

浅水中度振幅孤立波解的分支*

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摘要: 利用平面动力系统分支方法研究浅水中度振幅方程的定性行为和孤立波解.给出了系统在不同参数条件下的相图.获得了光滑孤立波、cuspon 解和周期波解的隐式表达式.对方程的光滑孤立波解、cuspon 解和周期波解进行了数值模拟.获得的结果完善了相关文献已有的结果.

关键词: 孤立波; 尖孤立波; 周期波; 分支

中图分类号: O357.1 **文献标志码:** A

doi: 10.3879/j.issn.1000-0887.2014.09.006

引言

大量自然现象的动力学过程可以通过非线性偏微分方程进行描述.这些偏微分方程有一类重要的解——行波解.这类解是方程的一种特殊解,解的范围可能是局部的,也可能是周期的,并且在传播过程中波形和波速保持不变.

人们用非线性模型描述自然现象,例如孤立波.其中,一个突出的例子是 Camassa-Holm 方程^[1].Camassa-Holm 方程是一个可积无穷维 Hamilton(哈密顿)系统,它的孤立波是孤子.依据 Johnson^d 的研究,Constantin 和 Lannes 推导出了浅水区域中的中度振幅波方程^[2]:

$$u_t + u_x + 6uu_x - 6u^2u_x + 12u^3u_x + u_{xxx} - u_{xxt} + 14uu_{xxx} + 28u_xu_{xx} = 0. \quad (1)$$

利用动力系统理论可以证明方程(1)孤立波解的存在性^[3].文献[4]则研究了方程(1)的孤立波解的轨道稳定性.

在过去的几年里,许多作者致力于研究行波模型解的动力学行为,并得到了一些有价值的结果^[5-10].本文将进一步研究方程(1)的定性行为和孤立波解.

令 $u(x, t) = \varphi(\xi) = \varphi(x - ct)$, 其中 c 为波速,代入方程(1)可得下列常微分方程:

$$\varphi_\xi(1 - c) + 6\varphi\varphi_\xi - 6\varphi^2\varphi_\xi + 12\varphi^3\varphi_\xi + (1 + c)\varphi_{\xi\xi\xi} + 14\varphi\varphi_{\xi\xi\xi} + 28\varphi_\xi\varphi_{\xi\xi} = 0. \quad (2)$$

将方程(2)关于 ξ 积分一次,并令积分常数为 0,则有下列方程:

$$\varphi(1 - c) + 3\varphi^2 - 2\varphi^3 + 3\varphi^4 + (1 + c)\varphi_{\xi\xi} + 14\varphi\varphi_{\xi\xi} + 7\varphi_\xi^2 = 0. \quad (3)$$

设 $\varphi_\xi = y$, 可知方程(3)等价于二维系统:

* 收稿日期: 2013-11-18; 修订日期: 2014-06-25

基金项目: 国家自然科学基金(11161013; 11361017); 广西自然科学基金(2014GXNSFBA118007); 广西高等学校优秀中青年骨干教师培养工程

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$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{(c-1)\varphi - 3\varphi^2 + 2\varphi^3 - 3\varphi^4 - 7y^2}{1+c+14\varphi}. \end{cases} \quad (4)$$

显然,方程(4)有首次积分

$$H(\varphi, y) = (1+c+14\varphi)y^2 + (1-c)\varphi^2 + 2\varphi^3 - \varphi^4 + \frac{6}{5}\varphi^5. \quad (5)$$

再令 $d\xi = (1+c+14\varphi)d\tau$, 则系统(4)可以变形为如下多项式微分系统:

$$\begin{cases} \frac{d\varphi}{d\tau} = (1+c+14\varphi)y, \\ \frac{dy}{d\tau} = (c-1)\varphi - 3\varphi^2 + 2\varphi^3 - 3\varphi^4 - 7y^2. \end{cases} \quad (6)$$

