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在分子晶体中的一类广义非线性 Schrödinger 方程组的初边值问题*

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摘要: 研究了在分子晶体中的一类广义非线性 Schrödinger 方程组的初边值问题. 应用先验估计的方法, 得到了整体解的存在性.

关 键 词: 非线性 Schrödinger 方程组; 初边值问题; 先验估计; 整体解
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引 言

在一维晶体[1-2]和 α- 螺旋生物分子[3]中产生的一类广义非线性 Schrödinger 方程组具有很强的物理意义, 在文献[1, 4] 中指出这类方程的经典形式为

$$i\varphi_t = \varphi - \alpha\varphi_x - \varphi_{xx} - \varphi|\varphi|^2.$$

它和通常的非线性 Schrödinger 方程不同在于多了一项: $\alpha\varphi_x$. 这一项在此类问题中是很本质的项, 不能忽略. 从数学上来说, 由于多了一项, Schrödinger 算子为非自共轭, 必须加以细心处理.

本文考虑如下一类广义非线性 Schrödinger 方程组的初边值问题:

$$iu_{jt} - [a(x)u_{jx}]_x + \alpha_j(x)u_{jx} + \beta(x)q(|u_1|^2 + |u_2|^2)u_j + h_{j1}(x,t)u_1 + h_{j2}(x,t)u_2 = 0 \quad (j = 1, 2), \tag{1}$$

$$u_1(x,0) = u_{01}(x),\ u_2(x,0) = u_{02}(x), \qquad x \in [0, 1], \tag{2}$$

$$u(0,t) = u(1,t) = 0, \tag{3}$$

其中 $a(x)$、$\alpha_j(x)$、$\beta(x)$、$h_{ij}(x,t)$ 为已知实值函数, $u(x,t)$ 为未知复值函数, $q(s) > 0$. 我们将证明上述问题整体解的存在性.

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1 先验估计

我们先对问题(1)至(3)的解做先验估计.

引理1 假设下列条件成立

1) $\alpha_j'(x), h_{ij}(x,t)$ 为有界函数, $i,j=1,2$;

2) $u_{0j}(x)\in L_2 \quad (j=1,2)$.

则有

$$\sum_{j=1}^{2}\|u_j\|^2\leqslant E_0. \tag{4}$$

其中 E_0 是与 u 无关的常数.

证明 令问题(1)与 u_1、u_2 做内积可得

$$(\mathrm{i}u_{jt},u_j)-((a(x)u_{jx})_x,u_j)+(\alpha_j u_{jx},u_j)+(\beta(x)q(|u_1|^2+|u_2|^2)u_j,u_j)+ \\ (h_{j1}u_1+h_{j2}u_2,u_j)=0,\qquad j=1,2, \tag{5}$$

其中

$$\mathrm{Re}(u_{jt},u_j)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_j\|^2,\ -((a(x)u_{jx})_x,u_j)=\int a(x)\left|\frac{\partial u_j}{\partial x}\right|^2\mathrm{d}x,$$

$$(\alpha_j u_{jx},u_j)=\int\alpha_j(x)u_{jx}u_j\mathrm{d}x=\frac{1}{2}\int\alpha_j(x)\frac{\partial}{\partial x}(u_j)^2\mathrm{d}x=-\frac{1}{2}\int\alpha_j'(x)(u_j)^2\mathrm{d}x,$$

$$(\beta(x)q(|u_1|^2+|u_2|^2)u_j,u_j)=\int\beta(x)q(|u_1|^2+|u_2|^2)|u_j|^2\mathrm{d}x,$$

$$|(h_{j1}u_1+h_{j2}u_2,u_j)|\leqslant M\left|\sum_{j=1}^{2}\|u_j\|^2\right|,$$

于是对式(5)取虚部可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{2}\|u_j\|^2\leqslant M_1\left|\sum_{j=1}^{2}\|u_j\|^2\right|.$$

由 Gronwall 不等式即得式(4). 证毕. □

引理2(Sobolev 不等式)[5] 令 $u\in L^q(\Omega)$, $D^m u\in L^r(\Omega)$, $q,r>1$, 有

$$\|D^j u\|_{L^p(\Omega)}\leqslant C\|u\|^{\alpha}_{W^{m,r}(\Omega)}\|u\|^{1-\alpha}_{L^q(\Omega)}. \tag{6}$$

其中 $\dfrac{1}{p}=\dfrac{j}{n}+\alpha\left(\dfrac{1}{r}-\dfrac{m}{n}\right)+(1-\alpha)\dfrac{1}{q}$, $\dfrac{j}{m}\leqslant\alpha<1$, $0\leqslant j<m$, $C>0$.

