{21}^{(i)\mathrm{T}}),\quad i=1,2. \tag{63}$$

结合上述推导结果,给出定理4.

定理4 给定 $\tilde{\boldsymbol{C}},\tilde{\boldsymbol{K}}\in R^{n\times n}$, 则问题Ⅱ存在唯一解,并且其解可以表示为

$$\hat{\boldsymbol{C}}=\boldsymbol{D}\begin{bmatrix}\hat{\boldsymbol{C}}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \hat{\boldsymbol{C}}_2\end{bmatrix}\boldsymbol{D}^{\mathrm{T}},\ \hat{\boldsymbol{K}}=\boldsymbol{D}\begin{bmatrix}\hat{\boldsymbol{K}}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \hat{\boldsymbol{K}}_2\end{bmatrix}\boldsymbol{D}^{\mathrm{T}}, \tag{64}$$

其中

$$\hat{\boldsymbol{C}}_i=\boldsymbol{U}^{(i)}\begin{bmatrix}-\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}(\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}_i+\boldsymbol{G}_i\boldsymbol{\Lambda})\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}\hat{\boldsymbol{H}}_i\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1} & \hat{\boldsymbol{C}}_{12}^{(i)}\\ \hat{\boldsymbol{C}}_{12}^{(i)\mathrm{T}} & \hat{\boldsymbol{C}}_{22}^{(i)}\end{bmatrix}\boldsymbol{U}^{(i)\mathrm{T}},\quad i=1,2,$$

$$\hat{\boldsymbol{K}}_i=\boldsymbol{U}^{(i)}\begin{bmatrix}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}_i\boldsymbol{\Lambda}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}\hat{\boldsymbol{H}}_i\boldsymbol{\Lambda}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1} & -\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i\hat{\boldsymbol{C}}_{12}^{(i)}\\ -\hat{\boldsymbol{C}}_{12}^{(i)\mathrm{T}}\boldsymbol{\Sigma}_i\boldsymbol{Q}^{(i)\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{Q}^{(i)}\boldsymbol{\Sigma}_i^{-1} & \hat{\boldsymbol{K}}_{22}^{(i)}\end{bmatrix}\boldsymbol{U}^{(i)\mathrm{T}},\quad i=1,2,$$

$\hat{\boldsymbol{H}}_i$ 为式(49)的形式,标量 $\varepsilon_{2s-1}^{(i)},\delta_{2s-1}^{(i)},s=1,2,\cdots,l,\xi_j^{(i)},j=2l+1,2l+2,\cdots,p,i=1,2$ 分别为 $\hat{\boldsymbol{H}}_i$ 的元素,对应具体数值为式(59)的形式,$\hat{\boldsymbol{C}}_{22}^{(i)},\hat{\boldsymbol{K}}_{22}^{(i)},\hat{\boldsymbol{C}}_{12}^{(i)}(i=1,2)$ 的具体数值由式(60)、(63)给出.

算例 对于自由度为 n 的振动系统,计算系统阻尼矩阵和刚度矩阵的最佳逼近双对称解的运算量约为 $18n^3$ FLOPS,主要集中在与自由度 n 有关的分块矩阵相乘环节,实际计算中利用矩阵的特殊结构算法的运算量可适当降低.

令 $n=6$, $p=3$, 给定 $\boldsymbol{\Lambda}=\mathrm{diag}\{-0.1376+1.2139\mathrm{i},-0.1376-1.2139\mathrm{i},-2.0458\}$, $\mathrm{i}^2=-1$, i 为虚数单位.

$$\boldsymbol{X}=\begin{bmatrix}0.1317+0.1493\mathrm{i} & 0.1317-0.1493\mathrm{i} & 0.2667\\ -0.0415-0.3656\mathrm{i} & -0.0415+0.3656\mathrm{i} & 0.0480\\ 0.0469-0.1529\mathrm{i} & 0.0469+0.1529\mathrm{i} & 0.1516\\ 0.0469-0.1529\mathrm{i} & 0.0469+0.1529\mathrm{i} & 0.1516\\ -0.0415-0.3656\mathrm{i} & -0.0415+0.3656\mathrm{i} & 0.0480\\ 0.1317+0.1493\mathrm{i} & 0.1317-0.1493\mathrm{i} & 0.2667\end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} 1.4610 & 0.9562 & 1.0528 & -0.1079 & -0.3893 & 0.1257 \\ 0.9562 & 0.8609 & 0.8139 & -0.4157 & -0.6351 & -0.3894 \\ 1.0531 & 0.8143 & 0.8817 & -0.1534 & -0.4157 & -0.1082 \\ -0.1085 & -0.4165 & -0.1537 & 0.8816 & 0.8140 & 1.0526 \\ -0.3890 & -0.6351 & -0.4160 & 0.8143 & 0.8609 & 0.9567 \\ 0.1256 & -0.3889 & -0.1082 & 1.0529 & 0.9565 & 1.4606 \end{bmatrix},$$

$$\tilde{K} = \begin{bmatrix} 0.5309 & 0.2508 & 0.4202 & -0.3449 & -0.0718 & -0.2397 \\ 0.2505 & 0.9502 & 0.6593 & 0.2244 & 0.5466 & -0.0713 \\ 0.4202 & 0.6598 & 0.7023 & -0.1574 & 0.2250 & -0.3449 \\ -0.3447 & 0.2252 & -0.1574 & 0.7020 & 0.6590 & 0.4204 \\ -0.0716 & 0.5468 & 0.2248 & 0.6594 & 0.9509 & 0.2513 \\ -0.2393 & -0.0718 & -0.3445 & 0.4205 & 0.2511 & 0.5312 \end{bmatrix},$$

