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Effects of Surface Stresses on Contact Problems of an Elastic Half Plane With a Circular Cavity*

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Abstract: In view of the importance of surface stress in controlling mechanical responses of nanoscale structures, the effects of surface stresses on the elastic field around a circular hole in an elastic half plane were analyzed. The complex variable function method was adopted to derive the fundamental solution to the contact problem. The deformation caused by the uniformly distributed traction on the plane surface and the surface stress along the cavity boundary was analyzed in detail. The results reveal strong size-dependence of the stress field and the surface deformation on the surface stress, and the surface displacement directly above the circular hole is a function of the surface stress.

Key words: surface stress; nanosized cavity; half plane; uniform load
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Introduction

The study of the mechanical behavior of inhomogeneous materials containing nanosized inclusions or holes has attracted great interest in the fields of material science, solid state physics and nanomechanics. Due to the large ratio of surface area to volume in nanocomposites, the surface effects need to be taken into account. For the increasing applications of nanostructured materials, researchers recently have considered the effects of surface energy on the deformation field of elastic materials containing nanoinclusions or nanoholes according to the theory of surface elasticity[1-8].

Based on the classical theory of elasticity, the deformation of an elastic semi-infinite plane containing heterogeneous structures has been well studied. However, the corresponding solution for nanosized inhomogeneities is still absent up to now. Sharma et al.[9] analyzed the size-dependent elastic stresses of nano-inhomogeneities by surface elasticity theory. Sharma and Ganti[10] formulated the size-dependent Eshelby tensor for nanoinclu-

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sions with surface energy. Jammes et al.[11-14] used a semi-analytical method to solve the problem of a half-plane, which contains, circular nanoinclusions. Zhao and Rajapaske[15] derived the fundamental solution of an elastic layer bonded to a rigid substrate with surface effects. In these papers, the Fourier integral transform was used to solve the non-classical boundary value problems with surface effects. However, none of them examined the effects of surface stresses on the deformation of an elastic half plane containing an inhomogeneity and subjected to external loadings on the surface of the half plane and along the cavity boundary.

The purpose of this paper is to analyze the effects of surface stresses on the deformation of an elastic half plane containing a circular hole near the free surface. A uniformly distributed loading is applied over an area of the free surface, and a surface tension is specified along the cavity boundary. The Fourier integral transform method, the complex variable function method and the superposition principle are used here to solve the deformation field.

1 Mathematical formulation

Consider a semi-infinite, isotropic elastic half plane as is shown in fig. 1. We refer to Cartesian coordinate system $O\text{-}xy$, where x -axis is along the free surface, and y -axis perpendicular to the free surface. The elastic half plane occupies the region of $y \leqslant 0$, and is subjected to a uniform loading P on the free surface of the half plane.

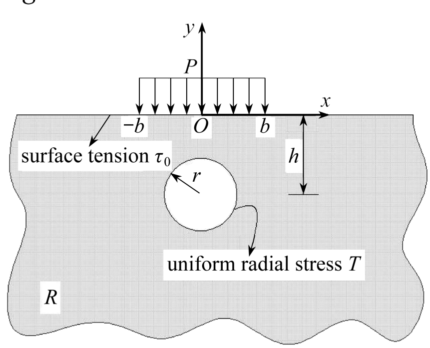

Fig. 1 The half plane with a circular cavity

The elastic half plane contains a circular hole of radius r , and the surface tension along the cavity boundary is T. The distance from the hole centre to the free surface of the elastic half plane is h ; and loading P is applied on the free surface of the elastic half plane within the range of $-b \leqslant x \leqslant b$. The shear modulus and Poisson's ratio of the elastic half-plane are μ and ν , respectively. The surface energy is τ_0.

