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有限位移理论的功的互等定理及其应用*

付宝连

(燕山大学 建筑工程与力学学院, 河北 秦皇岛 066004)

摘要： 提出了有限位移理论三维线弹性力学的功的互等定理.基于这一定理,导出了大挠度弯曲矩形板的功的互等定理.同时,应用简化矩形板的定理,直接得到了大挠度板条的功的互等定理.作为应用,计算了在均载作用下两端固定大挠度板条的弯曲和在均载作用下 4 边固定大挠度矩形板的弯曲.计算表明,根据弯曲薄板大挠度功的互等定理,大挠度弯曲矩形板可应用小挠度的相应基本解得以简单地解决.

关 键 词： 线弹性力学； 有限位移理论； 大挠度； 功的互等定理

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引 言

功的互等定理被认为是固体力学中的一个经典性能量原理.Maxwell(麦克斯威尔)首先在文献[1]中计算了一位移互等定理的具体算例.其次,Betti(贝蒂)在文献[2]中提出了功的互等定理,它有时又被称为 Betti 功的互等定理.尔后,Rayleigh(瑞雷)在文献[3]中将这一定理推广到有谐载作用的情况,Lamb(兰姆)在文献[4]中完成了动力学功的互等定理的一般研究.最后,Андреев(安德列夫)等在文献[5-9]中给出了该定理的一系列应用.

自 1980 年,本文作者发表了一些有关功的互等定理的理论研究与应用方面的文章.在理论研究方面,笔者在文献[10]中首先提出了功的互等定理与叠加原理等价的两个原理.其次,在文献[11]中提出了广义功的互等定理,这一定理适用于具有不同本构关系的两个变形体.在应用方面,在文献[12]中,首先推广功的互等定理于求解悬臂矩形板的弯曲.尔后,在文献[13-15]中,继续推广该定理于求解矩形板的振动.在文献[16-17]中进一步应用于线弹性力学的平面问题和空间问题从而形成一系统的方法,称之为功的互等法(RMW).

在文献[18-19]中,给出了修正的功的互等定理,在该定理中给出了功的互等定理的正确命题,因此,功的互等法(RMW)被置于严格和正确的理论基础上.这是一结构分析的新颖的和强有力的方法,这一方法的提出极大地扩展了功的互等定理的固有功能.

所有上述工作只局限于小位移理论问题.如所周知,有限位移理论能量原理的研究是长时期研究的重要课题.并且有关有限位移理论变分原理的研究获得了重要进展[20-23].然而,迄今为止,尚未发现有限位移理论的功的互等定理的任何研究工作.本文提出了有限位移理论线弹性力学的功的互等定理.基于该理论,导出了大挠度矩形板和板条的功的互等定理.同时,给出

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作者简介： 付宝连(1934—),男,辽宁辽阳人,教授(E-mail：ysufubaolian@163.com).

了一大挠度板条和大挠度矩形板的数值算例.这些算例表明,大挠度板的问题可应用相应小挠度问题的基本解得以简单地解决.

小位移理论的修正的功的互等定理和有限位移理论的功的互等定理一起形成一完整的功的互等理论.

1 基本方程

在本文中,我们将应用 Lagrange 方法.在该方法中,确定变形前物体一点的坐标被用于确定随后变形过程中点的位移.

在 Lagrange 方法中,在 Cartesian(笛卡尔)直角坐标系下,有限位移理论线弹性力学的基本方程可写成

$$[(\delta_{ki}+u_{k,i})\sigma_{kj}]_{,j}+F_i=0,\qquad x_i\in V,\tag{1}$$

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}),\qquad x_i\in V,\tag{2}$$

$$[(\delta_{ki}+u_{k,i})\sigma_{kj}n_j]=\bar{p}_i,\qquad x_i\in S_p,\tag{3}$$

$$u_i=\bar{u}_i,\qquad x_i\in S_u,\tag{4}$$

$$\sigma_{ij}=2Ge_{ij}+\lambda e_{ll}\delta_{ij},\qquad x_i\in V,\tag{5}$$

这里 σ_{ij} 是 Kirchhoff(克希霍夫)应力张量, e_{ij} 为 Green(格林)应变张量, u_i 是位移分量, δ_{ij} 是 Kronecker delta 符号,它可以表示为

$$\delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j,\end{cases}$$

G 和 λ 是弹性常数.

2 有限位移理论三维线弹性力学的功的互等定理

现考虑有限位移的两个线弹性体.它们具有相同的几何形状、尺寸和本构关系,但具有不相同的力的边界条件和位移边界条件.在各自外力作用下它们都处于真实状态.其中之一被称为第一弹性体,相应的力学量表以$(\ )_1$;另一弹性体称为第二弹性体,相应的力学量表以$(\ )_2$.

第一弹性体的应力分量在第二弹性体的相应应变分量上所做的总功等于

$$\begin{aligned}\bar{W}_{12}&\equiv\iiint_V\sigma_{1ij}e_{2ij}\mathrm{d}v=\iiint_V\sigma_{1ij}\frac{1}{2}(u_{2i,j}+u_{2j,i}+u_{2k,i}u_{2k,j})\mathrm{d}v=\\&\iiint_V\sigma_{1ij}(u_{2i,j}+u_{1k,i}u_{2k,j})\mathrm{d}v+\iiint_V\sigma_{1ij}\left(\frac{1}{2}u_{2k,i}u_{2k,j}-u_{1k,i}u_{2k,j}\right)\mathrm{d}v.\end{aligned}\tag{6}$$

应用第一弹性体的平衡方程和静力边界条件,应用第二弹性体的位移边界条件和应变-位移关系,并应用 Green 公式,则得

$$\begin{aligned}&\iiint_V\sigma_{1ij}(u_{2i,j}+u_{1k,i}u_{2k,j})\mathrm{d}v=\iiint_V[\sigma_{1ij}(\delta_{ki}+u_{1k,i})u_{2k,j}]\mathrm{d}v=\\&\iiint_V\{[(\delta_{ki}+u_{1k,i})\sigma_{1ij}u_{2k}]_{,j}-[(\delta_{ki}+u_{1k,i})\sigma_{1ij}]_{,j}u_{2k}\}\mathrm{d}v=\\&\iint_S[(\delta_{ki}+u_{1k,i})\sigma_{1ij}n_j]u_{2k}\mathrm{d}s-\iiint_V[(\delta_{ki}+u_{1k,i})\sigma_{1ij}]_{,j}u_{2k}\mathrm{d}v=\\&\iint_{S_{1p}}\bar{p}_{1i}u_{2i}\mathrm{d}s+\iint_{S_{1u}}p_{1i}u_{2i}\mathrm{d}s+\iiint_V F_{1i}u_{2i}\mathrm{d}v.\end{aligned}\tag{7}$$