1 系统(6)的相图分支

现在考虑多项式常微分系统(6)的动力学行为,系统(6)在水平轴 $O\varphi$ 上至多有两个奇点,分别记为 $P_1(0,0)$ 和 $P_2(\varphi_*, 0)$. 其中 φ_* 是多项式

$$P(\varphi) = 3\varphi^3 - 2\varphi^2 + 3\varphi + 1 - c \quad (7)$$

的唯一实根.事实上,可以判断 $P(\varphi)$ 的判别式总是正的,所以它没有重根.又因为 $P(\varphi)$ 的导函数 $P'(\varphi) = 9\varphi^2 - 4\varphi + 3$ 没有实根,所以 $P(\varphi)$ 没有极值.从而证明 $P(\varphi)$ 有唯一的实根.利用3次多项式 Cardano 公式 φ_* 可以表示为

$$\varphi_* = \frac{1}{18}q(c)^{1/3} - \frac{46}{9}q(c)^{-1/3} + \frac{2}{9}, \quad (8)$$

其中

$$q(c) = -1556 + 972c + 36\sqrt{2469 - 2334c + 729c^2}. \quad (9)$$

设

$$c_* = \frac{14}{9}(694 + 9\sqrt{46785})^{1/3} - \frac{2086}{9}(694 + 9\sqrt{46785})^{-1/3} - \frac{37}{9}. \quad (10)$$

如果 $-1 < c < c_*$, 则在直线 $\varphi = A = -(1+c)/14$ 上有两个临界点, $S^+(A, y_0)$ 和 $S^-(A, -y_0)$, 其中

$$y_0 = \frac{1}{1372}\sqrt{14875 - 8904c - 24038c^2 - 280c^3 - 21c^4}. \quad (11)$$

接下来分析 Jacobi 矩阵

$$\mathbf{M}(\varphi_e, y_e) = \begin{pmatrix} 14y_e & 1+c+14\varphi_e \\ c-1-6\varphi_e+6\varphi_e^2+12\varphi_e^3 & -14y_e \end{pmatrix},$$

其中 (φ_e, y_e) 是系统(6)的奇点.令 $J(\varphi_e, y_e) = \det \mathbf{M}(\varphi_e, y_e)$. 若 $J(\varphi_e, y_e) < 0$, 则奇点是鞍点; 若 $J(\varphi_e, y_e) > 0$, 则奇点是中心; 若 $J(\varphi_e, y_e) = 0$, 则奇点是退化奇点. 由

$$J(A, y_0) = J(A, -y_0) = -196y_0^2 < 0. \quad (12)$$

可知临界点 $S^+(A, y_0)$ 和 $S^-(A, -y_0)$ 都是鞍点. 再由 $P_1 = (0,0)$, 知

$$J(0,0) = (1+c)(1-c). \quad (13)$$

如果 $-1 < c < 1$, 则 $J(0,0) > 0$, $P_1(0,0)$ 是中心; 如果 $c < -1$ 或 $c > 1$, 则 $J(0,0) < 0$, $P_1(0,$

0) 是鞍点; 如果 $c = -1$ 或 $c = 1$, 则 $J(0,0) = 0, P_1(0,0)$ 是退化奇点.

当 $c = 1$ 时, 从方程 $(2 + 14\varphi)y = 0$ 中得到

$$\begin{cases} y = f(\varphi) = 0, \\ G(\varphi, f(\varphi)) = -3\varphi^2 + 2\varphi^3 - 3\varphi^4, \\ \Phi(\varphi) \equiv 0. \end{cases} \quad (14)$$

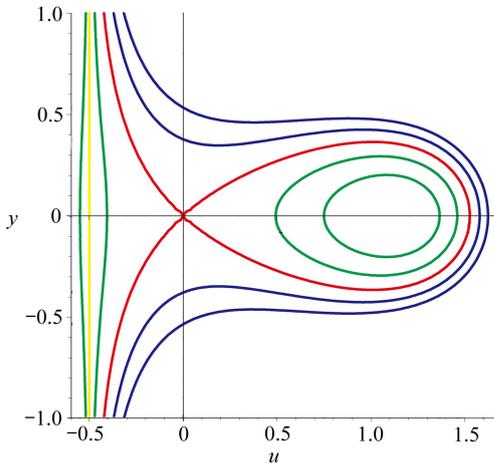
因此, $K = -3, k = 2, \Phi(\varphi) \equiv 0$. 根据文献[8], $P_1(0,0)$ 是系统(6)的尖点.

