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渗流耦合系统动边值问题特征差分方法及其应用*

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摘要: 油气运移、聚集史软件的功能是重建盆地发育中油气资源运移、聚集的历史, 它对合理评估油气资源的勘探和开发有重要的价值. 其数学模型是一组多层对流扩散耦合系统的动边值问题. 提出一类特征差分格式, 得到最佳阶误差估计结果. 对这一领域的模型分析、数值方法和软件研制均有重要的理论和实用价值.

关 键 词: 多层渗流耦合; 动边值问题; 特征差分; 误差估计; 油藏数值模拟
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引 言

近年来石油科学在石油有机地球化学、石油的生成、运移和聚集理论的研究取得重大的进展. 在评价一个盆地的油气资源时, 对于盆地发育史, 尤其是受热变化历史的了解是非常重要的. 运移聚集史软件的功能是重建油气盆地的运移、聚集历史, 它对于合理评估油气资源的勘探和开发有着重要的价值. 我们在国内外率先完成了多层运移- 聚集软件系统的研制, 并应用于胜利油田区域的油气资源评估①. 问题的数学模型是下述一组多层对流扩散耦合系统的动边值问题[1- 5]:

$$\phi_1(x,t)\frac{\partial u}{\partial t}+a(x,t)\frac{\partial u}{\partial x}-\frac{\partial}{\partial x}\left(K_1(x,t)\frac{\partial u}{\partial x}\right)+K_2(x,z,t)\frac{\partial w}{\partial z}\bigg|_{z=\mu(t)}=$$
$$Q_1(x,t,u),\qquad x\in\Omega(t);\ t\in J=(0,T], \tag{1a}$$

$$\phi_2(x,z,t)\frac{\partial w}{\partial t}=\frac{\partial}{\partial z}\left(K_2(x,z,t)\frac{\partial w}{\partial z}\right),$$
$$x\in\Omega(t);\ z\in\Theta(t)=(0,\mu(t));\ t\in J, \tag{1b}$$

$$\phi_3(x,t)\frac{\partial v}{\partial t}+b(x,t)\frac{\partial v}{\partial x}-\frac{\partial}{\partial x}\left(K_3(x,t)\frac{\partial v}{\partial x}\right)-K_2(x,z,t)\frac{\partial w}{\partial z}\bigg|_{z=0}=$$
$$Q_3(x,t,v),\qquad x\in\Omega(t);\ t\in J. \tag{1c}$$

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① 山东大学数学研究所, 胜利油田物探研究院: 石油运移聚集通道数值模拟技术研究, 2005. 10.

初始条件

$$\begin{cases} u(x,0)=\psi_1(x), & x\in\Omega(0),\\ w(x,z,0)=\psi_2(x,z), & x\in\Omega(0);z\in\Theta(0),\\ v(x,0)=\psi_3(x), & x\in\Omega(0). \end{cases} \tag{2}$$

边界条件是第一型的:

$$u(x,t)|_{\partial\Omega(t)}=0,\ w(x,z,t)|_{x\in\partial\Omega(t),z\in\Theta(t)}=0,\ v(x,t)|_{\partial\Omega(t)}=0, \tag{3a}$$

$$w(x,z,t)|_{z=\mu(t)}=u(x,t), w(x,z,t)|_{z=0}=v(x,t),$$

$$x\in\Omega(t);t\in J.\ (\text{内边界条件}) \tag{3b}$$

在渗流力学上, 待求的函数 u、w、v 为位势函数, $\partial u/\partial x$、$\partial v/\partial x$、$\partial w/\partial z$ 为Darcy 速度, $\phi_\alpha(\alpha=1,2,3)$ 为孔隙度函数, $K_1(x,t)$、$K_2(x,z,t)$ 和 $K_3(x,t)$ 为渗透率函数, $a(x,t)$、$b(x,t)$ 为相应的对流系数, 且满足条件:

$$a(x,t)|_{\partial\Omega(t)}=0,\quad b(x,t)|_{\partial\Omega(t)}=0. \tag{4}$$

$Q_1(x,t,u)$、$Q_3(x,t,v)$ 为产量项.