将式(7)代入式(6),得

$$\bar{W}_{12} \equiv \iiint_V \boldsymbol{\sigma}_{1ij} e_{2ij} \mathrm{d}v = \iint_{S_{1p}} \bar{p}_{1i} u_{2i} \mathrm{d}s + \iint_{S_{1u}} p_{1i} u_{2i} \mathrm{d}s + \iiint_V F_{1i} u_{2i} \mathrm{d}v + \iiint_V \boldsymbol{\sigma}_{1ij} \left(\frac{1}{2} u_{2k,i} u_{2k,j} - u_{1k,i} u_{2k,j} \right) \mathrm{d}v . \tag{8}$$

应用第二弹性体的平衡方程和静力边界条件,应用第一弹性体的位移边界条件和应变-位移关系,并应用 Green 公式,则得

$$\bar{W}_{21} \equiv \iiint_V \boldsymbol{\sigma}_{2ij} e_{1ij} \mathrm{d}v = \iint_{S_{2p}} \bar{p}_{2i} u_{1i} \mathrm{d}s + \iint_{S_{2u}} p_{2i} u_{1i} \mathrm{d}s + \iiint_V F_{2i} u_{1i} \mathrm{d}v + \iiint_V \boldsymbol{\sigma}_{2ij} \left(\frac{1}{2} u_{1k,i} u_{1k,j} - u_{2k,i} u_{1k,j} \right) \mathrm{d}v . \tag{9}$$

这里是第二弹性体的应力分量在第一弹性体相应应变分量上所做的总功.

Novoshilov(诺沃日洛夫)在文献[20]中指出,线弹性的应力-应变关系仍适用于大挠度和大转动的某些情况,故有

$$\bar{W}_{12} \equiv \iiint_V \boldsymbol{\sigma}_{1ij} e_{2ij} \mathrm{d}v = \iiint_V (2Ge_{1ij} + \lambda e_{1ll} \delta_{ij}) e_{2ij} \mathrm{d}v = \iiint_V (2Ge_{1ij} e_{2ij} + \lambda e_{ll} e_{2kk}) \mathrm{d}v = \iiint_V (2Ge_{2ij} + \lambda e_{2kk} \delta_{ij}) e_{1ij} \mathrm{d}v = \iiint_V \boldsymbol{\sigma}_{2ij} e_{1ij} \mathrm{d}v = \bar{W}_{21} . \tag{10}$$

将式(8)和式(9)代入式(10),最后得到

$$\iiint_V F_{1i} u_{2i} \mathrm{d}v + \iint_{S_{1p}} \bar{p}_{1i} u_{2i} \mathrm{d}s + \iint_{S_{1u}} p_{1i} u_{2i} \mathrm{d}s + \iiint_V \boldsymbol{\sigma}_{1ij} \left(\frac{1}{2} u_{2k,i} u_{2k,j} - u_{1k,i} u_{2k,j} \right) \mathrm{d}v = \iiint_V F_{2i} u_{1i} \mathrm{d}v + \iint_{S_{2p}} \bar{p}_{2i} u_{1i} \mathrm{d}s + \iint_{S_{2u}} p_{2i} u_{1i} \mathrm{d}s + \iiint_V \boldsymbol{\sigma}_{2ij} \left(\frac{1}{2} u_{1k,i} u_{1k,j} - u_{2k,i} u_{1k,j} \right) \mathrm{d}v . \tag{11}$$

式(11)被称为有限位移理论三维线弹性力学的功的互等定理.在导出式(11)的过程中看出,已经应用了两个弹性体的平衡方程、静力边界条件、位移边界条件、应变-位移关系和线弹性应力-应变关系 5 大类方程.易于知道,式(11)两边的对应项对于注脚“1”和“2”具有倒易性.

对于具有小位移的两个线性弹性体,式(11)简化为

$$\iiint_V F_{1i} u_{2i} \mathrm{d}v + \iint_{S_{1p}} \bar{p}_{1i} u_{2i} \mathrm{d}s + \iint_{S_{1u}} p_{1i} u_{2i} \mathrm{d}s = \iiint_V F_{2i} u_{1i} \mathrm{d}v + \iint_{S_{2p}} \bar{p}_{2i} u_{1i} \mathrm{d}s + \iint_{S_{2u}} p_{2i} u_{1i} \mathrm{d}s . \tag{12}$$

3 大挠度弯曲矩形板的功的互等定理

考虑没有角点挠度的两个大挠度矩形板.假设该两个矩形板具有和第 2 节相同的要求和约定.

由第一矩形板的弯矩、扭矩和中面力在第二矩形板相应的曲率、扭率和中面应变上所做的总功等于

$$\bar{W}_{12} = \int_0^a \int_0^b \left[-D \left(\frac{\partial^2 w_1}{\partial x^2} + \nu \frac{\partial^2 w_1}{\partial y^2} \right) \left(-\frac{\partial^2 w_2}{\partial x^2} \right) - D \left(\frac{\partial^2 w_1}{\partial y^2} + \nu \frac{\partial^2 w_1}{\partial x^2} \right) \left(-\frac{\partial^2 w_2}{\partial y^2} \right) + \right.$$

$$2D(1-\nu)\frac{\partial^2 w_1}{\partial x\partial y}\frac{\partial^2 w_2}{\partial x\partial y}\Bigg]\mathrm{d}x\mathrm{d}y+$$

$$\int_0^a\int_0^b\Bigg(C\Bigg\{\Bigg[\frac{\partial u_1}{\partial x}+\frac{1}{2}\Big(\frac{\partial w_1}{\partial x}\Big)^2\Bigg]+\nu\Bigg[\frac{\partial v_1}{\partial y}+\frac{1}{2}\Big(\frac{\partial w_1}{\partial y}\Big)^2\Bigg]\Bigg\}\times$$

$$\Bigg[\frac{\partial u_2}{\partial x}+\frac{1}{2}\Big(\frac{\partial w_2}{\partial x}\Big)^2\Bigg]+C\Bigg\{\Bigg[\frac{\partial v_1}{\partial y}+\frac{1}{2}\Big(\frac{\partial w_1}{\partial y}\Big)^2\Bigg]+\nu\Bigg[\frac{\partial u_1}{\partial x}+\frac{1}{2}\Big(\frac{\partial w_1}{\partial x}\Big)^2\Bigg]\Bigg\}\times$$

$$\Bigg[\frac{\partial v_2}{\partial y}+\frac{1}{2}\Big(\frac{\partial w_2}{\partial y}\Big)^2\Bigg]+\frac{1}{2}C(1-\nu)\Big(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}+\frac{\partial w_1}{\partial x}\frac{\partial w_1}{\partial y}\Big)\times$$

$$\Big(\frac{\partial u_2}{\partial y}+\frac{\partial v_2}{\partial x}+\frac{\partial w_2}{\partial x}\frac{\partial w_2}{\partial y}\Big)\Bigg)\mathrm{d}x\mathrm{d}y,\qquad(13)$$

这里 $D=Eh^3/[12(1-\nu^2)]$ 和 $C=Eh/(1-\nu^2)$.