当 $c = -1$ 时, 系统(6)可以化为如下形式:

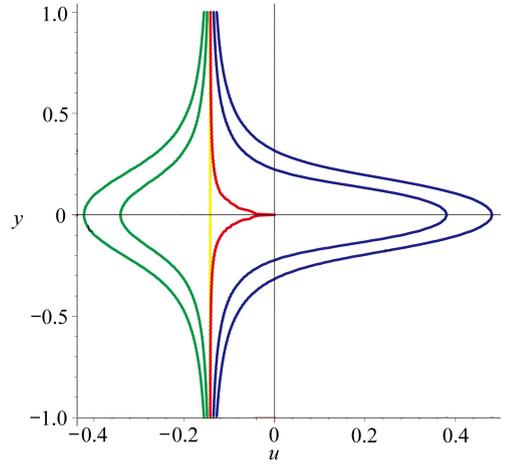
$$\begin{cases} \frac{d\varphi}{d\tau} = G(\varphi, y), \\ \frac{dy}{d\tau} = \varphi + F(\varphi, y), \end{cases} \quad (15)$$

其中

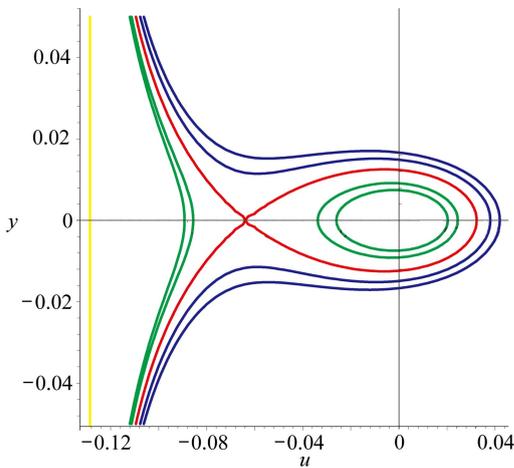
$$G(\varphi, y) = 14\varphi y, \quad F(\varphi, y) = (c - 2)\varphi - 3\varphi^2 + 2\varphi^3 - 3\varphi^4 - 7y^2. \quad (16)$$