在渗流力学上, 问题(1)~(4) 描述3 层油资源运移聚集模型问题, 方程(1a) 和(1c) 分别表示第1、第3 层并近似认为是水平流动, 而置于它们中间的层(弱渗透层) 仅有垂直流动. 这里 $\Omega(t)=\{x\mid \chi_1(t)\leqslant x\leqslant\chi_2(t),t\in J\}$, $\chi_1(t)$、$\chi_2(t)$ 是给定的已知函数, 对于 $t\in J$ 具有一阶连续的导函数, 并且 $\Omega(t)$ 是扩张型的, 其数学形式为: $\chi_1(t'')\leqslant\chi_1(t')$, $\chi_2(t'')\geqslant\chi_2(t')$, 对于任意的 $t'<t''$ 成立. $\partial\Omega(t)$ 表示其边界. 同样 $\Theta(t)=\{z\mid 0\leqslant z\leqslant\mu(t),t\in J\}$ 亦为扩张型的, 如图1 所示.

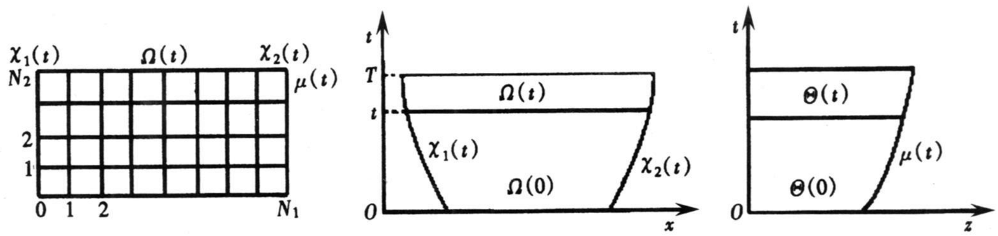

图1 区域 $\Omega(t)$, $\Theta(t)$ 示意图

对于对流扩散问题已有 Douglas 和 Russell 关于特征差分方法和特征有限元的著名工作[1,6-7], 它克服经典方法可能出现数值解的振荡和失真, 随后 Douglas 应用特征差分方法去解决不可压缩二相渗流驱动问题, 并应用能量数学方法得到了收敛性结果, 但没有得到最优阶 l^2 模误差估计, 且他必须假定问题是 Ω- 周期的[7]. 对于油资源运移聚集数值模拟中出现的一类多层渗流耦合系统的动边值问题, 本文就一维情况提出了适合并行计算的特征差分格式, 应用变分形式、能量方法、微分方程先验估计的理论和技巧, 在不需要问题是 Ω- 周期的假设条件下, 得到了收敛性的最佳阶 l^2 误差估计结果. 本文所提出的计算方法可以拓广到二维情况, 并已成功的应用此方法于多层油资源运移聚集数值模拟计算和工程实践中①. 到目前为止还未见这方面成果发表[1,4-5],②, 我们成功的解决了这一著名问题.

通常问题是正定的, 即满足:

(D) $0<\phi_*\leqslant\phi_\alpha\leqslant\phi^*$, $0<K_*\leqslant K_\alpha\leqslant K^*$, $\alpha=1,2,3$,

② R. E. Ewing. Mathematical Modeling and Simulation for Multiphase Flow in Porous Media, An International Workshop on Computation Physics: Fluid Flow and Transport in Porous Media, Beijing, 1999.

此处 ϕ_*、ϕ^*、K_* 和 K^* 均为正常数.

假定问题(1)~(4)的精确解满足正则性条件(R),即

$$\frac{\partial^2 u}{\partial \tau_1^2} \in L^\infty(L^\infty(\Omega(t))),\ u \in L^\infty(W^{4,\infty}(\Omega(t)));$$

$$(\mathrm{R})\ \frac{\partial^2 v}{\partial \tau_3^2} \in L^\infty(L^\infty(\Omega(t))),\ v \in L^\infty(W^{4,\infty}(\Omega(t))),$$

$$\frac{\partial^2 w}{\partial t^2} \in L^\infty(L^\infty(\Omega(t)\times\Theta(t))),\ w \in L^\infty(W^{4,\infty}(\Omega(t)\times\Theta(t))).$$

且 $Q_1(x,t,u)$、$Q_3(x,t,u)$ 在解的 ε_0 – 邻域满足Lipschitz连续条件,即存在常数 M,当 $|\varepsilon_i| \leqslant \varepsilon_0 (1 \leqslant i \leqslant 4)$ 时有

$$|Q_1(u(x,t)+\varepsilon_1) - Q_1(u(x,t)+\varepsilon_2)| \leqslant M|\varepsilon_1-\varepsilon_2|,\ (x,t)\in\Omega(t)\times J,$$

$$|Q_3(v(x,t)+\varepsilon_3) - Q_3(v(x,t)+\varepsilon_4)| \leqslant M|\varepsilon_3-\varepsilon_4|,\ (x,t)\in\Omega(t)\times J.$$

本文中记号 M 和 ε 分别表示普通的正常数和普通小的正数,在不同处有不同的含义.