利用第一矩形板的平衡方程和静力边界条件,利用第二矩形板的应变-位移关系和位移边界条件并利用 Green 公式,能把式(13)转换成

$$\bar{W}_{12}=\int_0^a\int_0^b\Bigg[q_1+\frac{\partial}{\partial x}\Big(N_{1x}\frac{\partial w_1}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(N_{1y}\frac{\partial w_1}{\partial y}\Big)+\frac{\partial}{\partial x}\Big(N_{1xy}\frac{\partial w_1}{\partial y}\Big)+\frac{\partial}{\partial y}\Big(N_{1xy}\frac{\partial w_1}{\partial x}\Big)\Bigg]\times$$

$$w_2\mathrm{d}x\mathrm{d}y+\int_0^a\int_0^b\Bigg[\frac{1}{2}N_{1x}\Big(\frac{\partial w_2}{\partial x}\Big)^2+\frac{1}{2}N_{1y}\Big(\frac{\partial w_2}{\partial y}\Big)^2+N_{1xy}\Big(\frac{\partial w_2}{\partial x}\Big)\Big(\frac{\partial w_2}{\partial y}\Big)\Bigg]\mathrm{d}x\mathrm{d}y-$$

$$\Big(\int_0^b M_{1x}\frac{\partial w_2}{\partial x}\mathrm{d}y\Big)_0^a+\Bigg[\int_0^b\Big(Q_{1x}+\frac{\partial M_{1xy}}{\partial y}\Big)w_2\mathrm{d}y\Bigg]_0^a-\Big(\int_0^a M_{1y}\frac{\partial w_2}{\partial y}\mathrm{d}x\Big)_0^b+$$

$$\Bigg[\int_0^a\Big(Q_{1y}+\frac{\partial M_{1xy}}{\partial x}\Big)w_2\mathrm{d}x\Bigg]_0^b+\Big[\int_0^b(N_{1x}u_2+N_{1xy}v_2)\mathrm{d}y\Big]_0^a+$$

$$\Big[\int_0^a(N_{1y}v_2+N_{1xy}u_2)\mathrm{d}x\Big]_0^b.\qquad(14)$$

应用相同的方法,我们能够得到第二矩形板的弯矩、扭矩和中面力在第一矩形板相应的曲率、扭率和中面应变上所做的总功等于

$$\bar{W}_{21}=\int_0^a\int_0^b\Bigg[-D\Big(\frac{\partial^2 w_2}{\partial x^2}+\nu\frac{\partial^2 w_2}{\partial y^2}\Big)\Big(-\frac{\partial^2 w_1}{\partial x^2}\Big)-D\Big(\frac{\partial^2 w_2}{\partial y^2}+\nu\frac{\partial^2 w_2}{\partial x^2}\Big)\Big(-\frac{\partial^2 w_1}{\partial y^2}\Big)+$$

$$2D(1-\nu)\frac{\partial^2 w_2}{\partial x\partial y}\frac{\partial^2 w_1}{\partial x\partial y}\Bigg]\mathrm{d}x\mathrm{d}y+$$

$$\int_0^a\int_0^b\Bigg(C\Bigg\{\Bigg[\frac{\partial u_2}{\partial x}+\frac{1}{2}\Big(\frac{\partial w_2}{\partial x}\Big)^2\Bigg]+\nu\Bigg[\frac{\partial v_2}{\partial y}+\frac{1}{2}\Big(\frac{\partial w_2}{\partial y}\Big)^2\Bigg]\Bigg\}\times$$

$$\Bigg[\frac{\partial u_1}{\partial x}+\frac{1}{2}\Big(\frac{\partial w_1}{\partial x}\Big)^2\Bigg]+C\Bigg\{\Bigg[\frac{\partial v_2}{\partial y}+\frac{1}{2}\Big(\frac{\partial w_2}{\partial y}\Big)^2\Bigg]+\nu\Bigg[\frac{\partial u_2}{\partial x}+\frac{1}{2}\Big(\frac{\partial w_2}{\partial x}\Big)^2\Bigg]\Bigg\}\times$$

$$\Bigg[\frac{\partial v_1}{\partial y}+\frac{1}{2}\Big(\frac{\partial w_1}{\partial y}\Big)^2\Bigg]+$$

$$\frac{1}{2}C(1-\nu)\Big(\frac{\partial u_2}{\partial y}+\frac{\partial v_2}{\partial x}+\frac{\partial w_2}{\partial x}\frac{\partial w_2}{\partial y}\Big)\Big(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}+\frac{\partial w_1}{\partial x}\frac{\partial w_1}{\partial y}\Big)\Bigg)\mathrm{d}x\mathrm{d}y=$$

$$\int_0^a\int_0^b\Bigg[q_2+\frac{\partial}{\partial x}\Big(N_{2x}\frac{\partial w_2}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(N_{2y}\frac{\partial w_2}{\partial y}\Big)+\frac{\partial}{\partial x}\Big(N_{2xy}\frac{\partial w_2}{\partial y}\Big)+\frac{\partial}{\partial y}\Big(N_{2xy}\frac{\partial w_2}{\partial x}\Big)\Bigg]\times$$

$$w_1\mathrm{d}x\mathrm{d}y+\int_0^a\int_0^b\left[\frac{1}{2}N_{2x}\left(\frac{\partial w_1}{\partial x}\right)^2+\frac{1}{2}N_{2y}\left(\frac{\partial w_1}{\partial y}\right)^2+N_{2xy}\left(\frac{\partial w_1}{\partial x}\right)\left(\frac{\partial w_1}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y-$$
$$\left(\int_0^b M_{2x}\frac{\partial w_1}{\partial x}\mathrm{d}y\right)_0^a+\left[\int_0^b\left(Q_{2x}+\frac{\partial M_{2xy}}{\partial y}\right)w_1\mathrm{d}y\right]_0^a-\left(\int_0^a M_{2y}\frac{\partial w_1}{\partial y}\mathrm{d}x\right)_0^b+$$
$$\left[\int_0^a\left(Q_{2y}+\frac{\partial M_{2xy}}{\partial x}\right)w_1\mathrm{d}x\right]_0^b+$$
$$\left[\int_0^b(N_{2x}u_1+N_{2xy}v_1)\mathrm{d}y\right]_0^a+\left[\int_0^a(N_{2y}v_1+N_{2xy}u_1)\mathrm{d}x\right]_0^b. \qquad (15)$$