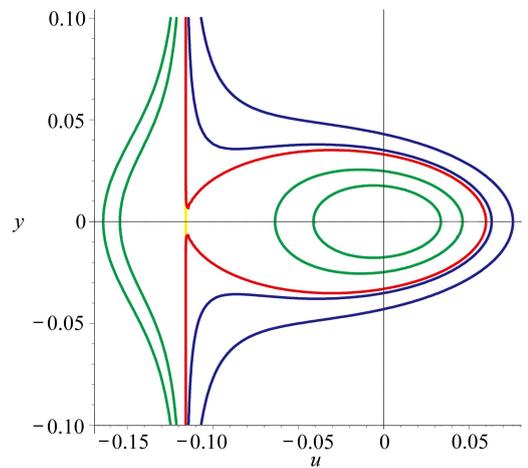
(a) $c > 1$



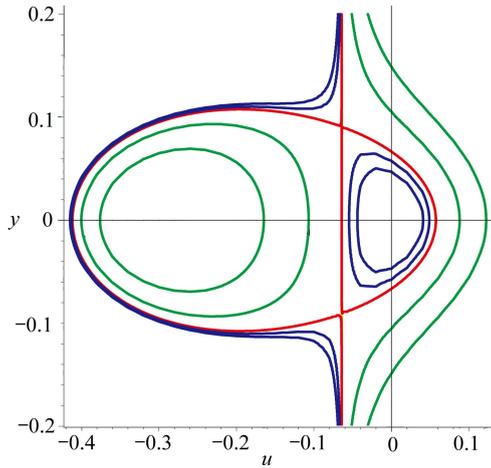
(b) $c = 1$



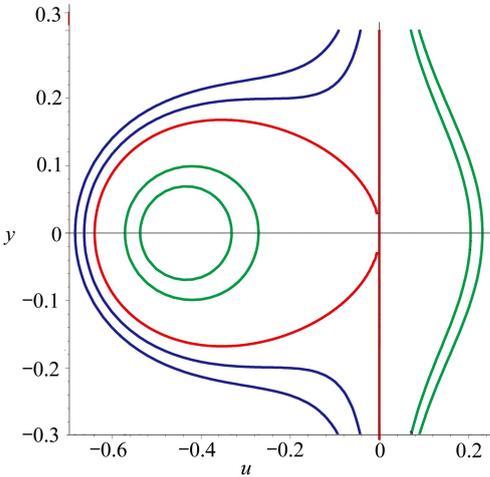
(c) $c^* < c < 1$



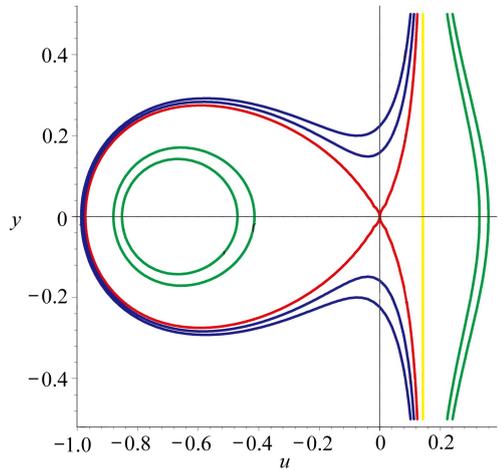
(d) $c = c^*$



(e) $-1 < c < c^*$



(f) $c = -1$



(g) $c < -1$

图1 系统(6)的相图

Fig.1 The phase portraits of system (6)

从方程 $\varphi + F(\varphi, y) = 0$ 推导出 $\varphi = -(7/2)\varphi^2 + \text{HOT}$, 因此

$$G(\varphi(y), y) = -49y^3 + \text{HOT}. \tag{17}$$

此时, $K = -49 < 0, k = 3$, 所以 $P_1(0, 0)$ 是一个幂零鞍点.

对于奇点 $P_2(\varphi_*, 0)$, 有等式

$$J(\varphi_*, 0) = (1 + c + 14\varphi_*)(c - 1 - 6\varphi_* + 6\varphi_*^2 + 12\varphi_*^3) \tag{18}$$

成立.在不同的参数条件下,用类似的方法可以判断出 $P_2(\varphi_*, 0)$ 的类型.综上,画出系统(6)的相图,如图1所示.

2 光滑孤立波、cuspon 与周期波解

由不变代数曲线 $H(\varphi, y) = 0$ 推出下式:

$$y^2 = \frac{\varphi^2 \left(c - 1 - 2\varphi + \varphi^2 - \frac{6}{5}\varphi^3 \right)}{1 + c + 14\varphi}. \tag{19}$$

化简后得

$$y^2 = \frac{6\varphi^2(B - \varphi) \left(\varphi^2 + \left(B - \frac{5}{6} \right) \varphi + B^2 - \frac{5}{6} B + \frac{5}{3} \right)}{5(1 + c + 14\varphi)} = F(\varphi), \quad (20)$$

其中

$$B = \frac{1}{18} g(c)^{1/3} - \frac{155}{18} g(c)^{-1/3} + \frac{5}{18}, \quad (21)$$

$$g(c) = -3655 + 2430c + 90\sqrt{2109 - 2193c + 729c^2}. \quad (22)$$

方程(20)隐含条件 $(B - \varphi)(1 + c + 14\varphi) \geq 0$. 通过方程(20) 还可以得出结论: $F(\varphi)$ 有一个单根 B 和一个二重根 0 . 根据 $F(\varphi)$ 的零点的不同类别, φ 有不同的定性行为.