1.1 Surface stress formulation

Owing to the large ratio of surface area to volume, the surface stress plays a key role in the stress field near the nanosized objects. Based on the surface elasticity theory, the equilibrium and the isotropic constitutive relations are

$$\sigma_{ij,j} = 0, \tag{1}$$

$$\sigma_{ij} = 2\mu\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right), \tag{2}$$

where σ_{ij} and ε_{ij} are the stress tensor and strain tensor of the half plane, respectively. Throughout the paper, Einstein's summation convention is adopted for all repeated Latin indices (1, 2, 3) and Greek indices (1, 2). The strain tensor is related to the displacement vector u_{ij} by

$$\varepsilon_{ij} = \frac{1}{2}(u_{ij,j} + u_{ij,i}). \tag{3}$$

Assume that the surface of the material adheres perfectly to its bulk without slipping. Then the equilibrium conditions on the surface are expressed as

$$\sigma_{\beta\alpha}n_\beta + \sigma^s_{\beta\alpha,\beta} = 0, \tag{4}$$

$$\sigma_{ij}n_in_j = \sigma^s_{\alpha\beta}\kappa_{\alpha\beta}, \tag{5}$$

where n_i denotes the normal to the surface, $\kappa_{\alpha\beta}$ is the curvature tensor of the surface, and $\sigma^s_{\alpha\beta}$ is the surface stress tensor. The surface stress tensor as a function of surface energy can be expressed as

$$\sigma^s_{\alpha\beta} = \tau_0\delta_{\alpha\beta} + \frac{\partial\tau_0}{\partial\varepsilon_{\alpha\beta}}. \tag{6}$$

The last term in eq. (6) demonstrates a variation of the surface energy density with respect to elastic strain, which is related to the stretch or compression of the surface atoms to accommodate to the bulk phase. If the change of the atomic spacing in deformation is infinitesimal, the contribution from the 2nd term to the surface stresses is negligibly small compared to the residual surface tension. In what follows, for simplicity, we neglect the contribution from the 2nd term in eq. (6). Then, the surface stresses are given by

$$\sigma^s_{\alpha\beta} = \tau_0\delta_{\alpha\beta}. \tag{7}$$

The boundary conditions on the free surface $y = 0$ are simplified to

$$\sigma_{xy} = 0, \tag{8}$$

$$-P - \sigma_{yy} = \frac{\tau_0}{R(x)}, \tag{9}$$

where $R(x)$ is the curvature radius of the deformed surface.

1.2 Application of the complex variable method

With the complex variable method (Muskhelishvili[16]), the elastic solution can be expressed in terms of 2 functions of $\phi(z)$ and $\psi(z)$, which are analytic in region R shown in fig. 1. The stress and the displacement components as functions of $\phi(z)$ and $\psi(z)$ are

$$\sigma_x + \sigma_y = 2\{\phi'(z) + \overline{\phi'(z)}\}, \tag{10}$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2\{\bar{z}\phi''(z) + \psi'(z)\}, \tag{11}$$

$$2\mu(u_x + iu_y) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \tag{12}$$

where $z = x + iy$ is the complex coordinate variable, $\kappa = 3 - 4\nu$ is for the plane strain and $\kappa = (3-\nu)(1+\nu)$ for the plane stress.

It's convenient to express the boundary condition of the cavity in terms of the integral of surface traction along the boundary

$$F(s) = F_1 + iF_2 = i\int_{s_0}^{s} (t_x + it_y)\,ds, \tag{13}$$

where s_0 is an arbitrary point of the cavity boundary, and t_x, t_y are the horizontal and vertical tractions, respectively. Thus the cavity boundary condition is

$$F_x + iF_y = -i[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}] + C, \tag{14}$$

where $F(s)$ is a given function of the coordinates along the cavity boundary, and C is an integration constant.

1.3 Conformal mapping

Region R in z-plane can be mapped conformally onto a ring in ξ-plane with $\xi = 1$ as the outer surface (standing for the transformed surface of the elastic half plane) and $\xi = \alpha$ ($\alpha \leqslant 1$) as the inner boundary of the ring (standing for the transformed surface of the circular hole), as is shown in fig. 2.