另一方面,易于确信,式(14)和式(15)的右端项彼此相等且它们都等于

$$\bar{W}_{12}=\bar{W}_{21}=\int_0^a\int_0^b\left(D\left[\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 w_2}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}\frac{\partial^2 w_2}{\partial y^2}-\nu\left(\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 w_2}{\partial y^2}+\frac{\partial^2 w_1}{\partial y^2}\frac{\partial^2 w_2}{\partial x^2}\right)\right]+\right.$$
$$2D(1-\nu)\frac{\partial^2 w_1}{\partial x\partial y}\frac{\partial^2 w_2}{\partial x\partial y}+$$
$$C\left\{\left[\frac{\partial u_1}{\partial x}+\frac{1}{2}\left(\frac{\partial w_1}{\partial x}\right)^2\right]\left[\frac{\partial u_2}{\partial x}+\frac{1}{2}\left(\frac{\partial w_2}{\partial x}\right)^2\right]+\right.$$
$$\left.\left[\frac{\partial v_1}{\partial y}+\frac{1}{2}\left(\frac{\partial w_1}{\partial y}\right)^2\right]\left[\frac{\partial v_2}{\partial y}+\frac{1}{2}\left(\frac{\partial w_2}{\partial y}\right)^2\right]\right\}+$$
$$C\nu\left\{\left[\frac{\partial u_1}{\partial x}+\frac{1}{2}\left(\frac{\partial w_1}{\partial x}\right)^2\right]\left[\frac{\partial v_2}{\partial y}+\frac{1}{2}\left(\frac{\partial w_2}{\partial y}\right)^2\right]+\right.$$
$$\left.\left[\frac{\partial v_1}{\partial y}+\frac{1}{2}\left(\frac{\partial w_1}{\partial y}\right)^2\right]\left[\frac{\partial u_2}{\partial x}+\frac{1}{2}\left(\frac{\partial w_2}{\partial x}\right)^2\right]\right\}+$$
$$\frac{1}{2}C(1-\nu)\left[\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}+\left(\frac{\partial w_1}{\partial x}\right)\left(\frac{\partial w_1}{\partial y}\right)\right]\times$$
$$\left.\left[\frac{\partial u_2}{\partial y}+\frac{\partial v_2}{\partial x}+\left(\frac{\partial w_2}{\partial x}\right)\left(\frac{\partial w_2}{\partial y}\right)\right]\right)\mathrm{d}x\mathrm{d}y. \qquad (16)$$

于是,我们最后得到

$$\int_0^a\int_0^b\left[q_1+\frac{\partial}{\partial x}\left(N_{1x}\frac{\partial w_1}{\partial x}\right)+\frac{\partial}{\partial y}\left(N_{1y}\frac{\partial w_1}{\partial y}\right)+\frac{\partial}{\partial x}\left(N_{1xy}\frac{\partial w_1}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{1xy}\frac{\partial w_1}{\partial x}\right)\right]w_2\mathrm{d}x\mathrm{d}y+$$
$$\int_0^a\int_0^b\left[\frac{1}{2}N_{1x}\left(\frac{\partial w_2}{\partial x}\right)^2+\frac{1}{2}N_{1y}\left(\frac{\partial w_2}{\partial y}\right)^2+N_{1xy}\left(\frac{\partial w_2}{\partial x}\right)\left(\frac{\partial w_2}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y-$$
$$\left(\int_0^b M_{1x}\frac{\partial w_2}{\partial x}\mathrm{d}y\right)_0^a+\left[\int_0^b\left(Q_{1x}+\frac{\partial M_{1xy}}{\partial y}\right)w_2\mathrm{d}y\right]_0^a-$$
$$\left(\int_0^a M_{1y}\frac{\partial w_2}{\partial y}\mathrm{d}x\right)_0^b+\left[\int_0^a\left(Q_{1y}+\frac{\partial M_{1xy}}{\partial x}\right)w_2\mathrm{d}x\right]_0^b+$$
$$\left[\int_0^b(N_{1x}u_2+N_{1xy}v_2)\mathrm{d}y\right]_0^a+\left[\int_0^a(N_{1y}v_2+N_{1xy}u_2)\mathrm{d}x\right]_0^b=$$
$$\int_0^a\int_0^b\left[q_2+\frac{\partial}{\partial x}\left(N_{2x}\frac{\partial w_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(N_{2y}\frac{\partial w_2}{\partial y}\right)+\right.$$
$$\left.\frac{\partial}{\partial x}\left(N_{2xy}\frac{\partial w_2}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{2xy}\frac{\partial w_2}{\partial x}\right)\right]w_1\mathrm{d}x\mathrm{d}y+$$

$$\int_0^a\int_0^b\left[\frac{1}{2}N_{2x}\left(\frac{\partial w_1}{\partial x}\right)^2+\frac{1}{2}N_{2y}\left(\frac{\partial w_1}{\partial y}\right)^2+N_{2xy}\left(\frac{\partial w_1}{\partial x}\right)\left(\frac{\partial w_1}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y-$$
$$\left(\int_0^b M_{2x}\frac{\partial w_1}{\partial x}\mathrm{d}y\right)_0^a+\left[\int_0^b\left(Q_{2x}+\frac{\partial M_{2xy}}{\partial y}\right)w_1\mathrm{d}y\right]_0^a-$$
$$\left(\int_0^a M_{2y}\frac{\partial w_1}{\partial y}\mathrm{d}x\right)_0^b+\left[\int_0^a\left(Q_{2y}+\frac{\partial M_{2xy}}{\partial x}\right)w_1\mathrm{d}x\right]_0^b+$$
$$\left[\int_0^b(N_{2x}u_1+N_{2xy}v_1)\mathrm{d}y\right]_0^a+\left[\int_0^a(N_{2y}v_1+N_{2xy}u_1)\mathrm{d}x\right]_0^b. \tag{17}$$

这就是大挠度线弹性弯曲矩形板的功的互等定理.

像第 2 节中式(11)一样,式(17)两边的相应项对于脚标“1”和“2”具有倒易性.

对于两相关小挠度矩形板,式(17)可被简化为

$$\int_0^a\int_0^b q_1w_2\mathrm{d}x\mathrm{d}y-\left(\int_0^b M_{1x}\frac{\partial w_2}{\partial x}\mathrm{d}y\right)_0^a-\left(\int_0^a M_{1y}\frac{\partial w_2}{\partial y}\mathrm{d}x\right)_0^b+$$
$$\left[\int_0^b\left(Q_{1x}+\frac{\partial M_{1xy}}{\partial y}\right)w_2\mathrm{d}y\right]_0^a+\left[\int_0^a\left(Q_{1y}+\frac{\partial M_{1xy}}{\partial x}\right)w_2\mathrm{d}x\right]_0^b=$$
$$\int_0^a\int_0^b q_2w_1\mathrm{d}x\mathrm{d}y-\left(\int_0^b M_{2x}\frac{\partial w_1}{\partial x}\mathrm{d}y\right)_0^a-\left(\int_0^a M_{2y}\frac{\partial w_1}{\partial y}\mathrm{d}x\right)_0^b+$$
$$\left[\int_0^b\left(Q_{2x}+\frac{\partial M_{2xy}}{\partial y}\right)w_1\mathrm{d}y\right]_0^a+\left[\int_0^a\left(Q_{2y}+\frac{\partial M_{2xy}}{\partial x}\right)w_1\mathrm{d}x\right]_0^b. \tag{18}$$

对于两大挠度板条,式(17)可被进一步简化为

$$(N_1u_2)_0^l+\int_0^l\frac{1}{2}N_1\left(\frac{\mathrm{d}w_2}{\mathrm{d}x}\right)^2\mathrm{d}x-\left(M_1\frac{\mathrm{d}w_2}{\mathrm{d}x}\right)_0^l+$$
$$(Q_1w_2)_0^l+\int_0^l\left(q_1+N_1\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}\right)w_2\mathrm{d}x=$$
$$(N_2u_1)_0^l+\int_0^l\frac{1}{2}N_2\left(\frac{\mathrm{d}w_1}{\mathrm{d}x}\right)^2\mathrm{d}x-\left(M_2\frac{\mathrm{d}w_1}{\mathrm{d}x}\right)_0^l+$$
$$(Q_2w_1)_0^l+\int_0^l\left(q_2+N_2\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}\right)w_1\mathrm{d}x. \tag{19}$$