1 二阶特征差分格式

取 $\Delta t = T/L$,对 $\Omega^n = \Omega(t^n)$ 利用等距剖分,$\chi_1(t^n) = x_0^n < x_1^n < \cdots < x_{N_1}^n = \chi_2(t^n)$,在 x 方向每个时间层节点数为 N_1+1,对 $\Theta^n = \Theta(t^n)$ 同样采用等距剖分,$0 = z_0^n < z_1^n < \cdots < z_{N_2}^n = \mu(t^n)$. 引入记号:$\omega^n = \left\{x = x_i \mid i = 0,1,\cdots,N_1; x_0^n = \chi_1(t^n), x_{N_1}^n = \chi_2(t^n)\right\}$ 是 $[\chi_1(t^n), \chi_2(t^n)]$ 上的网格.

$$h^n = \frac{\chi_2(t^n) - \chi_1(t^n)}{N_1},\ h_i^n = h^n,\ 1 \leqslant i \leqslant N_1 - 1;$$

$$h_0^n = h_{N_1}^n = \frac{h^n}{2},\ h = \max_{0\leqslant n\leqslant L} h^n.$$

$$\omega = \left\{x_i \mid i = 1,2,\cdots,N_1-1\right\},\ \omega = \left\{x_i \mid i = 0,1,\cdots,N_1\right\},$$

$$\omega^+ = \left\{x_i \mid i = 1,\cdots,N_1\right\},\ {}^+\omega = \left\{x_i \mid i = 0,1,2,\cdots,N_1-1\right\}.$$

类似地引入记号:

$$\theta^n = \left\{z = z_j^n \mid j = 0,1,\cdots,N_2; z_0^n = 0, z_{N_2}^n = \mu(t^n)\right\}$$

是 $[0, \mu(t^n)]$ 上的网格,

$$k^n = \mu(t^n)/N_2,\ k_j^n = k^n,\ 1 \leqslant j \leqslant N_2 - 1,\ k_0^n = k_{N_2}^n = k^n/2,\ k = \max_{0\leqslant n\leqslant L} k^n,$$

类似的定义 θ、θ^+ 和 ${}^+\theta$. 为了得到高精度的计算格式,对方程(1a)在 $(x, z_{N_2-1/2}, t^{n+1})$ 点展开,得

$$\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{N_2-1/2}^{n+1} =$$

$$\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{N_2}^{n+1} - \frac{k^{n+1}}{2}\left[\frac{\partial}{\partial z}K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{N_2}^{n+1} + O((k^{n+1})^2),$$

于是有

$$\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{N_2}^{n+1} =$$

$$\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{N_2-1/2}^{n+1} + \frac{k^{n+1}}{2}\left[\phi_2(x,z,t)\frac{\partial w}{\partial t}\right]_{N_2}^{n+1} + O((k^{n+1})^2).$$

类似地对方程(1c)在 $(x, z_{1/2}, t^{n+1})$ 点展开,可得

$$\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_0^{n+1}=$$

$$\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{1/2}^{n+1}-\frac{k^{n+1}}{2}\left[\phi_2(x,z,t)\frac{\partial w}{\partial t}\right]_0^{n+1}+O((k^{n+1})^2),$$

故在点 $(x, \mu(t^{n+1}), t^{n+1})$ 有

$$\phi_1(x,t)\frac{\partial u}{\partial t}+a(x,t)\frac{\partial u}{\partial x}-\frac{\partial}{\partial x}\left(K_1(x,t)\frac{\partial u}{\partial x}\right)+\frac{k^{n+1}}{2}\left[\phi_2(x,\mu(t),t)\frac{\partial u}{\partial t}\right]+$$

$$\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{N_2-1/2}+O(k^2)=Q_1(u),$$

即

$$\hat{\phi}_1(x,t,k)\frac{\partial u}{\partial t}+a(x,t)\frac{\partial u}{\partial x}-\frac{\partial}{\partial x}\left(K_1(x,t)\frac{\partial u}{\partial x}\right)=$$

$$-\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{N_2-1/2}+Q_1(u)+O(k^2),\tag{5a}$$

此处　$\hat{\phi}_1(x,t,k)=\phi_1(x,t)+(k^{n+1}/2)\phi_1(x,\mu(t),t)$.