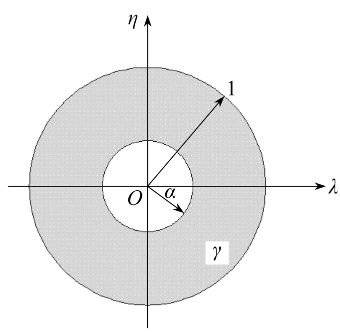

Fig. 2 The half plane by conformal transformation

With the conformal mapping method, 3 points $z_1 = 0$, $z_2 = -(h-r)i$, $z_3 = -(h+r)i$ in z-plane can be mapped to 3 points $\xi_1 = i$, $\xi_2 = \alpha i$, $\xi_3 = -\alpha i$ in ξ-plane by

$$\frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2} = \frac{\xi - \xi_1}{\xi - \xi_2} : \frac{\xi_3 - \xi_1}{\xi_3 - \xi_2}. \tag{15}$$

Thus the proper conformal mapping is

$$z = \omega(\xi) = -im\frac{1+\xi}{1-\xi}, \tag{16}$$

where $m = h(1-\alpha^2)/(1+\alpha^2)$. The governing geometrical parameter is the ratio of the radius to the depth of the hole, which is expressed as

$$\frac{r}{h} = \frac{2\alpha}{1+\alpha^2}. \tag{17}$$

Transformation function $\omega(z)$ is analytic in the ring bounded by circles $|\xi| = 1$ and $|\xi| = \alpha$. Thus, functions $\phi(z)$ and $\psi(z)$ become ones of ξ as

$$\phi(z) = \phi(\omega(\xi)) = \Phi(\xi), \tag{18}$$

$$\psi(z) = \psi(\omega(\xi)) = \Psi(\xi). \tag{19}$$

This means that they can be represented by the Laurent series expansion.

1.4 Principle of superposition

The problem can be solved with 2 problems denoted by Ⅰ and Ⅱ below according to the principle of superposition[4]. Superscripts Ⅰ and Ⅱ refer to the field variables associated with the solutions to problems Ⅰ and Ⅱ, respectively.

$$\sigma_{ij} = \sigma_{ij}^{\mathrm{I}} + \sigma_{ij}^{\mathrm{II}}, \tag{20}$$

$$u_i = u_i^{\mathrm{I}} + u_i^{\mathrm{II}}. \tag{21}$$

1.4.1 Problem Ⅰ: a homogeneous half plane under a distributed surface pressure

In case Ⅰ, uniform distributed pressure P acts over region $|x| \leqslant b$. We can obtain the stresses[5] as

$$\sigma_{xx}^{\mathrm{I}}(x,y) = \frac{2P}{\pi}\int_0^\infty \left(\frac{y\zeta - 1}{s\zeta + 1}\right)\frac{\sin(b\zeta)}{\zeta}\cos(x\zeta)\mathrm{e}^{-y\zeta}\mathrm{d}\zeta, \tag{22}$$

$$\sigma_{yy}^{\mathrm{I}}(x,y) = -\frac{2P}{\pi}\int_0^\infty \left(\frac{1 + y\zeta}{s\zeta + 1}\right)\frac{\sin(b\zeta)}{\zeta}\cos(x\zeta)\mathrm{e}^{-y\zeta}\mathrm{d}\zeta, \tag{23}$$

$$\sigma_{xy}^{\mathrm{I}}(x,y) = -\frac{2P}{\pi}\int_0^\infty \frac{\sin(b\zeta)}{(s\zeta + 1)}\frac{\sin(x\zeta)}{\zeta}y\zeta\mathrm{e}^{-y\zeta}\mathrm{d}\zeta, \tag{24}$$

where $s = \tau_0(1-\nu)/\mu$. The corresponding displacements are expressed as

$$u^{\mathrm{I}}(x,y) = \frac{P}{\pi\mu}\int_0^\infty \frac{y\zeta + 2\nu - 1}{(s\zeta + 1)\zeta^2}\sin(b\zeta)\sin(x\zeta)\mathrm{e}^{-y\zeta}\mathrm{d}\zeta + C_1, \tag{25}$$

$$w^{\mathrm{I}}(x,y) = \frac{P}{\pi\mu}\int_0^\infty \frac{2(1-\nu) + y\zeta}{(s\zeta + 1)\zeta^2}\sin(b\zeta)\cos(x\zeta)\mathrm{e}^{-y\zeta}\mathrm{d}\zeta + C_2. \tag{26}$$