4 应 用

4.1 均载两端固定大挠度板条的弯曲

考虑大挠度板条功的互等定理的一个应用.假设,在一单位集中载荷作用下小挠度简支板条取为第一板条,即取为基本系统,如图 1 所示.大挠度实际系统考虑为第二板条,如图 2(a)所示.为简单计,实际系统力学量的注脚“2”被省略.在这种情况下式(19)简化成为

$$w(\xi)=\int_0^l\left(q+N\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)w_1(x,\xi)\mathrm{d}x+M_0\left(\frac{\mathrm{d}w_1}{\mathrm{d}x}\right)_{x=0}-M_0\left(\frac{\mathrm{d}w_1}{\mathrm{d}x}\right)_{x=1}. \tag{20}$$

现在,我们将应用式(20)去计算图 2(a)所示两固定端板条的弯曲.

解除图 2(a)板条两固定端的弯曲约束,并用两个弯矩 M_0 代替它们,于是得到一两端简支的板条,该板条在两端受弯矩 M_0 的作用和均匀分布的载荷 q 的作用,如图 2(b)所示.

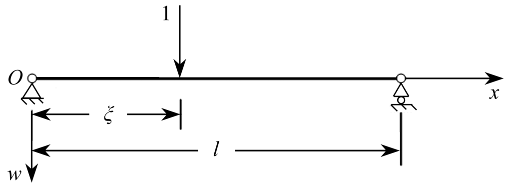
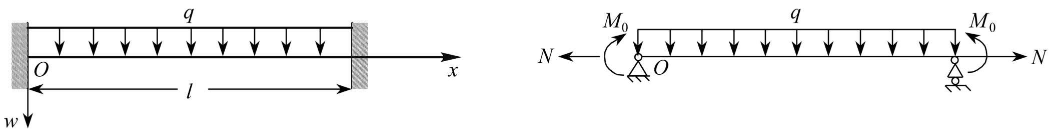


图 1　基本系统

Fig. 1　The basic system



（a）两端固定的板条　　（b）板条的实际系统

（a）The plate strip with 2 ends clamped　　（b）The actual system of the plate strip

图 2　板条系统

Fig. 2　The system of the plate strip

该板条的数值参数在表 1 中给出.

表 1　计算板条的参数

Table 1　The parameters of the plate strip calculated

parameter	length l /mm	thickness h /mm	load density q /MPa	Young's modulus E /MPa	Poisson's ratio ν
value	1 300	12	0.14	210 000	0.3

对于如图 1 所示基本系统,基本解被表示为

$$w_1(x,\xi)=\sum_{m=1,2}^{\infty}\frac{2l^3}{D(m\pi)^4}\sin\frac{m\pi x}{l}\sin\frac{m\pi\xi}{l}. \tag{21}$$

它的两端转角为

$$\left(\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)_{x=0}=\sum_{m=1,2}^{\infty}\frac{2l^2}{D(m\pi)^3}\sin\frac{m\pi\xi}{l}, \tag{22}$$

$$\left(\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)_{x=l}=\sum_{m=1,2}^{\infty}\frac{(-1)^m 2l^2}{D(m\pi)^3}\sin\frac{m\pi\xi}{l}. \tag{23}$$

假设图 2(b)所示板条的挠曲轴方程为

$$\begin{cases} w(\xi)=\displaystyle\sum_{m=1,3}^{\infty}A_m\sin\frac{m\pi\xi}{l}, \\ w(x)=\displaystyle\sum_{m=1,3}^{\infty}A_m\sin\frac{m\pi x}{l}. \end{cases} \tag{24}$$

将式(21)~(24)代到式(20)中,并进行计算,则得

$$w(\xi)=\sum_{m=1,3}^{\infty}\frac{1}{(m\pi)^2+Nl^2/D}\left[\frac{4ql^4}{D(m\pi)^3}+\frac{4M_0l^2}{D(m\pi)}\right]\sin\frac{m\pi\xi}{l}. \tag{25}$$

该问题的边界条件为

$$\left(\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)_{\xi=0}=0, \tag{26}$$

$$\frac{Nl(1-\nu^2)}{Eh}=\frac{1}{2}\int_0^l\left(\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)^2\mathrm{d}\xi. \tag{27}$$

将式(25)代入到边界条件(26)和(27),得到它们的执行方程为

$$\sum_{m=1,3}^{\infty}\frac{1}{(m\pi)^2+Nl^2/D}\left[\frac{4ql^3}{D(m\pi)^2}+\frac{4M_0l}{D}\right]=0, \tag{28}$$

$$\frac{Nl(1-\nu^2)}{Eh}=\frac{l}{4}\sum_{m=1,3}^{\infty}\frac{1}{[(m\pi)^2+Nl^2/D]^2}\left[\frac{4ql^3}{D(m\pi)^2}+\frac{4M_0l}{D}\right]^2. \tag{29}$$

引入轴向力因子

$$u^2=\frac{N}{D}\frac{l^2}{4} \tag{30}$$

和横向载荷因子

$$u_c=\lg(10^4\sqrt{U_0}),\ U_0=\frac{E^2h^8}{q^2l^8(1-\nu^2)^2}, \tag{31}$$

并且由式(28)和式(29)消去 M_0, 得到

$$U_0=\frac{1\,728}{u^2}\sum_{m=1,3}^{\infty}\frac{1}{[(m\pi)^2+4u^2]^2}\left[\frac{1}{(m\pi)^2}-\frac{\sum\limits_{m=1,3}^{\infty}\frac{1}{(m\pi)^2[(m\pi)^2+4u^2]}}{\sum\limits_{m=1,3}^{\infty}\frac{1}{(m\pi)^2+4u^2}}\right]^2. \tag{32}$$

由方程(32),利用迭代法可算出轴向力因子 u,尔后 $N,M_0,M(x)$ 和 $w(x)$ 都能被计算出来.

轴向力因子 u 和相应的横向载荷因子 u_c 在表2中给出.

