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带 Riemann-Stieltjes 积分条件的三阶边值问题的单调正解*

张海娥

(唐山学院 基础教学部, 河北 唐山 063000)

摘要: 研究了一类带 Riemann-Stieltjes 积分条件的非线性三阶非局部边值问题,将边值问题正解存在性的研究转化为扰动 Hammerstein 积分方程的研究,通过构造 Green(格林)函数及讨论其性质,运用推广的 Leggett-Williams 型不动点定理,得到了至少存在 3 个和 $2n-1$ 个正解的存在性准则, 所得结果推广和改进了最近文献中的结果,并充分反映了非线性项含导数对正解存在性研究的影响.主要结果由实例加以阐述.

关 键 词: Leggett-Williams 不动点定理; 正解; 非局部; 积分条件

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引 言

三阶微分方程起源于应用数学和物理学的许多不同领域, 譬如, 带有固定或变化横截面的屈曲梁的挠度、三层梁、电磁波、地球引力吹积的涨潮等.在流体力学中,三阶微分方程边值问题出现在耗散流与表面流的研究中,许多工程实际问题可用三阶微分方程边值问题准确描述,见文献[1-2].

Riemann-Stieltjes 积分边界条件包括多点和 Riemann 积分边界条件作为其特殊情况.在热传导、化学工程、地下水流、热弹性等应用力学或物理等领域中许多问题都抽象为带积分条件的边值问题[3-4].近些年来, 带积分条件的三阶非局部边值问题受到了学者的广泛关注,见文献[5-11]及其参考文献, 但是带 Riemann-Stieltjes 积分边界条件的边值问题研究结果还较少, 特别指出 Webb 等对此类问题做出了突出成果[12-13].众所周知,当边值问题的非线性项含有导数时,将使锥上的压缩或拉伸不易实现,对其正解研究造成困难.故本文将研究一类非线性项含导数的带 Riemann-Stieltjes 积分条件的三阶边值问题(boundary value problem 简记为 BVP)

$$\begin{cases} u'''(t)+f(t,u(t),u'(t))=0, \quad 0<t<1, \\ u(0)=0, \\ au'(0)-bu''(0)=\alpha[u], \\ cu'(1)+du''(1)=\beta[u], \end{cases} \tag{1}$$

其中

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作者简介: 张海娥(1981—),女,河北廊坊人,副教授,硕士(E-mail: haiezhang@126.com).

$$f \in C([0,1]\times \mathbf{R}_+\times \mathbf{R}_+,\mathbf{R}_+),\ \alpha[u]=\int_0^1 u(t)\mathrm{d}A(t),\ \beta[u]=\int_0^1 u(t)\mathrm{d}B(t)$$

是 $C[0,1]$ 上由 Riemann-Stieltjes 积分给出的的线性泛函,且 a,b,c,d 非负,满足 $\rho := ac+ad+bc>0$.文献[14]利用单调迭代技巧研究了该问题,获得了至少存在一个单调正解的结果.本文研究 BVP(1)至少 3 个或 $2n-1$ 个正解的存在性,并举例阐述了主要结果,所用工具为推广的 Leggett-Williams 型定理,如下:

定理 1[15] 设 E 为 Banach 空间, P 为 E 中的锥且给定 $0<r_1<b<d\leqslant r_2, 0<L_1\leqslant L_2$. 设 α_1,α_2 为 P 上的非负连续凸泛函使得对 $u\in P$, $\|u\|\leqslant M_1\max\{\alpha_1(u),\alpha_2(u)\}$ 且

$$P(\alpha_1,r;\alpha_2,L)\neq 0,\qquad \forall r,L>0.$$

γ 为 P 上的非负连续凹泛函,使得对所有的 $u\in \bar{P}(\alpha_1,r_2;\alpha_2,L_2)$,有 $\gamma(u)\leqslant\alpha_1(u)$ 成立.令 T: $\bar{P}(\alpha_1,r_2;\alpha_2,L_2)\to\bar{P}(\alpha_1,r_2;\alpha_2,L_2)$ 是全连续算子,假定下述条件满足:

(a1) $\{u\in\bar{P}(\alpha_1,d;\alpha_2,L_2;\gamma,b)\mid\gamma(u)>b\}\neq 0$ 且 $\gamma(Tu)>b$, $\forall u\in\bar{P}(\alpha_1,d;\alpha_2,L_2;\gamma,b)$;