应用定理 4 的方法,可得问题Ⅱ的最佳逼近解为

$$\hat{C} = \begin{bmatrix} 0.7778 & 1.3816 & 1.5088 & -0.6209 & -0.7848 & 0.8427 \\ 1.3816 & 1.3026 & -1.9289 & 2.2819 & -1.0563 & -0.7848 \\ 1.5088 & -1.9289 & -4.8537 & 5.6667 & 2.2819 & -0.6209 \\ -0.6209 & 2.2819 & 5.6667 & -4.8537 & -1.9289 & 1.5088 \\ -0.7848 & -1.0563 & 2.2819 & -1.9289 & 1.3026 & 1.3816 \\ 0.8427 & -0.7848 & -0.6209 & 1.5088 & 1.3816 & 0.7778 \end{bmatrix},$$

$$\hat{K} = \begin{bmatrix} 3.4644 & 1.7171 & -2.0291 & 2.2233 & -1.6034 & -3.2132 \\ 1.7171 & 2.0043 & -1.8884 & 2.9515 & -0.6073 & -1.6034 \\ -2.0291 & -1.8884 & -7.1031 & 7.3265 & 2.9515 & 2.2233 \\ 2.2233 & 2.9515 & 7.3265 & -7.1031 & -1.8884 & -2.0291 \\ -1.6034 & -0.6073 & 2.9515 & -1.8884 & 2.0043 & 1.7171 \\ -3.2132 & -1.6034 & 2.2233 & -2.0291 & 1.7171 & 3.4644 \end{bmatrix},$$

经验证,得

$$X\Lambda^2 + \hat{C}X\Lambda + \hat{K}X = 10^{-14} \times \begin{bmatrix} 0.2207 & -0.1166 & 0.0111 \\ 0.2637 & -0.0222 & -0.1443 \\ -0.3969 & 0.1332 & 0.2331 \\ -0.3303 & 0.0722 & 0.2665 \\ 0.2526 & -0.0333 & -0.1388 \\ 0.1860 & 0.1110 & -0.0555 \end{bmatrix}.$$

3 结　　论

本文为了使用试验数据最优修正对称系统的双对称阻尼矩阵和刚度矩阵,将文献[4-5]中对称矩阵在列满秩和行满秩情况下的结论进行了一般推广(定理 1),而已有文献中的结论是定理 1 的特殊情况(推论 1 与推论 2);定理 2 给出二次特征值反问题的双对称矩阵解有解的

条件及解的一般表示式;定理 3 考虑了矩阵 $\boldsymbol{X}_1,\boldsymbol{X}_2$ 为列满秩时问题Ⅰ的求解问题.

在问题Ⅰ定理的推导和问题Ⅱ最佳逼近解的求解过程中,将分块对角矩阵(频率矩阵) $\boldsymbol{\Lambda}$ 分解为对角矩阵 $\boldsymbol{\Omega}$ 与正交矩阵 $\boldsymbol{V}$ 的乘积(见式(8)),通过引理 2~引理 4 的提出和证明,简化了定理 1 的证明过程,与已有文献的定理证明方法不同;通过引理 6 的提出和证明,证明了定理 4 的结论,证明过程中避免了求解方程组、矩阵的 Hadamard 乘积和过多使用奇异值分解.

定理 4 考虑了矩阵 $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ 为列满秩时问题Ⅱ的求解问题, 对于其它情况 ($\mathrm{rank}(\boldsymbol{X}_i)=r_i$, $i=1,2$), 可利用定理 2 给出问题Ⅱ的最佳逼近双对称矩阵解.

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Bisymmetric Damping and Stiffness Matrices Calibration With Test Data of Vibration Systems

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Abstract: The problem of bisymmetric damping and stiffness matrices calibration with test data of vibration systems was discussed. Based on the eigen equation as well as bisymmetry of the damping and stiffness matrices, existence and uniqueness of the solution to the problem was studied by means of the theory and method for the inverse algebraic quadratic eigenvalue problem. A new method for the calibration of damping and stiffness matrices was presented. According to the properties of bisymmetric matrices, the bisymmetric solution to the matrix equation was studied. The general expression of the bisymmetric solution was obtained. Moreover, the related optimal approximation problem of any related matrix was addressed and the solution given. The damping and stiffness matrices calibrated with the method not only satisfy the quadratic eigen equation, but also are the unique bisymmetric matrix solution. A numerical example proves efficiency of the present method.

Key words: structural model; inverse problem; calibration; damping matrix; stiffness matrix; bisymmetric matrix

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