表2 横向载荷因子和轴向力因子之间的关系

Table 2 The relation between the factor of transverse load and the factor of axial force

		u_c						
		0.7	0.8	0.9	1.0	1.1	1.2	1.3
u	ref. [24]	11.999 87	11.042 84	10.154 93	9.330 84	8.565 63	7.854 67	7.193 64
	the paper	11.949 19	10.995 36	10.110 37	9.288 92	8.526 10	7.817 28	7.158 17
		1.4	1.5	1.6	1.7	1.8	1.9	2.0
u	ref. [24]	6.578 51	6.005 49	5.471 04	4.971 86	4.504 86	4.067 25	3.656 46
	the paper	6.544 74	5.973 21	5.440 06	4.941 98	4.475 91	4.039 06	3.628 91
		2.1	2.2	2.3	2.4	2.5	2.6	2.7
u	ref. [24]	3.270 28	2.906 88	2.564 92	2.243 67	1.943 12	1.664 06	1.408 01
	the paper	3.243 26	2.880 33	2.538 85	2.218 17	1.918 37	1.640 38	1.385 76
		2.8	2.9	3.0	3.1	3.2	3.3	3.4
u	ref. [24]	1.176 84	0.972 22	0.794 96	0.644 57	0.519 26	0.416 36	0.332 78
	the paper	1.156 44	0.954 02	0.779 15	0.631 15	0.508 10	0.407 21	0.325 35
		3.5	3.6	3.7	3.8	3.9	4.0	
u	ref. [24]	0.265 39	0.211 35	0.168 16	0.133 71	0.106 28	0.084 46	
	the paper	0.259 41	0.206 56	0.164 33	0.136 60	0.103 85	0.825 20	

沿着 x 轴向的挠度值和弯矩值分别在表 3 和表 4 中给出.与表 2、表 3 和表 4 相应的曲线分别示于图 3、图 4 和图 5 中.

表 3 沿着 x 轴的挠度分布

Table 3 The distribution of deflection along axis x

		x /mm						
		0	50	100	150	200	250	300
$w(x)$ /mm	ref. [24]	0	0.344 936	1.346 893	2.770 873	4.431 831	6.186 869	7.926 222
	the paper	0	0.413 734	1.466 129	2.923 105	4.604 125	6.370 121	8.114 184
		350	400	450	500	550	600	650
$w(x)$ /mm	ref. [24]	9.566 073	11.042 650	12.307 580	13.324 730	14.068 140	14.520 610	14.672 460
	the paper	9.754 419	11.228 460	12.489 300	13.502 130	14.242 070	14.692 400	14.843 560
		700	750	800	850	900	950	1 000
$w(x)$ /mm	ref. [24]	14.520 610	14.068 140	13.324 730	12.307 580	11.042 650	9.566 073	7.926 222
	the paper	14.692 400	14.242 070	13.502 130	12.489 300	11.228 460	9.754 419	8.114 184
		1 050	1 100	1 150	1 200	1 250	1 300	
$w(x)$ /mm	ref. [24]	6.186 869	4.431 831	2.770 873	1.346 893	0.344 936	0	
	the paper	6.370 121	4.604 125	2.923 105	1.466 129	0.413 734	0	

表 4 沿着 x 轴的弯矩分布

Table 4 The distribution of bending moment along axis x

		x /mm						
		0	50	100	150	200	250	300
$M(x)$ / (MN·mm)	ref. [24]	−12 422.40	−8 413.31	−5 319.02	−2 932.54	−1 094.20	318.99	1 401.57
	the paper	−12 760.90	−8 188.74	−5 915.92	−3 429.90	−1 191.79	453.01	1 457.32
		350	400	450	500	550	600	650
$M(x)$ / (MN·mm)	ref. [24]	2 225.98	2 847.37	3 307.31	3 636.58	3 857.19	3 983.92	4 025.25
	the paper	2 093.13	2 675.75	3 285.89	3 768.48	3 974.32	3 957.81	3 916.27
		700	750	800	850	900	950	1 000
$M(x)$ / (MN·mm)	ref. [24]	3 983.92	3 857.19	3 636.58	3 307.31	2 847.37	2 225.98	1 401.57
	the paper	3 957.81	3 974.32	3 768.48	3 285.89	2 675.75	2 093.13	1 457.32
		1 050	1 100	1 150	1 200	1 250	1 300	
$M(x)$ / (MN·mm)	ref. [24]	318.99	−1 094.20	−2 932.54	−5 319.02	−8 413.31	−12 422.40	
	the paper	453.01	−1 191.79	−3 429.90	−5 915.92	−8 188.74	−12 760.90	

4.2 均载 4 边固定大挠度矩形板的弯曲

考虑一均载 4 边固定大挠度矩形板的弯曲,如图 6 所示,求其挠曲面方程.

为求此大挠度板的解,首先考虑一在一单位集中载荷作用下小挠度 4 边固定矩形板作为基本系统,如图 7 所示.

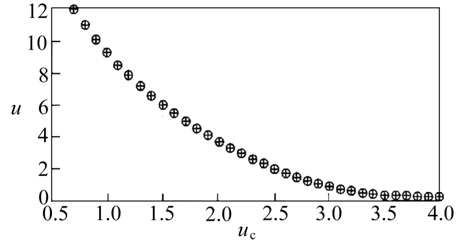


图 3 u 和 u_c 的关系

Fig. 3 The relation between u and u_c

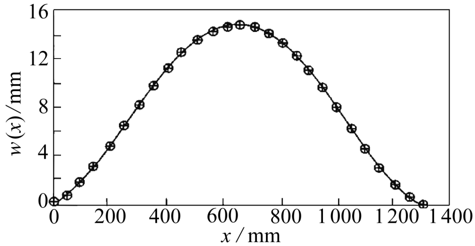


图 4 沿着 x 轴的挠度分布

Fig. 4 The distribution of deflection along axis x

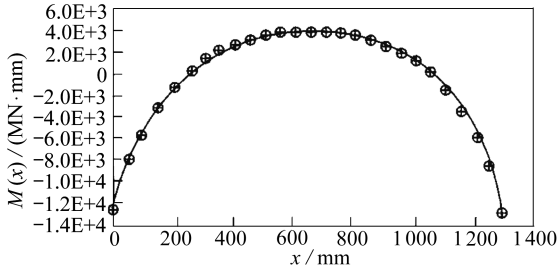


图 5 沿着 x 轴的弯矩分布

Fig. 5 The distribution of bending moment along axis x

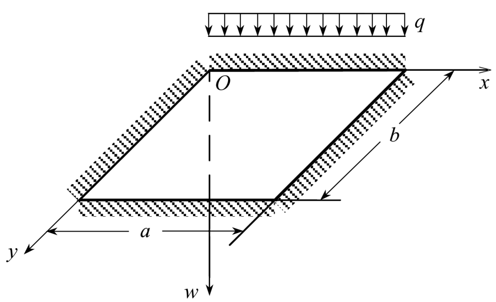


图 6 均载大挠度 4 边固定弯曲矩形板实际系统

Fig. 6 The actual system of bending rectangular plate with 4 edges fixed of large deflection under load uniformly distributed

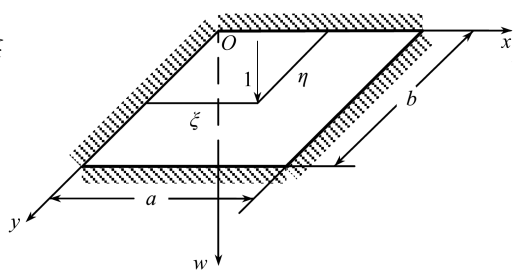


图 7 在一集中载荷作用下小挠度 4 边固定弯曲矩形板基本系统

Fig. 7 The basic system of bending rectangular plate with 4 edges fixed of small deflection under an unit concentrated load