引理3 在引理1的条件下, 同时满足

1) $|q(s)|\geqslant 0$;

2) $a(x)\geqslant a_0>0$; $h_{ij}(x,t)$ 为实值函数, 且 $h_{ij}=h_{ji}$, $i,j=1,2$;

3) h_{ijt}、$\alpha_j(x)$、$a(x)$、$a'(x)$ 均为有界函数, $i,j=1,2$;

4) $u_{0j}(x)\in H^1$, $j=1,2$.

则有

$$\sum_{j=1}^{2}\|u_{jx}\|^2\leqslant E_1, \tag{7}$$

其中 E_1 为常数.

证明 问题(1)与 u_{1t}、u_{2t} 做内积可得

$$(\mathrm{i}u_{jt},u_{jt})-((a(x)u_{jx})_x,u_{jt})+(\alpha_j(x)u_{jx},u_{jt})+(\beta(x)q(|u_1|^2+$$

$$|u_2|^2)u_j, u_{jt}) + \left(\sum_{l=1}^{2} h_{jl}u_l, u_{jt}\right) = 0, \qquad j = 1, 2, \tag{8}$$

其中

$$-\mathrm{Re}((a(x)u_{jx})_x, u_{jt}) = \mathrm{Re}\int a(x)u_{jx}\bar{u}_{jxt}\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int a(x)|u_{jx}|^2\mathrm{d}x,$$

$$(\alpha_j u_{jx}, u_{jt}) = \int \alpha_j(x)u_{jx}\bar{u}_{jt}\mathrm{d}x =$$

$$\mathrm{i}\int \alpha_j(x)u_{jx}\cdot[(a(x)\bar{u}_{jx})_x - \alpha_j(x)\bar{u}_{jx} -$$

$$\beta(x)q(|u_1|^2+|u_2|^2)\bar{u}_j - \sum_{l=1}^{2} h_{jl}\bar{u}_l]\,\mathrm{d}x =$$

$$\mathrm{i}\int \alpha_j(x)a(x)u_{jx}\bar{u}_{jxx}\mathrm{d}x - \mathrm{i}\int \alpha_j(x)a'(x)(u_{jx})^2\mathrm{d}x -$$

$$\mathrm{i}\int |\alpha_j(x)|^2|u_{jx}|^2\mathrm{d}x - \mathrm{i}\int \alpha_j(x)\beta(x)q(|u_1|^2+|u_2|^2)\bar{u}_j u_{jx}\mathrm{d}x -$$

$$\mathrm{i}\int \alpha_j(x)u_{jx}\left(\sum_{l=1}^{2} h_{jl}\bar{u}_l\right)\mathrm{d}x,$$

$$-\mathrm{i}\int \alpha_j(x)a(x)u_{jx}\bar{u}_{jxx}\mathrm{d}x = -\frac{\mathrm{i}}{2}\int\left(\alpha_j(x)a(x)\frac{\partial}{\partial x}(u_{jx})^2\right)\mathrm{d}x =$$

$$-\frac{\mathrm{i}}{2}\int\left(\alpha_j(x)a(x)\right)_x(u_{jx})^2\mathrm{d}x.$$

故由引理 2 可得

$$\left|\mathrm{i}\int \alpha_j(x)a(x)u_{jx}\bar{u}_{jxx}\mathrm{d}x\right| \leqslant M_1\|u_{jx}\|^2,$$

$$\left|-\mathrm{i}\int \alpha_j(x)\beta(x)q(|u_1|^2+|u_2|^2)\bar{u}_j u_{jx}\mathrm{d}x\right| \leqslant$$

$$M_2\left(\int|q(|u_1|^2+|u_2|^2)u_j|^2\mathrm{d}x + \int|u_{jx}|^2\mathrm{d}x\right) \leqslant$$

$$M_3\left(\sum_{l=1}^{2}\|u_{jx}\|^2\right),$$

$$\left|-\mathrm{i}\int \alpha_j(x)u_{jx}\left(\sum_{l=1}^{2} h_{jl}\bar{u}_l\right)\mathrm{d}x\right| \leqslant$$

$$M_4\left(\sum_{j=1}^{2}\|u_{jx}\|^2 + \sum_{j=1}^{2}\|u_j\|^2\right) \sum_{j=1}^{2}\mathrm{Re}(\beta(x)q(|u_1|^2+|u_2|^2)u_j, u_{jt}) =$$

$$\frac{1}{2}\sum_{j=1}^{2}\int\beta(x)q(|u_1|^2+|u_2|^2)\frac{\partial}{\partial t}|u_j|^2\mathrm{d}x =$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\beta(x)\alpha(|u_1|^2+|u_2|^2)\mathrm{d}x,$$