(a2) $\alpha_1(Tu)<r_1$, $\alpha_2(Tu)<L_1$, 对 $\forall u\in\bar{P}(\alpha_1,r_1;\alpha_2,L_1)$;

(a3) $\forall u\in\bar{P}(\alpha_1,d;\alpha_2,L_2;\gamma,b)$ 与 $\alpha_1(Tu)>b$,总有 $\gamma(Tu)>b$,

则 T 至少有 3 个不动点 $u_1,u_2,u_3\in\bar{P}(\alpha_1,r_2;\alpha_2,L_2)$ 满足 $u_1\in P(\alpha_1,r_1;\alpha_2,L_1)$, $u_2\in P(\alpha_1,r_2;\alpha_2,L_2;\gamma,b)$ 且 $u_3\in P(\alpha_1,r_2;\alpha_2,L_2)\setminus(\bar{P}(\alpha_1,r_1;\alpha_2,L_1)\cup\bar{P}(\alpha_1,r_2;\alpha_2,L_2;\gamma,b))$.

1 预备引理

根据文献[12-13]的相关结果, BVP(1)的研究可以转化为扰动 Hammerstein 积分方程的研究, 即

$$u(t)=\gamma(t)\alpha[u]+\delta(t)\beta[u]+\int_0^1 G(t,s)f(s,u(s),u'(s))\mathrm{d}s, \tag{2}$$

$\gamma(t),\delta(t)$ 线性无关且

$$\begin{cases}-\gamma'''(t)=0,\ \gamma(0)=0,\ a\gamma'(0)-b\gamma''(0)=1,\ c\gamma'(1)+d\gamma''(1)=0,\\ -\delta'''(t)=0,\ \delta(0)=0,\ a\delta'(0)-b\delta''(0)=0,\ c\delta'(1)+d\delta''(1)=1,\end{cases}$$

这意味着 $\gamma(t)=(2ct+2dt-ct^2)/(2\rho)$ 和 $\delta(t)=(at^2+2bt)/(2\rho)$, $t\in[0,1]$.对 $t\in[\theta,1]$, $\gamma(t)\geqslant c_1\|\gamma\|_\infty$ 和 $\delta(t)\geqslant c_2\|\delta\|_\infty$($\|\cdot\|_\infty$ 是 $C[0,1]$ 上通常的上确界范数), 其中 $c_1=(2c\theta+2d\theta-c\theta^2)/(c+2d)$ 和 $c_2=(a\theta^2+2b\theta)/(a+2b)$; $G(t,s)$ 是 $\alpha[u]=\beta[u]=0$ 时的 Green 函数.定义算子 T 为

$$Tu(t):=\gamma(t)\alpha[u]+\delta(t)\beta[u]+\int_0^1 G(t,s)f(s,u(s),u'(s))\mathrm{d}s.$$

如文献[14]中的定理 2.3,若 u 是 T 的不动点,则 u 是 S 的不动点,本文 S 定义如下:

$$\begin{aligned}Su(t):=&\frac{\gamma(t)}{D}\left((1-\beta[\delta])\int_0^1\kappa_A(s)f(s,u(s),u'(s))\mathrm{d}s+\right.\\ &\left.\alpha[\delta]\int_0^1\kappa_B(s)f(s,u(s),u'(s))\mathrm{d}s\right)+\\ &\frac{\gamma(t)}{D}\left(\beta[\gamma]\int_0^1\kappa_A(s)f(s,u(s),u'(s))\mathrm{d}s+\right.\end{aligned}$$

$$(1-\alpha[\gamma])\int_0^1\kappa_B(s)f(s,u(s),u'(s))\mathrm{d}s\Big)+$$

$$\int_0^1 G(t,s)f(s,u(s),u'(s))\mathrm{d}s=\int_0^1 G_S(t,s)f(s,u(s),u'(s))\mathrm{d}s,$$

其中核 G_S 是 BVP(1)的 Green 函数.

本文剩余部分做如下假设:

(H1) A,B 是具有有界变差的函数, 且 $\kappa_A(s),\kappa_B(s),s\in[0,1]$, 其中

$$\kappa_A(s):=\int_0^1 G(t,s)\mathrm{d}A(t),\ \kappa_B(s):=\int_0^1 G(t,s)\mathrm{d}B(t);$$

(H2) $0\leqslant\alpha[\gamma],\ \beta[\delta]<1,\ \alpha[\delta],\ \beta[\gamma]\geqslant 0$ 且 $D:=(1-\alpha[\gamma])(1-\beta[\delta])-\alpha[\delta]\beta[\gamma]>0$.