On free surface $y = 0$, the stresses and displacements are given by

$$\sigma_{xx}^{\mathrm{I}}(x,0) = \sigma_{yy}^{\mathrm{I}}(x,0) = -\frac{2P}{\pi}\int_0^\infty \frac{\sin(t)}{t}\left(\frac{s}{b}t + 1\right)^{-1}\cos\left(\frac{x}{b}t\right)\mathrm{d}t, \tag{27}$$

$$u^{\mathrm{I}}(x,0) = -\frac{P(1-2\nu)b}{\pi\nu}\int_0^\infty \frac{\sin(t)}{t^2}\left(\frac{s}{b}t + 1\right)^{-1}\sin\left(\frac{x}{b}t\right)\mathrm{d}t, \tag{28}$$

$$w^{\mathrm{I}}(x,0) = \frac{2P(1-\nu)b}{\pi\mu}\int_0^\infty \frac{\sin(t)}{t^2}\left(\frac{s}{b}t + 1\right)^{-1}\left[\cos\left(\frac{x}{b}t\right) - \cos\left(\frac{r_0}{b}t\right)\right]\mathrm{d}t. \tag{29}$$

1.4.2 Problem Ⅱ: a circular hole embedded in the elastic half plane under a uniform radial stress of magnitude T on the cavity boundary

To solve problem Ⅱ, the proper complex potential based on the Laurent series is introduced as

$$\Phi^{\mathrm{II}} = \Phi^{\mathrm{II}}(\xi) = \sum_{k=0}^{\infty} a_k\xi^k + \sum_{k=1}^{\infty} b_k\xi^k, \tag{30}$$

$$\Psi^{\mathrm{II}} = \Psi^{\mathrm{II}}(\xi) = \sum_{k=0}^{\infty} c_k\xi^k + \sum_{k=1}^{\infty} d_k\xi^k, \tag{31}$$

where a_k, b_k, c_k, d_k are coefficients to be determined.

Stress functions $\phi(\xi)$ and $\psi(\xi)$ can be easily found to be

$$\frac{\phi^{\mathrm{II}}(\xi)}{L} = -2\mathrm{i}(1+\alpha^2) + 2\mathrm{i}\xi + \frac{2\mathrm{i}\alpha^2}{\xi}, \tag{32}$$

$$\frac{\psi^{\mathrm{II}}(\xi)}{L} = -3\mathrm{i}(1+\alpha^2) + 2\mathrm{i}\alpha^2\xi + \mathrm{i}\xi^2 + \frac{2\mathrm{i}}{\xi} + \frac{\mathrm{i}\alpha^2}{\xi^2}, \tag{33}$$

where

$$L=\frac{\alpha^2 Th}{(1-\alpha^2)(1-\alpha^4)}.$$

Substituting eq. (16) into eq. (32) and (33), we obtain stress functions $\phi^{\mathrm{II}}(z)$ and $\psi^{\mathrm{II}}(z)$ in z-plane as

$$\phi^{\mathrm{II}}(z)=-\frac{4zhAB^2}{z^2A^2+h^2B^2}L-4\mathrm{i}\frac{h^2AB^2}{z^2A^2+h^2B^2}L, \tag{34}$$

$$\psi^{\mathrm{II}}(z)=1+\frac{8zh^3AB^4}{(z^2A^2+h^2B^2)^2}L-\mathrm{i}A\frac{5z^2A^2+h^2B^2+6z^2h^2A^2B^2}{z^2A^2+h^2B^2}L. \tag{35}$$