类似的在点 $(x, 0, t^{n+1})$ 有

$$\hat{\phi}_3(x,t,k)\frac{\partial v}{\partial t}+b(x,t)\frac{\partial v}{\partial x}-\frac{\partial}{\partial x}\left(K_3(x,t)\frac{\partial v}{\partial x}\right)=$$

$$\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{1/2}+Q_3(v)+O(k^2),\tag{5b}$$

此处　$\hat{\phi}_3(x,t,k)=\phi_3(x,t)+(k^{n+1}/2)\phi_2(x,0,t)$.

对方程(1a), 这一流动实际上沿着迁移的特征方向, 用特征线方法处理方程(5a)的一阶双曲部分, 具有很高的精确度, 对时间 t 可用大步长计算[1,7], 记

$$\psi_1=(\hat{\phi}_1^2+a^2)^{1/2},\quad \frac{\partial}{\partial \tau_1}=\psi_1^{-1}\left\{\hat{\phi}_1\frac{\partial}{\partial t}+a\frac{\partial}{\partial x}\right\},$$

则式(5a)可写为下述形式:

$$\psi_1\frac{\partial u}{\partial \tau_1}-\frac{\partial}{\partial x}\left(K_1(x,t)\frac{\partial u}{\partial x}\right)=-\left[K_2(x,z,t)\frac{\partial w}{\partial z}\right]_{N_2-1/2}+$$

$$Q_1(x,t,u)+O(k^2),\qquad x\in\Omega(t),\ t\in J^n=(t^n,t^{n+1}].\tag{6}$$

对方程(6), 考虑逼近 $\psi_1\partial u/\partial\tau_1$, 在这里向后差商沿着在 (x_i^{n+1}, t^{n+1}) 的 τ_1 的特征线方向,

$$\left(\psi_1\frac{\partial u}{\partial \tau_1}\right)_{(x_i^{n-1},t^{n+1})}\approx\hat{\phi}_1(x_i^{n+1})\frac{u^{n+1}(x_i^{n+1})-u^n(x_i^{n+1}-a(x_i^{n+1},t^{n+1})\Delta t/(\hat{\phi}_1(x_i^{n+1})))}{\Delta t}.$$

记 $\hat{u}_i^n=u^n(\hat{x}_{1,i}^n)$, $\hat{x}_{1,i}^n=x_i^{n+1}-a(x_i^{n+1},t^{n+1})\Delta t/(\hat{\phi}_1(x_i^{n+1}))$.

方程(6)的特征差分格式:

$$\hat{\phi}_1(x_i^{n+1})\frac{U_i^{n+1}-\hat{U}_i^n}{\Delta t}-\delta_x(K_1(x_i^{n+1},t^{n+1})\delta_x U^{n+1})_i=$$

$$-K_2(x_i^{n+1},z_{N_2-1/2}^{n+1},t^{n+1})\delta_z w_{i,N_2}^{n+1}+Q_1(x_i^{n+1},t^{n+1},\hat{U}_i^n),\qquad 0<i<N_1,\tag{7a}$$

此处 $\hat{\phi}_1(x_i^{n+1})=\hat{\phi}_1(x_i^{n+1},t^{n+1})$, $\delta_x(K_1(x_i^{n+1},t^{n+1})\delta_x U^{n+1})_i=(h^{n+1})^{-2}[K_1(x_{i+1/2}^{n+1},t^{n+1})(U_{i+1}^{n+1}-U_i^{n+1})-K_1(x_{i-1/2}^{n+1},t^{n+1})(U_i^{n+1}-U_{i-1}^{n+1})]$, $\delta_z W_{i,N_2}^{n+1}=(k^{n+1})^{-1}(W_{i,N_2}^{n+1}-W_{i,N_2-1}^{n+1})$. 设 $\{U_i^n, i=0,1,\cdots,N_1\}$ 是差分网点值, $U^n(x)$ 是网点值的二次插值函数, 即由 U_i^n 及其相邻点 U_{i-1}^n、U_{i+1}^n 网点值所决定. 记

$$\hat{U}_i^n = U^n(\hat{x}_{1,i}^n), \hat{x}_{1,i}^n = x_i^n - a(x_i^n, t^n)\Delta t/(\hat{\phi}_1(x_i^n)).$$