引理 1[14] 令 $\rho:=ac+ad+bc>0$,存在可测函数 $\Phi(s)$ 使得 $G(t,s)$ 满足:

$$G(t,s)\leqslant\Phi(s),\qquad t\in[0,1],\ s\in[0,1];$$
$$G(t,s)\geqslant c_3\Phi(s),\qquad t\in[\theta,1],\ s\in[0,1],$$

其中

$$c_3=\frac{\rho\int_0^\theta\Phi(\tau)\mathrm{d}\tau}{(a+b)(c+d)},\qquad 0<\theta<1.$$

引理 2[14] 令 $c_0=\min\{c_1,c_2,c_3\}$, Green 函数 $G_S(t,s)$ 满足

(i) $G_S(t,s)\leqslant\Phi_1(s),\qquad t\in[0,1],\ s\in[0,1];$

(ii) $G_S(t,s)\geqslant c_0\Phi_1(s),\qquad t\in[\theta,1],\ s\in[0,1];$

(iii) $0\leqslant\dfrac{\partial G_S(t,s)}{\partial t}\leqslant\Phi_2(s),\qquad t,s\in[0,1]\times[0,1],$

其中

$$\Phi_1(s):=\frac{\|\gamma\|_\infty}{D}((1-\beta[\delta])\kappa_A(s)+\alpha[\delta]\kappa_B(s))+$$
$$\frac{\|\delta\|_\infty}{D}(\beta[\gamma]\kappa_A(s)+(1-\alpha[\gamma])\kappa_B(s))+\Phi(s),\qquad s\in[0,1],$$

$$\Phi_2(s):=\frac{\|\gamma'\|_\infty}{D}((1-\beta[\delta])\kappa_A(s)+\alpha[\delta]\kappa_B(s))+$$
$$\frac{\|\delta'\|_\infty}{D}(\beta[\gamma]\kappa_A(s)+(1-\alpha[\gamma])\kappa_B(s))+\Phi(s),\qquad s\in[0,1].$$

令 $E=C^1[0,1]$ 赋予其范数 $\|u\|=\max\{\|u\|_\infty,\|u'\|_\infty\}$,其中 $\|u\|_\infty$ 是 $C[0,1]$ 上通常的上确界范数.令 $P=\{u\in E:u(t)\geqslant 0,t\in[0,1]\}$,则 E 是 Banach 空间, P 为 E 中的锥.

引理 3[14] $T,S:P\to P$ 是紧算子.

引理 4[14] T 和 S 在 P 中有相同的不动点.

2 主要结果

对 $\forall u\in P$, 定义泛函

$$\alpha_1(u)=\max_{t\in[0,1]}|u(t)|,\ \alpha_2(u)=\max_{t\in[0,1]}|u'(t)|,\ \gamma(u)=\min_{t\in[0,1]}|u(t)|,$$

则 $\alpha_1,\alpha_2:P\to[0,+\infty)$ 是非负连续凸泛函且 γ 是非负连续凹泛函,均满足定理 1 的条件.记

$$M=\int_0^1\Phi_1(s)\,\mathrm{d}s,\ N=\int_0^1\Phi_2(s)\,\mathrm{d}s.$$

定理 2 设存在常数

$$0<r_1<b<\frac{b}{c_0}\leqslant r_2,\ 0<L_1<L_2$$

使得

$$\frac{b}{c_0M}\leqslant\min\left\{\frac{r_2}{M},\frac{L_2}{N}\right\}.$$

假设如下条件满足:

(b1) $f(t,u,v)<\min\left\{\frac{r_1}{M},\frac{L_1}{N}\right\},\qquad (t,u,v)\in[0,1]\times[0,r_1]\times[-L_1,L_1]$;

(b2) $f(t,u,v)>\frac{b}{c_0M},\qquad (t,u,v)\in[0,1]\times\left[b,\frac{b}{c_0}\right]\times[-L_2,L_2]$;

(b3) $f(t,u,v)<\min\left\{\frac{r_2}{M},\frac{L_2}{N}\right\},\qquad (t,u,v)\in[0,1]\times[0,r_2]\times[-L_2,L_2]$,