假设该基本系统基本解的容许位移为

$$w_1(x,y;\xi,\eta) = A\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}. \tag{33}$$

该板的总势能为

$$\begin{aligned}\Pi_p = {} & \int_0^a\int_0^b \frac{D}{2}\left\{\left(\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 w_1}{\partial y^2}-\left(\frac{\partial^2 w_1}{\partial x\partial y}\right)^2\right]\right\}\mathrm{d}x\mathrm{d}y - \\ & A\sin^2\frac{\pi\xi}{a}\sin^2\frac{\pi\eta}{b} = \\ & \frac{(3b^4+3a^4+2a^2b^2)\pi^4 D}{8a^3b^3}A^2 - A\sin^2\frac{\pi\xi}{a}\sin^2\frac{\pi\eta}{b}.\end{aligned} \tag{34}$$

对总势能式(34)取 A 的变分极值,则得

$$\delta\Pi_p = \frac{(3b^4+3a^4+2a^2b^2)\pi^4 D}{4a^3b^3}A\delta A - \delta A\sin^2\frac{\pi\xi}{a}\sin^2\frac{\pi\eta}{b} = 0. \tag{35}$$

由变分法基本预备定理,则得

$$\frac{(3b^4+3a^4+2a^2b^2)\pi^4 D}{4a^3b^3}A - \sin^2\frac{\pi\xi}{a}\sin^2\frac{\pi\eta}{b} = 0. \tag{36}$$

解方程式(36),则得

$$A = \frac{4a^3b^3}{(3b^4+3a^4+2a^2b^2)\pi^4 D}\sin^2\frac{\pi\xi}{a}\sin^2\frac{\pi\eta}{b}. \tag{37}$$

设

$$C \equiv \frac{4a^3b^3}{(3b^4+3a^4+2a^2b^2)\pi^4 D}, \tag{38}$$

最后则得

$$w_1(x,y;\xi,\eta) = C\sin^2\frac{\pi\xi}{a}\sin^2\frac{\pi\eta}{b}\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}, \tag{39}$$

这就是小挠度 4 边固定矩形板基本系统的基本解.

在图 7 基本系统小挠度弯曲矩形板和图 6 实际系统大挠度弯曲矩形板之间应用功的互等定理式(17),则得

$$w(\xi,\eta) = \int_0^a\int_0^b\left(q + h\frac{\partial^2\Phi}{\partial^2 y}\frac{\partial^2 w}{\partial^2 x} + h\frac{\partial^2\Phi}{\partial^2 x}\frac{\partial^2 w}{\partial^2 y} - 2h\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)w_1(x,y;\xi,\eta)\,\mathrm{d}x\mathrm{d}y. \tag{40}$$

如进一步假设

$$\begin{cases} w(x,y) = f_1\sin^2\dfrac{\pi\xi}{a}\sin^2\dfrac{\pi\eta}{b}, \\ w(\xi,\eta) = f_1\sin^2\dfrac{\pi x}{a}\sin^2\dfrac{\pi y}{b}. \end{cases} \tag{41a,b}$$

将式(41b)代入大挠度板的协调方程式

$$\frac{1}{E}\nabla^4\varphi = \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{42}$$

中,则得

$$\begin{aligned} \frac{1}{E}\nabla^4\Phi &= f_1^2\frac{\pi^4}{a^2b^2}\left(\sin^2\frac{2\pi x}{a}\sin^2\frac{2\pi y}{b} - 4\cos\frac{2\pi x}{a}\sin^2\frac{\pi x}{a}\cos\frac{2\pi y}{b}\sin^2\frac{\pi y}{b}\right) = \\ &\frac{f_1^2\pi^4}{2a^2b^2}\left(\cos\frac{2\pi x}{a} - \cos\frac{4\pi x}{a} + \cos\frac{2\pi y}{b} - \cos\frac{4\pi y}{b} - 2\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b} + \right. \\ &\left.\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{b} + \cos\frac{4\pi x}{a}\cos\frac{2\pi y}{b}\right). \end{aligned} \tag{43}$$

设齐次方程 $\nabla^4\Phi/E = 0$ 的解为 Φ_1, 则有

$$\Phi_1 = \frac{1}{2}p_x y^2 + \frac{1}{2}p_y x^2. \tag{44}$$

设特解形式为

$$\Phi_2 = D_1\cos\frac{2\pi x}{a} + D_2\cos\frac{2\pi y}{b} + D_3\cos\frac{4\pi x}{a} + D_4\cos\frac{4\pi y}{b} + D_5\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b} + D_6\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{b} + D_7\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{b}. \tag{45}$$

将式(45)代入式(43)中,比较其两边同类项系数,则得

$$\begin{cases} D_1 = \dfrac{Ea^2}{32b^2}f_1^2,\ D_2 = \dfrac{Eb^2}{32a^2}f_1^2,\ D_3 = -\dfrac{Ea^2}{512b^2}f_1^2,\ D_4 = \dfrac{Eb^2}{512a^2}f_1^2, \\ D_5 = -\dfrac{Ea^2b^2}{16(a^2+b^2)^2}f_1^2,\ D_6 = \dfrac{Ea^2b^2}{32(b^2+4a^2)^2}f_1^2,\ D_7 = \dfrac{Ea^2b^2}{32(4b^2+a^2)^2}f_1^2. \end{cases} \tag{46}$$

由式(44)~(46)可得

$$\Phi = \Phi_1 + \Phi_2 = Ef_1^2\left\{\frac{1}{32}\left(\frac{a^2}{b^2}\cos\frac{2\pi x}{a} + \frac{b^2}{a^2}\cos\frac{2\pi y}{b}\right) - \frac{1}{512}\left(\frac{a^2}{b^2}\cos\frac{4\pi x}{a} - \frac{b^2}{a^2}\cos\frac{4\pi y}{b}\right) + \frac{a^2b^2}{32}\left[\frac{1}{(a^2+4b^2)^2}\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{b} + \frac{1}{(4a^2+b^2)^2}\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{b}\right] - \frac{a^2b^2}{16(a^2+b^2)^2}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\right\} + \frac{1}{2}p_xy^2 + \frac{1}{2}p_yx^2. \tag{47}$$

并进而可求出

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} = p_x + Ef_1^2\pi^2\left\{-\frac{1}{8a^2}\cos\frac{2\pi y}{b} + \frac{1}{32a^2}\cos\frac{4\pi y}{b} - \frac{a^2}{8}\left[\frac{1}{(a^2+4b^2)^2}\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{b} + \frac{4}{(4a^2+b^2)^2}\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{b}\right] + \frac{a^2}{4(a^2+b^2)^2}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\right\}, \tag{48}$$

$$\sigma_y = \frac{\partial^2\Phi}{\partial x^2} = p_y + Ef_1^2\pi^2\left\{-\frac{1}{8b^2}\cos\frac{2\pi x}{a} + \frac{1}{32b^2}\cos\frac{4\pi x}{a} - \frac{b^2}{8}\left[\frac{4}{(a^2+4b^2)^2}\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{b} + \frac{1}{(4a^2+b^2)^2}\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{b}\right] + \frac{b^2}{4(a^2+b^2)^2}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\right\}, \tag{49}$$

$$-\tau_{xy} = \frac{\partial^2\Phi}{\partial x\partial y} = \frac{Eab\pi^2}{4}f_1^2\left[-\frac{1}{(a^2+b^2)^2}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b} + \frac{1}{(a^2+4b^2)^2}\sin\frac{4\pi x}{a}\sin\frac{2\pi y}{b} + \frac{1}{(4a^2+b^2)^2}\sin\frac{2\pi x}{a}\sin\frac{4\pi y}{b}\right]. \tag{50}$$