边界条件

$$U_0^{n+1} = U_{N_1}^{n+1} = 0. \tag{7b}$$

类似的, 方程(1c)的特征差分格式:

$$\hat{\phi}_3(x_i^{n+1})\frac{V_i^{n+1} - \hat{V}_i^n}{\Delta t} - \delta_x(K_3(x_i^{n+1}, t^{n+1})\delta_x V^{n+1})_i =$$

$$K_2(x_i^{n+1}, z_{1/2}^{n+1}, t^{n+1})\delta_z W_{i,0}^{n+1} + Q_3(x_i^{n+1}, t^{n+1}, \hat{V}_i^n), \qquad 0 < i < N_1, \tag{8a}$$

此处

$$\hat{\phi}_3(x_i^{n+1}) = \hat{\phi}_3(x_i^{n+1}, t^{n+1}),$$

$$\delta_x(K_3(x_i^{n+1}, t^{n+1})\delta_x V^{n+1})_i = (h^{n+1})^{-2}[K_2(x_{i+1/2}^{n+1}, t^{n+1})(V_{i+1}^{n+1} - V_i^{n+1}) -$$

$$K_3(x_{i-1/2}^{n+1}, t^{n+1})(V_i^{n+1} - V_{i-1}^{n+1})],$$

$$\delta_z W_{i,0}^{n+1} = (k^{n+1})^{-1}(W_{i,1}^{n+1} - W_{i,0}^{n+1}).$$

$V^n(x)$ 是网点值的二次插值函数, 即由 V_i^n 及其相邻点 V_{i-1}^n、V_{i+1}^n 网点值所决定. 同样记

$$\hat{V}_i^n = V^n(\hat{x}_{3,i}^n), \hat{x}_{3,i}^n = x_i^n - b(x_i^n, t^n)\Delta t/(\hat{\phi}_3(x_i^n)).$$

边界条件

$$V_0^{n+1} = V_{N_1}^{n+1} = 0. \tag{8b}$$

对方程(1b)的差分格式:

$$\hat{\phi}_2(x_i^{n+1}, t^{n+1})\frac{W_{ij}^{n+1} - W_{ij}^n}{\Delta t} = \delta_z(K_2(x, z, t)\delta_z W)_{ij}^{n+1},$$

$$0 < i < N_1; 0 < j < N_2, \tag{9a}$$

此处

$$\delta_z(K_2(x, z, t)\delta_z W)_{ij}^{n+1} = (k^{n+1})^{-2}[K_2(x_i^{n+1}, z_{j+1/2}^{n+1}, t^{n+1})(W_{i,j+1}^{n+1} - W_{ij}^{n+1}) -$$

$$K_2(x_i^{n+1}, z_{j-1/2}^{n+1}, t^{n+1})(W_{ij}^{n+1} - W_{i,j-1}^{n+1})].$$

边界条件

$$\begin{cases} W_{0,j}^{n+1} = W_{N_1,j}^{n+1} = 0, & 0 < j < N_2, \\ W_{i,N_2}^{n+1} = U_i^{n+1}, \quad W_{i,0}^{n+1} = V_i^{n+1}, & 0 < i < N_1. \end{cases} \quad (\text{内边界条件}) \tag{9b}$$

特征差分格式(7)、(8)和(9)的计算程序: 在实际计算时, (7a)的 $\delta_z W_{i,N_2}^{n+1}$ 近似的取为 $\delta_z W_{i,N_2}^n$, 式(8a)中的 $\delta_z W_{i,0}^{n+1}$ 近似的取为 $\delta_z W_{i,0}^n$. 若已知时刻 $t = t^n$ 的差分解 $\{U_i^n, W_{ik}^n, V_i^n\}$ 时, 寻求下一时刻 $t = t^{n+1}$ 的 $\{U_i^{n+1}, W_{ik}^{n+1}, V_i^{n+1}\}$. 首先由式(7)应用追赶法计算出 $\{U_i^{n+1}\}$, 同时可并行的由式(8)应用追赶法计算出 $\{V_i^{n+1}\}$, 最后由式(9)用追赶法计算出 $\{W_{ij}^{n+1}\}$, 由正定性条件(D), 此差分解存在且唯一.

注意 此处设 $y = x - a(x, t)\Delta t/\hat{\phi}_1 = g(x)$, $z = x - b(x, t)\Delta t/\hat{\phi}_3 = f(x)$, 当 Δt 适当小时, 条件 $a(x, t)|_{\partial\Omega(t)} = 0$, $b(x, t)|_{\partial\Omega(t)} = 0$, 得知 $y = g(x)$, $z = f(x)$ 分别同胚映射 $\Omega(t)$ 为自身, 故得知 $\hat{x}_{1,i}^n$、$\hat{x}_{3,i}^n$ 仍然属于 $\Omega(t^n)$. 这里将 $\Omega-$ 周期条件去掉.