则 BVP(1)至少存在 3 个正解 u_1,u_2 和 u_3 满足

$$\max_{t\in[0,1]}u_1(t)<r_1,\ \max_{t\in[0,1]}|u_1'(t)|<L_1,$$
$$b<\min_{t\in[0,1]}u_2(t)\leqslant\max_{t\in[0,1]}u_2(t)<r_2,\ \max_{t\in[0,1]}|u_2'(t)|<L_2$$

且

$$r_1<\max_{t\in[0,1]}u_3(t)<r_2,\ \min_{t\in[0,1]}u_3(t)<b,\ \max_{t\in[0,1]}|u_3'(t)|<L_2.$$

证明 首先,如果 $u\in\bar{P}(\alpha_1,r_2;\alpha_2,L_2)$,则 $\alpha_1(u)\leqslant r_2,\alpha_2(u)\leqslant L_2$, 结合条件(b3)可得

$$f(t,u,v)<\min\left\{\frac{r_2}{M},\frac{L_2}{N}\right\},\qquad t\in[0,1].$$

因此,

$$\alpha_1(Tu)=\max_{t\in[0,1]}|Tu(t)|=$$
$$\max_{t\in[0,1]}\left|\int_0^1G_S(t,s)f(s,u(s),u'(s))\,\mathrm{d}s\right|\leqslant\frac{r_2}{M}\int_0^1\Phi_1(s)\,\mathrm{d}s=r_2,$$
$$\alpha_2(Tu)=\max_{t\in[0,1]}|(Tu(t))'|=$$
$$\max_{t\in[0,1]}\left|\int_0^1\frac{\partial G_S(t,s)}{\partial t}f(s,u(s),u'(s))\,\mathrm{d}s\right|\leqslant\frac{L_2}{N}\int_0^1\Phi_2(s)\,\mathrm{d}s=L_2.$$

上式表明 $T:\bar{P}(\alpha_1,r_2;\alpha_2,L_2)\to P(\alpha_1,r_2;\alpha_2,L_2)$,类似可证 $T:\bar{P}(\alpha_1,r_1;\alpha_2,L_1)\to P(\alpha_1,r_1;\alpha_2,L_1)$, 故定理 1 的条件(a2)满足.接下来证

$$P\left(\alpha_1,\frac{b}{c_0};\alpha_2,L_2;\gamma,b\right)\neq 0\ \text{且}\ \gamma(Tu)>b,\forall u\in\bar{P}\left(\alpha_1,\frac{b}{c_0};\alpha_2,L_2;\gamma,b\right).$$

事实上,

$$\frac{b+b/c_0}{2}\in P\left(\alpha_1,\frac{b}{c_0};\alpha_2,L_2;\gamma,b\right).$$

此外,

$$\forall u \in \bar{P}\left(\alpha_1, \frac{b}{c_0}; \alpha_2, L_2; \gamma, b\right), 有 \frac{b}{c_0} \geqslant \alpha_1(u) \geqslant u(t) \geqslant \gamma(u) \geqslant b, t \in [0,1].$$

因此,由(b2)可知

$$\gamma(Tu) = \min_{t\in[\theta,1]} |Tu(t)| =$$

$$\min_{t\in[\theta,1]} \left| \int_0^1 G_S(t,s) f(s,u(s),u'(s))\mathrm{d}s \right| > \frac{b}{c_0 M}\int_0^1 c_0 \Phi_1(s)\mathrm{d}s = b,$$

则定理 1 的(a1)满足.

最后,证明定理 1 的(a3)满足.设 $\forall u \in \bar{P}(\alpha_1, r_2; \alpha_2, L_2; \gamma, b)$ 与 $\alpha_1(Tu) > b/c_0$,

$$\gamma(Tu) = \min_{t\in[\theta,1]} |Tu(t)| \geqslant c_0 \left| \int_0^1 \Phi_1(s) f(s,u(s),u'(s))\mathrm{d}s \right| \geqslant$$

$$c_0 \max_{t\in[0,1]} |Tu(t)| = c_0 \alpha_1(Tu) > c_0 \frac{b}{c_0} = b.$$

综上所述,定理 1 的所有假设条件满足,因此 T 至少存在 3 个不动点,即 BVP(1)至少存在 3 个正解.