由挠度 w 引起 x 方向的伸长为

$$\lambda_1 = \frac{1}{2}\int_0^a\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x = -\frac{\pi^2}{32a}f_1^2\left(4\cos\frac{2\pi y}{b} - \cos\frac{4\pi y}{b} - 3\right). \tag{51}$$

另一方面,由于中面应力所引起的在 x 方向的伸长量为

$$\lambda_2 = \int_0^a\varepsilon_x\mathrm{d}x = \frac{1}{E}\int_0^a(\sigma_x - \nu\sigma_y)\mathrm{d}x =$$

$$
-\frac{\pi^2}{32a}f_1^2\left(4\cos\frac{2\pi y}{b}-\cos\frac{4\pi y}{b}\right)+\frac{a}{E}(p_x-\nu p_y). \tag{52}
$$

令 λ_1 和 λ_2 相等,则得 x 方向的边界条件为

$$
p_x-\nu p_y=\frac{3E\pi^2}{32a^2}f_1^2, \tag{53}
$$

同理,可得 y 方向的边界条件为

$$
p_y-\nu p_x=\frac{3E\pi^2}{32b^2}f_1^2. \tag{54}
$$

由式(53)和式(54)可解得

$$
\begin{cases}
p_x=\dfrac{3E\pi^2f_1^2}{32(1-\nu^2)}\left(\dfrac{1}{a^2}+\dfrac{\nu}{b^2}\right),\\
p_y=\dfrac{3E\pi^2f_1^2}{32(1-\nu^2)}\left(\dfrac{\nu}{a^2}+\dfrac{1}{b^2}\right).
\end{cases} \tag{55}
$$

表 5 均载大挠度 4 边固定矩形板在 $x=125$ mm 中线上的挠度分布

Table 5 The deflection distribution at $x=125$ mm middle line of the rectangular plate of large deflection with 4 edges fixed under load uniformly distributed

		y/mm					
		0	50	55	60	65	70
w(y)/mm	ANSYS	0	5.251 4	5.662 4	6.127 7	6.547 9	6.823 6
	the paper	0	4.132 2	4.778 6	5.409 1	6.008 1	6.561 0
		75	80	85	90	95	100
w(y)/mm	ANSYS	7.155 5	7.344 2	7.490 2	7.594 2	7.656 4	7.774 6
	the paper	7.054 1	7.475 2	7.813 9	8.062 0	8.213 4	8.264 2
		105	110	115	120	125	130
w(y)/mm	ANSYS	7.656 4	7.594 2	7.490 2	7.344 2	7.155 5	6.823 6
	the paper	8.213 4	8.062 0	7.813 9	7.475 2	7.054 1	6.561 0
		135	140	145	150	200	
w(y)/mm	ANSYS	6.547 9	6.127 7	5.662 4	5.251 4	0	
	the paper	6.008 1	5.409 1	4.778 6	4.132 2	0	

将式(39)、(41)、(48)~(50)和式(55)代入式(40)中,并比较两边同类项的系数,则得

$$
f_1=\frac{abqC}{4}-\frac{Eh\pi^4C}{32}\left[\frac{17}{32}\left(\frac{b}{a^3}+\frac{a}{b^3}\right)+\frac{ab}{(a^2+b^2)^2}+\frac{ab}{4(a^2+4b^2)^2}+\frac{ab}{4(4a^2+b^2)^2}+\frac{9}{16(1-\nu^2)}\left(\frac{b}{a^3}+\frac{a}{b^3}+\frac{2\nu}{ab}\right)\right]f_1^3. \tag{56}
$$

用迭代法求解式(56),然后将 f_1 代入式(41)即得挠曲面方程.

均载大挠度 4 边固定矩形板的原始数据参数列于表 1;本文算得的 $x=a/2=125$ mm 剖面上挠度的分布结果与用 ANSYS 程序的计算结果比较,见表 5 和图 8.

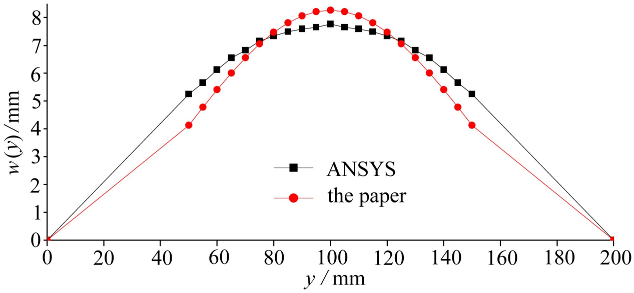


图 8 均载大挠度四边固定矩形板在 $x = 125$ mm 中线上的挠度分布曲线

Fig. 8 The curve of deflection distribution at $x = 125$ mm middle line of the rectangular plate of large deflection with 4 edges fixed under load uniformly distributed

5 结 论

1）本文首次提出有限位移理论三维线弹性力学功的互等定理的一般公式.

2）基于该一般公式,导出了大挠度矩形板的功的互等定理.

3）简化大挠度矩形板的功的互等定理,得到了大挠度板条的功的互等定理.

4）作为应用,计算了在均载作用下两端固定大挠度板条的弯曲和在均载作用下 4 边固定大挠度矩形板的弯曲.计算过程表明,计算是简单和有效的.

5）小位移理论的修正的功的互等定理和有限位移理论的功的互等定理一起构成了一完整的功的互等理论体系.

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The Reciprocal Theorem for the Finite Displacement Theory and Its Applications

FU Bao-lian

(*College of Architectural Engineering and Mechanics*, *Yanshan University*, *Qinhuangdao*, *Hebei* 066004, *P.R.China*)

Abstract: The reciprocal theorem of 3D linear elasticity for the finite displacement theory was proposed. On the basis of the theorem, the reciprocal theorem of rectangular plates of large defection was derived. Meanwhile, the reciprocal theorem of plate strips of large deflection was directly obtained through simplification of the theorem of the rectangular plates. For applications, the bending of a plate strip of large deflection with 2 ends fixed under uniformly distributed load and the bending of a rectangular plate of large deflection with 4 edges fixed under uniformly distributed load were calculated. The calculation shows, on the basis of the reciprocal theorem of bending thin plates of large deflection, the bending rectangular plates of large deflection can easily be sovled with the aid of the basic solution corresponding to the small-deflection case.

Key words: linear elasticity; finite displacement theory; large deflection; reciprocal theorem