2 收敛性分析

首先定义网格函数空间 H_h 的内积[8-11], 对一维网络区域, 设 $v(x)$ 与 $u(x)$ 是给定在 ω 上的网络函数, 用下面的公式定义离散内积:

$$(v, u)_\omega = \sum_{i=0}^{N_1} v(x_i)u(x_i)h_i,\quad (v, u)_{\omega^+} = \sum_{i=0}^{N_1} v(x_i)u(x_i)h_i,$$

$$(v, u)_{^+\omega} = \sum_{i=0}^{N_1-1} v(x_i)u(x_i)h_i.$$

容易证明 $(v, u)_\omega = [(v, u)_{\omega^+} + (v, u)_{^+\omega}]/2$.

对二维网格区域, 设 $f(x, z)$ 与 $g(x, z)$ 是给定在 $\omega\times\theta$ 上的网格函数, 用下面的公式定义离散内积:

$$\langle f, g\rangle_{\omega\times\theta} = \sum_{0\leqslant i\leqslant N_1}\sum_{0\leqslant j\leqslant N_2} f(x_i, z_j)g(x_i, z_j)h_ik_j,$$

$$\langle f, g]_{w\times\theta,1} = \sum_{0\leqslant j\leqslant N_2}\sum_{1\leqslant i\leqslant N_1} f(x_i, z_j)g(x_i, z_j)h_ik_j,$$

$$\langle f, g]_{w\times\theta,2} = \sum_{0\leqslant i\leqslant N_1}\sum_{1\leqslant j\leqslant N_2} f(x_i, z_j)g(x_i, z_j)h_ik_j.$$

同样有 $\langle f, g\rangle_{\omega\times\theta} = [\langle f, g]_{w\times\theta,1} + \langle f, g]_{w\times\theta,2}]/2$, 在通常情况下, 为了简便, 将下标省略.

关于格式(7)~(9)的收敛性分析, 设 u、v、w 为问题(1)~(4)的精确解, U、V、W 为格式(7)~(9)的差解, 记误差函数为 $\xi = u - U$, $\zeta = v - V$, $\omega = w - W$.

首先讨论方程(7), 由方程(7)和(1a) $(t = t^{n+1})$, 可得下述关于误差函数的误差方程:

$$\begin{aligned}
&\hat{\varphi}_{1,i}^{n+1}\frac{\xi_i^{n+1} - (u^n(\hat{x}_{1,i}^n) - \hat{U}_i^n)}{\Delta t} - \delta_x(K_1(x_i^{n+1}, t^{n+1})\delta_x\xi^{n+1})_i = \\
&\quad - K_2(x_i^{n+1}, z_{N_2-1/2}^{n+1}, t^{n+1})\delta_z\omega_{i,N_2}^{n+1} + Q_1(x_i^{n+1}, t^{n+1}, u_i^{n+1}) - \\
&\quad Q_1(x_i^{n+1}, t^{n+1}, \hat{U}_i^n) + \varepsilon_{1,i}^{n+1},\qquad 0 < i < N_1,
\end{aligned}\tag{10a}$$

$$\begin{cases}\xi_0^{n+1} = \xi_{N_1}^{n+1} = 0, \\ \xi_i^{n+1} = \omega_{i,N_2}^{n+1},\qquad 0 < i < N_1,\end{cases}\tag{10b}$$

此处 $|\varepsilon_{1,i}^{n+1}| \leqslant M\left\{\left\|\dfrac{\partial^2 u}{\partial\tau_1^2}\right\|_{L^\infty(J^n, L^\infty)}, \|u\|_{L^\infty(J^n, W^{4,\infty})}, \|w\|_{L^\infty(J^n, W^{4,\infty})}\right\}\left\{h^2 + \Delta t\right\}$. $J^n = [t^n, t^{n+1}]$ 和 $\tau_1 = \tau_{1,i}^{n+1}$. 特征方向在$(x_i^{n+1}, t^{n+1})$ 处, 函数 $\xi^n(x)$ 是按节点值$\left\{\xi_i^n\right\}$的分片二次插值函数, $\hat{\xi}_i^n = \xi^n(\hat{x}_i^n)$, 则有

$$\begin{aligned}
&\xi_i^{n+1} - (u^n(\hat{x}_{1,i}^n) - \hat{U}_i^n) = \\
&\quad(\xi_i^{n+1} - \hat{\xi}_i^n) - (u^n(\hat{x}_{1,i}^n) - u^n(\hat{x}_{1,i}^n)) + (I - I_2)u^n(\hat{x}_{1,i}^n),
\end{aligned}\tag{11}$$