推论 1 令 n 为正整数,设存在常数

$$0 < r_1 < b_1 < \frac{b_1}{c_0} \leqslant r_2 < b_2 < \frac{b_2}{c_0} \leqslant \cdots \leqslant r_n,\ 0 < L_1 \leqslant L_2 \leqslant L_3 \leqslant \cdots \leqslant L_n,$$

使得

$$\frac{b_i}{c_0 M} \leqslant \min\left\{\frac{r_{i+1}}{M}, \frac{L_{i+1}}{N}\right\} \qquad (i = 1,2,\cdots,n-1).$$

假设如下条件满足:

(d1) $f(t,u,v) < \min\left\{\frac{r_i}{M}, \frac{L_i}{N}\right\}$,

$$(t,u,v) \in [0,1] \times [0,r_i] \times [-L_i, L_i],\ i = 1,2,\cdots,n;$$

(d2) $f(t,u,v) > \frac{b_i}{c_0 M}$,

$$(t,u,v) \in [0,1] \times \left[b_i, \frac{b_i}{c_0}\right] \times [-L_{i+1}, L_{i+1}],\ i = 1,2,\cdots,n-1,$$

则 BVP(1)至少存在 $2n - 1$ 个正解.

证明 利用数学归纳法证明.

首先,当 $n = 1$, 由(d1)可知,

$$T:\bar{P}(\alpha_1, r_2; \alpha_2, L_2) \to P(\alpha_1, r_2; \alpha_2, L_2) \subset \bar{P}(\alpha_1, r_2; \alpha_2, L_2),$$

由 Schauder 不动点定理可知,BVP(1)在 $P(\alpha_1, r_2; \alpha_2, L_2)$ 中至少存在 1 个正解.

其次,假设 $n = k$ 结论仍然成立,往证 $n = k + 1$ 结论亦成立.

假设存在 $r_i (i = 1,2,\cdots,k+1)$ 和 $b_i (i = 1,2,\cdots,k)$ 满足

$$0 < r_1 < b_1 < \frac{b_1}{c_0} \leqslant r_2 < b_2 < \frac{b_2}{c_0} \leqslant \cdots \leqslant r_{k+1},$$

使得对 $i = 1,2,\cdots,k+1$, 有

$$f(t,u,v) < \min\left\{\frac{r_i}{M}, \frac{L_i}{N}\right\},$$

$$(t,u,v)\in[0,1]\times[0,r_i]\times(-L_i,L_i),\ i=1,2,\cdots,k,$$

$$f(t,u,v)>\frac{b_i}{c_0M},\qquad (t,u,v)\in[0,1]\times\left[b_i,\frac{b_i}{c_0}\right]\times[-L_{i+1},L_{i+1}],$$

则 BVP(1) 在 $P(\alpha_1,r_k;\alpha_2,L_k)$ 中存在至少 $2k-1$ 个正解 $u_i(i=1,2,\cdots,2k-1)$. 同时,由定理 2,BVP(1) 至少在 $P(\alpha_1,r_{k+1};\alpha_2,L_{k+1})$ 中存在 3 个正解 u,x 和 y 满足

$$u\in P(\alpha_1,r_k;\alpha_2,L_k),\ x\in P(\alpha_1,r_{k+1};\alpha_2,L_{k+1};\gamma,b)$$

且

$$y\in P(\alpha_1,r_{k+1};\alpha_2,L_{k+1})\backslash(\bar{P}(\alpha_1,r_k;\alpha_2,L_k)\cup\bar{P}(\alpha_1,r_{k+1};\alpha_2,L_{k+1};\gamma,b)).$$

显然, x,y 不同于 $u_i(i=1,2,\cdots,2k-1)$, 因此,BVP(1) 至少存在 $2k+1$ 个正解,得证.

3 算　　例

考虑边值问题

$$\begin{cases}u'''(t)+f(t,u(t),u'(t))=0, & 0<t<1,\\ u(0)=0,\\ u'(0)=\alpha[u],\\ u'(1)=\beta[u],\end{cases}\tag{3}$$

其中

$$\alpha[u]=\int_0^1(1-s)u(s)\mathrm{d}s,\ \beta[u]=\int_0^1 su(s)\mathrm{d}s.$$

$$f(t,u,v)=\begin{cases}\cos t+\dfrac{1}{2}u^2+\left(\dfrac{|v|}{20}\right)^2, & 0\leqslant u\leqslant 24,\\ \cos t+\dfrac{1}{2}(25-u)u^2+\left(\dfrac{|v|}{20}\right)^2, & 24<u\leqslant 25,\\ \cos t+\dfrac{1}{2}(u-25)u^2+\left(\dfrac{|v|}{20}\right)^2, & 25<u<26,\\ \cos t+318+\left(\dfrac{|v|}{20}\right)^2, & u\geqslant 26.\end{cases}$$