Substituting eq. (34) and (35) into eq. (10), (11) and (12), and given $y=0$, the surface stresses are simplified to

$$\frac{\sigma_{xx}^{\mathrm{II}}(x,0)}{L}=8hAB^2\frac{x^2A^2-h^2B^2}{(z^2A^2+h^2B^2)^2}+8h^3AB^4\frac{3x^2A^2-h^2B^2}{(z^2A^2+h^2B^2)^3}, \tag{36}$$

$$\frac{\sigma_{yy}^{\mathrm{II}}(x,0)}{L}=8hAB^2\frac{3x^2h^2A^2B^2-x^4A^4+x^2A^2-h^2B^2}{(x^2A^2+h^2B^2)^2}+8h^2A^2B^4\frac{h^2B^2-3x^2A^2}{(x^2A^2+h^2B^2)^3}. \tag{37}$$

The corresponding displacements are expressed as

$$\frac{u^{\mathrm{II}}(x,0)}{(A-1)Th}=-4x\frac{(\kappa h+2h-1)h^2B^2+(\kappa h+1)x^2A^2}{(x^2A^2+h^2B^2)^2}, \tag{38}$$

$$\frac{w^{\mathrm{II}}(x,0)}{(A-1)Th}=\frac{(2+3B^2-4A)x^2A^2B^{-2}+(3B^2+A-5)h^2-(x^2A^2B^{-2}-h^2)^2+8xh^2A^3}{(x^2A^2+h^2B^2)^2}-\frac{4\kappa h^2B^2(A-1)^{-1}}{x^2A^2+h^2B^2}, \tag{39}$$

where $A=1+\alpha^2$, $B=1-\alpha^2$.

2 Numerical calculation

It is instructive to examine the effects of the surface stresses on the stresses or displacements on the free surface and compare them with those out of the classical elasticity theory. In what follows, we set $P=T$, $\mu=14.5$ GPa[17], $\nu=0.4$, $\tau_0=0.1\ \mathrm{J/m^2}$, $h/r=1.5$ and $b/r=3$.

As is shown in eq. (27), the values of σ_{xx} and σ_{yy} depend on parameter s/b. Thus the stresses obtained from the superposition of eq. (27), (36) and (37) are functions of the size of the circular hole and the surface stresses on the half plane, where we set $x/r=1$.

According to Wang and Feng[5], the absolute values of s/b for metals are about 0.0 to 2.0, where solution $s/b=0$ is consistent with the classical elastic results. When loading size b is comparable to parameter s, i.e., of the order of nanometers, the effects of surface stresses will be prominent. It is also found from fig. 3 and fig. 4 that the normal and tangential stresses change smoothly, which is different from the 2 lower stresses predicted by the classical elasticity theory. In addition, the normal stress goes up after a rapid decline, while the tangential stress goes down slowly after a rapidly decline around $r=0.2$.

Fig. 5 and fig. 6 show the various stresses on the surfaces for $s/b = 1$ with different coordinates in the half plane. It can be seen that the stresses drastically declines with r when r is small enough. In fig. 5, the normal stress changes gently with r around $r = 0.2$, while the tangential stress reduces slowly around $r = 0.3$.