接下来, 利用平面相图分析技术并结合上述结论, 考虑以下 3 种情况.

第 1 种 光滑孤立波

当 $c > 1$ 时, 方程(1) 有一个光滑孤立波解. 该解对应于连接鞍点 $P_1(0, 0)$ 的同宿轨 (见图 2(a)). 由方程(20) 得到

$$y^2 = \frac{3\varphi^2(B - \varphi) \left(\varphi^2 + \left(B - \frac{5}{6} \right) \varphi + B^2 - \frac{5}{6} B + \frac{5}{3} \right)}{35(\varphi - A)}. \quad (23)$$

这个光滑孤立波解由积分方程

$$\int_{\varphi}^B \frac{\sqrt{z - A} dz}{z \sqrt{(B - z) \left(z^2 + \left(B - \frac{5}{6} \right) z + B^2 - \frac{5}{6} B + \frac{5}{3} \right)}} = \sqrt{\frac{3}{35}} \int_{\xi}^0 d\xi \quad (24)$$

确定. 通过引入一个新参数变量 χ , 得到上述光滑孤立波解的参数表示如下:

$$\varphi(\chi) = \frac{Bq + Ap - (Ap - Bq) \operatorname{cn}(\chi, k)}{p + q + (p - q) \operatorname{cn}(\chi, k)}, \quad (25)$$

$$\xi(\chi) = -\sqrt{\frac{35}{3pq}} \left[\frac{q(A - B)}{Ap - Bq} \chi - \frac{2Apq(A - B)}{((pA)^2 - (qB)^2)(1 - \alpha^2)} I_1(\varphi(\chi)) \right], \quad (26)$$

其中

$$I_1(\varphi) = \frac{1}{1 - \alpha^2} \left(\Pi \left(\arccos(\operatorname{cn} \chi), \frac{\alpha}{\alpha^2 - 1}, k \right) - \alpha f_1 \right), \quad (27)$$

$$f_1(\varphi) = \frac{1}{2} \sqrt{\frac{\alpha^2 - 1}{k^2 + k'^2 \alpha^2}} \ln \left| \frac{\sqrt{k^2 + k'^2 \alpha^2} \operatorname{dn} \chi + \sqrt{\alpha^2 - 1} \operatorname{sn} \chi}{\sqrt{k^2 + k'^2 \alpha^2} \operatorname{dn} \chi - \sqrt{\alpha^2 - 1} \operatorname{sn} \chi} \right|, \quad (28)$$

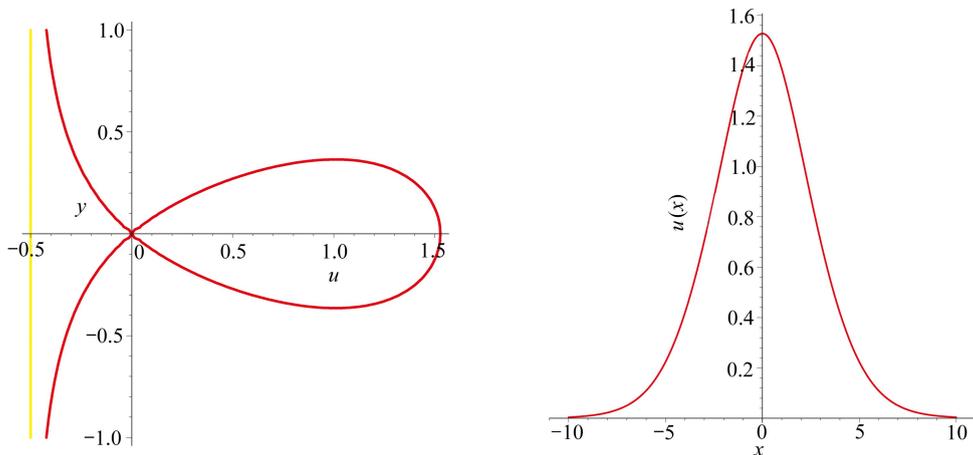
$$\operatorname{cn} \chi = \frac{(B - \varphi)q - (\varphi - A)p}{(B - \varphi)q + (\varphi - A)p}, \quad p^2 = (B - b_1)^2 + a_1^2, \quad q^2 = (A - b_1)^2 + a_1^2,$$