对应得到 $\gamma(t)=(2t-t^2)/2$ 和 $\delta(t)=t^2/2$. 简单计算表明

$$\alpha[\gamma]=\frac{1}{8},\ \alpha[\delta]=\frac{1}{24},\ \beta[\gamma]=\frac{5}{24},\ \beta[\delta]=\frac{1}{8},$$

$$D=(1-\alpha[\gamma])(1-\beta[\delta])-\alpha[\delta]\beta[\gamma]=\frac{109}{144},$$

$$\kappa_A(s)=\int_0^1G(t,s)(1-t)\mathrm{d}t=\frac{s}{8}-\frac{s^2}{4}+\frac{s^3}{6}-\frac{s^4}{24},$$

$$\kappa_B(s)=\int_0^1G(t,s)t\mathrm{d}t=\frac{5s}{24}-\frac{s^2}{4}+\frac{s^4}{24},$$

$$\Phi_1(s)=\frac{265s}{218}-\frac{145s^2}{109}+\frac{13s^3}{109}-\frac{s^4}{218},\ \Phi_2(s)=\frac{156s}{109}-\frac{181s^2}{109}+\frac{26s^3}{109}-\frac{s^4}{109},$$

且取

$$\theta=\frac{1}{2},\ r_1=1,\ b=2,\ r_2=100,\ L_1=10,\ L_2=300,$$

通过计算表明

$$M=\int_0^1\Phi_1(s)\mathrm{d}s\approx 0.133\ 69,\ N=\int_0^1\Phi_2(s)\mathrm{d}s\approx 0.219\ 877,\ c_0=\frac{1}{12},$$

$$\min\left\{\frac{r_1}{M},\frac{L_1}{N}\right\}\approx 7.48,\ \min\left\{\frac{r_2}{M},\frac{L_2}{N}\right\}\approx 748,\ \frac{b}{c_0M}\approx 180,$$

且满足

$$\frac{b}{c_0M}\leqslant\min\left\{\frac{r_2}{M},\frac{L_2}{N}\right\},$$

因此,$f(t,u,v)$ 满足

(b1) $f(t,u,v)<7.48,\quad (t,u,v)\in[0,1]\times[0,1]\times[-10,10]$;

(b2) $f(t,u,v)>180,\quad (t,u,v)\in[0,1]\times[2,24]\times[-300,300]$;

(b3) $f(t,u,v)<748,\quad (t,u,v)\in[0,1]\times[0,100]\times[-300,300]$,

则定理 2 的条件均满足,BVP(1)至少存在 3 个正解.

4 结　　论

当边值问题的非线性项含有导数时,将使锥上的“压缩”或“拉伸”不易实现,对寻求其正解造成困难.利用推广后的 Leggett-Williams 型不动点定理,只需增加 1 个有界的非负连续凸泛函,即可证明正解的存在性.所得结果丰富了边值问题解的定性理论的相关成果.接下来,可将该不动点定理进一步推广,使之能适用于非线性项含各阶导数或任意阶导数的边值问题,所得结果将更有意义.

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Multiple Monotone Positive Solutions to 3rd-Order Boundary Value Problems Involving Riemann-Stieltjes Integral Conditions

ZHANG Hai-e

(*Department of Basic Science*, *Tangshan College*, *Tangshan*, *Hebei* 063000, *P.R.China*)

Abstract: A class of 3rd-order nonlocal boundary value problems (BVPs) with Riemann-Stieltjes integral conditions were studied. The existence of positive solutions to BVPs was explored via perturbed Hammerstein integral equations. Through the construction of the Green functions and discussion on their properties, the existence criterion for at least 3 or $2n-1$ positive solutions was obtained by means of the generalization of the Leggett-Williams fixed point theorem. The results generalize and improve some known results of the latest literatures, and fully reflect the influence of nonlinear terms involving derivatives on the existence of positive solutions. An example was also included to illustrate the main results.

Key words: Leggett-Williams fixed point theorem; positive solution; nonlocal; integral condition