$$\mathrm{Re}\left[\int h_{11}u_1\bar{u}_{1t}\mathrm{d}x + \int h_{12}u_2\bar{u}_{1t}\mathrm{d}x + \int(h_{21}u_1 + h_{22}u_2)\bar{u}_{2t}\mathrm{d}x\right] =$$

$$\frac{1}{2}\int h_{11}\frac{\partial}{\partial t}|u_1|^2\mathrm{d}x + \frac{1}{2}\int h_{22}\frac{\partial}{\partial t}|u_2|^2\mathrm{d}x + \mathrm{Re}\int h_{12}\frac{\partial}{\partial t}(u_2\bar{u}_1)\mathrm{d}x =$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int h_{11}|u_1|^2\mathrm{d}x - \frac{1}{2}\int h_{11t}|u_1|^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int h_{22}|u_2|^2\mathrm{d}x -$$

$$\frac{1}{2}\int h_{22t}|u_2|^2\mathrm{d}x + \mathrm{Re}\frac{\mathrm{d}}{\mathrm{d}t}\int h_{12}u_2\bar{u}_1\mathrm{d}x - \mathrm{Re}\int h_{12t}u_2\bar{u}_1\mathrm{d}x,$$

其中 $\alpha'(x) = q(x)$. 于是, 式(8) 取虚部并对 j 求和可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{2}\int a(x)|u_{jx}|^2\mathrm{d}x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\beta(x)\alpha(|u_1|^2+|u_2|^2)\mathrm{d}x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(h_{11}|u_1|^2+h_{22}|u_2|^2)\mathrm{d}x+\mathrm{Re}\frac{\mathrm{d}}{\mathrm{d}t}\int h_{12}u_2u_1\mathrm{d}x\leqslant M_5\left(\sum_{j=1}^{2}\|u_{jx}\|^2+M_6\right).$$

对上式关于 t 积分可得

$$\sum_{j=1}^{2}\int a(x)|u_{jx}(T)|^2\mathrm{d}x+\int\beta(x)\alpha(|u_1(T)|^2+|u_2(T)|^2)\mathrm{d}x+\int(h_{11}|u_1(T)|^2+h_{22}|u_2(T)|^2)\mathrm{d}x+2\mathrm{Re}\int h_{12}u_2(T)u_1(T)\mathrm{d}x\leqslant M_7\left(\int_0^T\sum_{j=1}^{2}\|u_{jx}(t)\|^2\mathrm{d}t+M_8\right). \tag{9}$$

由于 $a(x)\geqslant a_0>0$, 且由 Sobolev 不等式, Poincare 不等式及引理条件有

$$\left|\int\beta(x)\alpha(|u_1|^2+|u_2|^2)\mathrm{d}x\right|\leqslant C_1\int|\alpha(|u_1|^2+|u_2|^2)|\mathrm{d}x\leqslant C_2\int|q(s)[2u_1u_{1x}+2u_2u_{2x}]|\mathrm{d}x\leqslant C_3\int[u_1^2+u_2^2+u_{1x}^2+u_{2x}^2]\mathrm{d}x\leqslant M_9(\|u_{1x}\|^2+\|u_{2x}\|^2),$$

其中 B、α 为正常数. 故由式(9)及 Gronwall 不等式, 即得式(7). 证毕.

由 Sobolev 不等式, 我们有

推论 $\sum_{j=1}^{2}\|u_j\|_\infty\leqslant E_2$, E_2 为正常数.

引理 4 在引理 3 的条件下, 且 $u_{0j}(x)\in H^2, j=1,2$, 则有

$$\|u_{jt}(T)\|^2\leqslant E_3, \tag{10}$$

其中 E_3 为正常数.

证明 令 $v_j=u_{jt}$, 问题(1) 关于 t 微商, 并与 u_{1t}、u_{2t} 做内积得

$$(\mathrm{i}v_{jt},v_j)-((a(x)v_{jx})_x,v_j)+(\alpha_j(x)v_{jx},v_j)+(\beta(x)(q(|u_1|^2+|u_2|^2)u_j)_t,v_j)+(h_{j1t}u_1+h_{j1}v_1+h_{j2t}u_2+h_{j2}v_2,v_j)=0. \tag{11}$$

由引理 1,

$$\mathrm{Re}(v_{jt},v_j)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_j\|^2,$$

$$-((a(x)v_{jx})_x,v_j)=\int a(x)|v_{jx}|^2\mathrm{d}x,$$

$$(\beta(x)(q(s)u_j)_t,v_j)=(\beta(x)q(s)v_j,v_j)+\left(\beta(x)q'(s)\frac{\partial}{\partial t}(|u_1|^2+|u_2|^2)u_j,v_j\right).$$