$$b_1 = \frac{1}{2} \left(B - \frac{5}{6} \right), \quad a_1^2 = \frac{1}{4} \left(3B^2 - \frac{5}{3} B + \frac{215}{36} \right),$$

$$k^2 = \frac{(B - A)^2 - (p - q)^2}{4pq}, \quad \alpha = \frac{Ap - Bq}{Ap + Bq},$$

这里 $\operatorname{sn} \chi$ 和 $\operatorname{dn} \chi$ 是 Jacobi 椭圆函数.

应用数值模拟的方法获得了光滑孤立波解 $\varphi(\xi)$ 的波形图, 见图 2(b).

(a) $c > 1$ 时的代数曲线 $H(\varphi, y) = 0$ (a) The algebraic curve $H(\varphi, y) = 0$ for $c > 1$

(b) 光滑孤立波的图形

(b) Profile of the smooth solitary wave

图2 方程(1)的代数曲线与光滑孤立波

Fig.2 The algebraic curve and smooth solitary wave of equation (1)

第2种 cuspon

当 $c < -1$ 时,有

$$B < 0 < A, \varphi(0) = A, 0 \leq \varphi < A.$$

此时不变代数曲线 $H(\varphi, y) = 0$ (见图 3(a)) 确定一个 cuspon 解,且解满足

$$\varphi(0) = A, \lim_{\xi \rightarrow \pm\infty} \varphi(\xi) = 0, \varphi'(-0) = +\infty, \varphi'(0) = -\infty.$$

这个 cuspon 解由积分方程

$$\int_{\varphi}^A \frac{\sqrt{A-z} dz}{z \sqrt{(z-B) \left(z^2 + \left(B - \frac{5}{6} \right) z + B^2 - \frac{5}{6} B + \frac{5}{3} \right)}} = \sqrt{\frac{3}{35}} \int_{\xi}^0 d\xi \quad (29)$$

确定.通过类似光滑孤立波解的计算,获得了 cuspon 解的参数表达式:

$$\varphi(\chi) = \frac{Aq + Bp - (Bp - Aq) \operatorname{cn}(\chi, k)}{p + q + (p - q) \operatorname{cn}(\chi, k)}, \quad (30)$$

$$\xi(\chi) = -\sqrt{\frac{35}{3pq}} \left[\frac{q(B-A)}{Bp - Aq} \chi - \frac{2Bpq(B-A)}{((pB)^2 - (qA)^2)(1-\alpha^2)} I_2(\varphi(\chi)) \right], \quad (31)$$

$$I_2(\varphi) = \frac{1}{1-\alpha^2} \left(\Pi \left(\arccos(\operatorname{cn} \chi), \frac{\alpha}{\alpha^2 - 1}, k \right) - \alpha f_2 \right), \quad (32)$$

$$f_2(\varphi) = \frac{1}{2} \sqrt{\frac{\alpha^2 - 1}{k^2 + k'^2 \alpha^2}} \ln \left| \frac{\sqrt{k^2 + k'^2 \alpha^2} \operatorname{dn} \chi + \sqrt{\alpha^2 - 1} \operatorname{sn} \chi}{\sqrt{k^2 + k'^2 \alpha^2} \operatorname{dn} \chi - \sqrt{\alpha^2 - 1} \operatorname{sn} \chi} \right|, \quad (33)$$