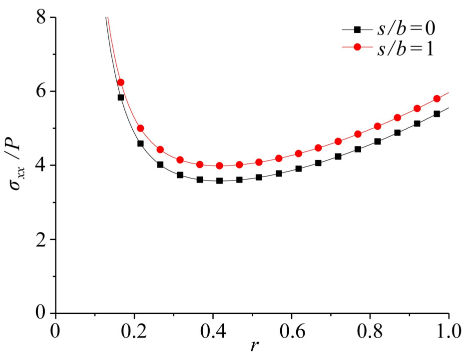

Fig. 3 Distribution of the normal stress

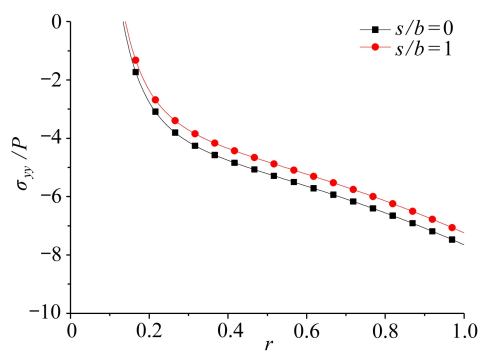

Fig. 4 Distribution of the tangential stress

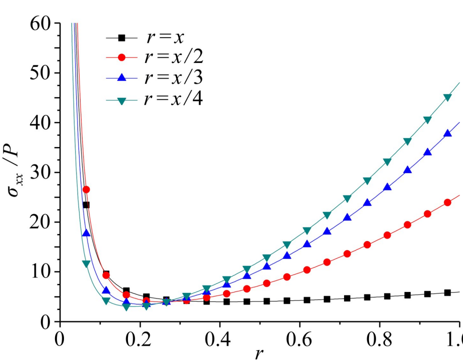

Fig. 5 Distribution of the normal stress on the surfaces of different coordinates

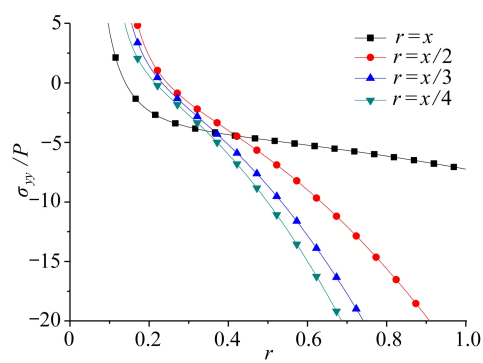

Fig. 6 Distribution of the tangential stress on the surfaces of different coordinates

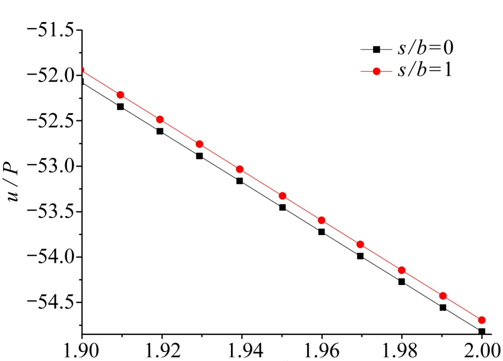

Fig. 7 Distribution of the normal displacements

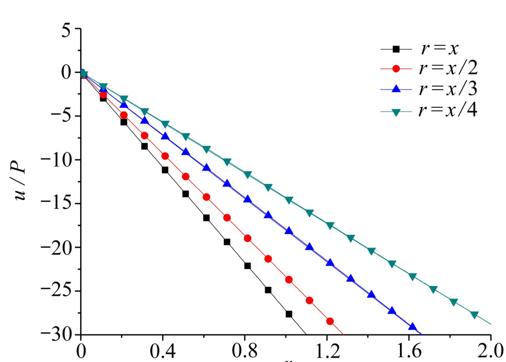

Fig. 8 Distribution of the normal displacements

Fig. 7 shows the normal displacements for 2 different values of s/b. We can see that they both decline linearly with r. The result from the classical elasticity theory is only slightly lower than that from the nanosized one.

Fig. 8 compares the normal displacements for different coordinates on the free surface. It can be seen that the displacements decrease linearly with r, and the slope of the deformed surface for $s > 0$ is continuous everywhere.

3 Summary

Based on the theory of surface elasticity and the complex variable method, the effects of surface stresses on the deformation of an elastic half plane containing a circular hole were analyzed, as the nanosized surface of the elastic half plane was subjected to a surface loading. It is found that the surface elasticity theory illuminates some interesting characteristics of contact problems at the nanoscale, which are distinctly different from those of the classical solutions. Therefore, for nanosized contact problems of half planes with cavities, the effects of surface stresses should be carefully considered.

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带有圆柱形孔洞弹性半空间的表面效应分析

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摘要： 针对表面应力在纳米结构控制机械响应中的重要性，利用复变函数的基本方法，研究了含有圆柱形孔洞的弹性半空间的表面应力问题.将含有缺陷的接触问题分解为均匀介质的接触问题和无外载荷的非均匀介质的接触问题两部分进行分析.结果显示：接触表面的应力和位移有很强的尺寸依赖性，同时表面位移可以用表面应力函数表示.

关 键 词： 表面应力； 纳米孔洞； 半空间； 均布载荷
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