由引理 3 的推论可得

$$\left(\beta(x)u_jq'(s)\frac{\partial}{\partial t}(|u_1|^2+|u_2|^2),v_j\right)\leqslant\|\beta(x)q'(|u_1|^2+|u_2|^2)\|_\infty\cdot\|u_j\|_\infty[\|u_1\|_\infty(\|v_1\|^2+\|v_j\|^2)+\|u_2\|_\infty(\|v_2\|^2+\|v_j\|^2)],$$

故

$$|(\beta(x)(q(s)u_j)_t,v_j)|\leqslant C_4[\|v_1\|^2+\|v_2\|^2],$$

$$(h_{j1t}u_1+h_{j1}v_1+h_{j2t}u_2+h_{j2}v_2,\ v_j)\leqslant M_{10}\Big[\sum_{j=1}^{2}\|v_j\|^2\Big]+M_{11}.$$

于是对式(11)取虚部,且对 j 求和可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{j=1}^{2}\|v_j\|^2\Big)\leqslant M_{12}\Big(\sum_{j=1}^{2}\|v_j\|^2\Big)+M_{13}.$$

由 Gronwall 不等式,引理假设及方程(1)知 $\|v_j(0)\|^2$ 有界,即得式(10). 证毕. □

2 整体解的存在性

由前面的的先验估计,易证问题(1)至(3)的局部解存在. 下面我们给出问题(1)至(3)整体解的存在性定理.

定理 1 若满足以下条件:

1) $a(x)$、$\alpha_j(x)$、$\beta(x)$、$q(s)$、$h_{ij}(x,t)$ 为实值函数, $x\in[0,1]$;

2) $a(x)$、$a'(x)$、$\alpha_j(x)$、$\alpha_j'(x)$、$\beta(x)$、$q(s)$、$h_{ij}(x,t)$、$h_{ijt}(x,t)$ 为有界函数, $x\in[0,1]$;

3) $q(s)>0$, $a(x)\geqslant a_0>0$, $h_{ij}(x,t)=h_{ji}(x,t)$, $u_{0j}(x)\in H^2$.

则初边值问题(1)至(3)的广义解 $u_j(x,t)\in L^\infty(0,T;H^2)$, $u_{jt}(x,t)\in L^\infty(0,T;H^2)(j=1,2)$ 存在.

定理 2 若满足引理 1 的条件,且 $q(s)\in C^1$, $s\in[0,\infty]$, 则定解问题(1)至(3)存在的整体解是唯一的.

证明 设有 2 组解: $u_j^\alpha, u_j^\beta(j=1,2)$, 令 $\omega_j=u_j^\alpha-u_j^\beta$, 则 ω_j 满足方程

$$\mathrm{i}\,\omega_{jt}-[a(x)\omega_{jx}]_x+\alpha_j(x)\omega_{jx}+\beta(x)R(u_j^\alpha,u_j^\beta)+h_{j1}\omega_1+h_{j2}\omega_2=0, \tag{12}$$

其中

$$R(u_1^\alpha,u_2^\alpha,u_1^\beta,u_2^\beta)=q(|u_1^\alpha|^2+|u_2^\alpha|^2)u_i^\alpha-q(|u_1^\beta|^2+|u_2^\beta|^2)u_i^\beta=$$
$$q'(z)[|u_1^\alpha|^2-|u_1^\beta|^2+|u_2^\alpha|^2-|u_2^\beta|^2]u_i^\alpha+q(|u_1^\beta|^2+|u_2^\beta|^2)[u_i^\alpha-u_i^\beta],$$

其中 z 在 $|u_1^\alpha|^2+|u_2^\alpha|^2$ 和 $|u_1^\beta|^2+|u_2^\beta|^2$ 之间. 设 $\|u_j^\alpha\|_\infty\leqslant M$, $\|u_j^\beta\|_\infty\leqslant M$. 由(12)式乘以 ω_j 作内积取虚部, 并对 j 求和易得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{2}\|\omega_j\|^2\leqslant C\Big(\sum_{j=1}^{2}\|\omega_j\|^2\Big),$$

由此即得

$$\sum_{j=1}^{2}\|\omega_j\|^2\leqslant C_1\Big(\sum_{j=1}^{2}\|\omega_j\|^2\Big)=0,$$

故

$$u_j^\alpha=u_j^\beta,\qquad j=1,2.$$ □

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Initial-Boundary Value Problem of One Class Nonlinear Schrödinger Equations Described in Molecular Crystals

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Abstract: The initial-boundary value problem of one class nonlinear Schrödinger equations described in molecular crystals is studied. Furthermore, the existence of the global solution is obtained by means of interpolation inequality and a priori estimation.

Key words: nonlinear Schrödinger equation; initial-boundary value problem; a priori estimate; global solution