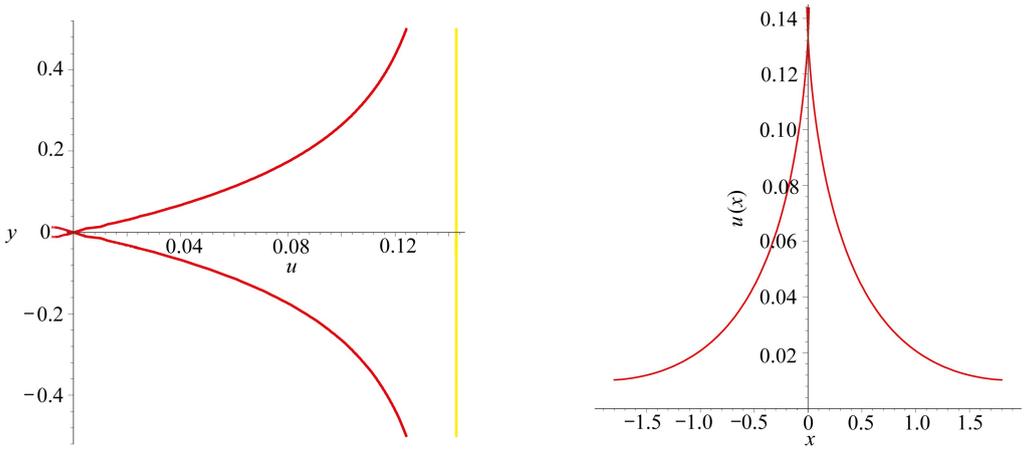
这里

$$\operatorname{cn} \chi = \frac{(A - \varphi)q - (\varphi - B)p}{(A - \varphi)q + (\varphi - B)p}, p^2 = (A - b_1)^2 + a_1^2, q^2 = (B - b_1)^2 + a_1^2,$$

$$b_1 = \frac{1}{2} \left(B - \frac{5}{6} \right), a_1^2 = \frac{1}{4} \left(3B^2 - \frac{5}{3} B + \frac{215}{36} \right),$$

$$k^2 = \frac{(A - B)^2 - (p - q)^2}{4pq}, \alpha = \frac{Bp - Aq}{Bp + Aq}.$$

同样利用数值模拟方法得出 cuspon 解 $\varphi(\xi)$ 的波形图, 见图 3(b).