此处 I 是恒等算子, I_2 是分段二次插值算子, 由问题的正则性条件, 并且由于实际上 $x_1(t)$、$x_2(t)$ 变化是很慢的, 可以假定 $x_i'(t) \approx O(\Delta t)(i = 1, 2)$. 则有

$$\begin{aligned}
&\hat{\varphi}_{1,i}^{n+1}\frac{\xi_i^{n+1} - \hat{\xi}_i^n}{\Delta t} - \delta_x(K_1(x_i^{n+1}, t^{n+1})\delta_x\xi^{n+1})_i \leqslant \\
&\quad - K_2(x_i^{n+1}, z_{N_2-1/2}^{n+1}, t^{n+1})\delta_z\omega_{i,N_2}^{n+1} + M\left\{|\hat{\xi}_i^n|^2 + h^2 + \Delta t\right\} - \\
&\quad\frac{\hat{\varphi}_{1,i}^{n+1}(I - I_2)u^n(\hat{x}_{1,i}^n)}{\Delta t} + \varepsilon_{1,i}^{n+1},\qquad 0 < i < N_1.
\end{aligned}\tag{12}$$

我们假定剖分参数满足下述限制性条件:

$$\Delta t = O(h^2).\tag{13}$$

注意到

$$h^{n+1} - h^n = \frac{1}{N_1}\left\{[\chi_2(t^{n+1}) - \chi_1(t^{n+1})] - [\chi_2(t^n) - \chi_1(t^n)]\right\} =$$
$$\frac{1}{N_1}\left\{[\chi_2(t^{n+1}) - \chi_2(t^n)] - [\chi_1(t^{n+1}) - \chi_1(t^n)]\right\} =$$
$$\frac{\Delta t}{N_1}\left\{\chi_2'(t^{n+1/2}) - \chi_1'(\tilde{\tilde{t}}^{n+1/2})\right\} = O\left\{h(\Delta t)^2\right\}, \tag{14}$$

此处 $t^n < t^{n+1/2}$、$\tilde{\tilde{t}}^{n+1/2} < t^{n+1}$. 对式(12)乘以 $\xi_i^{n+1}h_i^{n+1}$ 并求和, 依次分析诸项, 将式(12)左端第1项分解为

$$\hat{\phi}_{1,i}^{n+1}\frac{\xi_i^{n+1} - \hat{\xi}_i^n}{\Delta t} = \hat{\phi}_{1,i}^{n+1}\frac{\xi_i^{n+1} - \xi_i^n}{\Delta t} + \hat{\phi}_{1,i}^{n+1}\frac{\xi_i^n - \hat{\xi}_i^n}{\Delta t},$$

我们可得

$$\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}\left(\frac{\xi_i^{n+1} - \xi_i^n}{\Delta t}\right)\xi_i^{n+1}h_i^{n+1} \geqslant \frac{1}{2\Delta t}\left\{\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}(\xi_i^{n+1})^2h_i^{n+1} - \sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}(\xi_i^n)^2h_i^{n+1}\right\} =$$
$$\frac{1}{2\Delta t}\left\{\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}(\xi_i^{n+1})^2h_i^{n+1} - \sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n}(\xi_i^n)^2h_i^{n}\right\} + O\left\{\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n}(\xi_i^n)^2h_i^{n}\right\}. \tag{15}$$

对于项 $\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}\frac{\xi_i^n - \hat{\xi}_i^n}{\Delta t}\xi_i^{n+1}h_i^{n+1}$, 注意到

$$\xi_i^n - \hat{\xi}_i^n = -\int_{x_i^n}^{\hat{x}_i^n}\frac{\partial}{\partial x}\xi^n\mathrm{d}\sigma = -\int_{x_i^n}^{x_i^n - a(x_i^n, t^n)\Delta t/\hat{\phi}_{1,i}^n}\frac{\partial \xi^n}{\partial x}\mathrm{d}\sigma \leqslant$$
$$M\Delta t\max\left\{|\delta_x\xi_p^n| : |x_p^n - x_i^n| \leqslant h^n + M\Delta t\right\}.$$