(a) $c < -1$ 时的代数曲线 $H(\varphi, y) = 0$

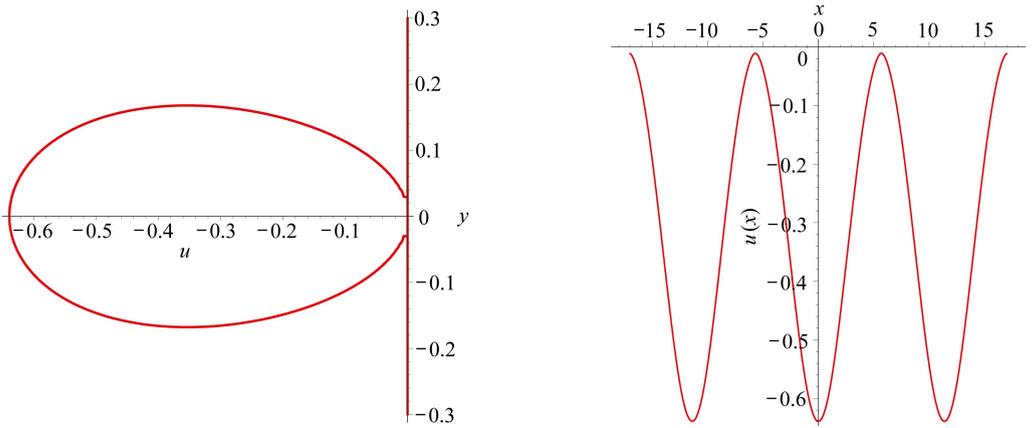
(b) Cuspon 的图形

(a) The algebraic curve $H(\varphi, y) = 0$ for $c < -1$

(b) Profile of the cuspon

图 3 方程(1)的代数曲线与 cuspon

Fig.3 The algebraic curve and cuspon of equation (1)



(a) $c = -1$ 时的代数曲线 $H(\varphi, y) = 0$

(b) 周期波的图形

(a) The algebraic curve $H(\varphi, y) = 0$ for $c = -1$

(b) Profile of the periodic wave

图 4 方程(1)的代数曲线与周期波

Fig.4 The algebraic curve and periodic wave of equation (1)

第 3 种 周期波

如果 $c = -1$, 从不变代数曲线 $H(\varphi, y) = 0$ (见图 4(a)), 推出

$$y^2 = \frac{3}{35} \varphi(B - \varphi) \left(\varphi^2 + \left(B - \frac{5}{6} \right) \varphi - \frac{5}{3B} \right), \tag{34}$$

其中

$$B = -\frac{1}{18} (6\,085 + 270\sqrt{559})^{1/3} + \frac{155}{18} (6\,085 + 270\sqrt{559})^{-1/3} + \frac{5}{18}. \tag{35}$$

对系统(4)的第一个方程进行积分, 则有

$$\int_B^\varphi \frac{dz}{\sqrt{z(B-z)\left(z^2 + \left(B - \frac{5}{6}\right)z - \frac{5}{3B}\right)}} = \sqrt{\frac{3}{35}} \int_0^\xi d\xi, \quad (36)$$

得出下面的周期波解:

$$\varphi(\xi) = \frac{aB \left(1 + \operatorname{cn} \left(\sqrt{\frac{3ab}{35}} \xi, k \right) \right)}{(a+b) + (a-b) \operatorname{cn} \left(\sqrt{\frac{3ab}{35}} \xi, k \right)}, \quad (37)$$

其中 $\operatorname{cn}(x, k)$ 是 Jacobi 椭圆函数且

$$a^2 = -\frac{5}{3B}, \quad b^2 = 2B^2 - \frac{5}{6}B - \frac{5}{3B}, \quad k^2 = \frac{B^2 - (a-b)^2}{4ab}. \quad (38)$$

该周期波解的图形见图 4(b).

3 结 论

本文利用动力系统分支方法研究了一个浅水中度振幅波方程,通过行波变换,将偏微分方程转化成常微分方程,进一步转换成一个平面动力系统.分析了系统在不同参数条件下的相图,获得光滑孤立波、cuspon 解和周期波解存在的条件.Geyer^[3]研究了解的存在性,Mutlubas 和 Geyer^[4]分析了孤立波解的轨道稳定性,但他们都没有获得方程的精确解,本文根据系统的相图,获得了光滑孤立波、cuspon 解和周期波解的参数表达式.

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Bifurcations of Solitary Wave Solutions to the Shallow Water Equation of Moderate Amplitude

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Abstract: The qualitative behavior and solitary wave solutions to the model equation for shallow water waves of moderate amplitude were studied with the bifurcation method for dynamic systems. The phase portraits of the system were given under different parametric conditions. The implicit expressions of the smooth solitary waves, cuspons and periodic wave solutions were obtained. Numerical simulations were conducted for the smooth solitary waves, cuspons and periodic wave solutions to the model equation. The results show that the presented findings improve the related previous conclusions.

Key words: solitary wave; cuspon wave; periodic wave; bifurcation

Foundation item: The National Natural Science Foundation of China(11161013;11361017)