于是有

$$\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}\left(\frac{\xi_i^n - \hat{\xi}_i^n}{\Delta t}\right)\xi_i^{n+1}h_i^{n+1} \leqslant \varepsilon\sum_{i=0}^{N_1-1}|\delta_x\xi_i^n|^2h^n + M\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}(\xi_i^{n+1})^2h_i^{n+1}, \tag{16}$$

$$-\sum_{i=0}^{N_1}\delta_x(K_1(x_i^{n+1}, t^{n+1})\delta_x\xi^{n+1})_i\xi_i^{n+1}h_i^{n+1} =$$
$$\sum_{i=1}^{N_1}K_{1,i-1/2}^{n+1}(\delta_x\xi_i^{n+1})^2h^{n+1} = \sum_{i=0}^{N_1-1}K_{1,i-1/2}^{n+1}(\delta_x\xi_i^{n+1})^2h^{n+1}, \tag{17}$$

$$\sum_{i=0}^{N_1}\varepsilon_{1,i}^{n+1}\xi_i^{n+1}h_i^{n+1} \leqslant M\left\{(\Delta t)^2 + h^4 + \sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}(\xi_i^{n+1})^2h_i^{n+1}\right\}. \tag{18}$$

由式(12)、(15)~(18)可得:

$$\frac{1}{2\Delta t}\left\{(\hat{\phi}_1^{n+1}\xi^{n+1}, \xi^{n+1})_\omega - (\hat{\phi}_1^n\xi^n, \xi^n)_\omega\right\} +$$
$$\frac{1}{2}\left\{(K_1^{n+1}\delta_x\xi^{n+1}, \delta_x\xi^{n+1})_{\omega^+} + (K_1^n\delta_x\xi^{n+1}, \delta_x\xi^{n+1})_{\omega^+}\right\} \leqslant$$
$$-\sum_{i=0}^{N_1}K_2(x_i^{n+1}, z_{N_2-1/2}^{n+1}, t^{n+1})\delta_z\omega_{i,N_2}^{n+1}\xi_i^{n+1}h_i^{n+1} +$$
$$M\left\{\|\xi^{n+1}\|_0^2 + \|\xi^n\|_0^2 + h^4 + (\Delta t)^2\right\} +$$
$$\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}\frac{(I - I_2)u^n(\hat{x}_{1,i}^n)}{\Delta t}\xi_i^{n+1}h_i^{n+1} + \varepsilon\|\delta_x\xi^{n+1}\|_0^2. \tag{19}$$

对于上式右端第3项, 应用 Peano 核定理, 由文献[6]可得估计

$$\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}\frac{(I-I_2)u^n(\hat{x}_{1,i}^n)}{\Delta t}\xi_i^{n+1}h_i^{n+1}\leqslant M\left\{h^4+\sum_{i=0}^{N_1}\hat{\phi}_{1,i}^{n+1}(\xi_i^{n+1})^2h_i^{n+1}\right\}. \tag{20}$$

对于式(19),应用式(20)并注意到 $K_1(x,t)$ 正定性条件可得:

$$\frac{1}{2\Delta t}\left\{\|(\hat{\phi}_1^{n+1})^{1/2}\xi^{n+1}\|_0^2-\|(\hat{\phi}_1^n)^{1/2}\xi^n\|_0^2\right\}+\frac{K_*}{4}\left\{\|\delta_x\xi^{n+1}\|_0^2+\|\delta_x\xi^{n+1}\|_0^2\right\}\leqslant$$
$$-\sum_{i=0}^{N_1}K_2(x_i^{n+1},z_{N_2-1/2}^{n+1},t^{n+1})\delta_z\omega_{i,N_2}^{n+1}\xi_i^{n+1}h_i^{n+1}+M\left\{\|\xi^{n+1}\|_0^2+\|\xi^n\|_0^2+h^4+(\Delta t)^2\right\}. \tag{21}$$

其次,类似的讨论方程(8),由方程(8)和(1c) $(t=t^{n+1})$,可得下述关于误差函数 ζ 的误差方程:

$$\hat{\phi}_{3,i}^{n+1}\frac{\zeta_i^{n+1}-(v^n(\overset{\frown}{x}_{3,i}^n)-\hat{V}_i^n)}{\Delta t}-\delta_x(K_3(x_i^{n+1},t^{n+1})\delta_x\zeta^{n+1})_i=$$
$$K_2(x_i^{n+1},z_{1/2}^{n+1},t^{n+1})\delta_z\omega_{i,0}^{n+1}+Q_3(x_i^{n+1},t^{n+1},v_i^{n+1})-$$
$$Q_3(x_i^{n+1},t^{n+1},\hat{V}_i^n)+\varepsilon_{3,i}^{n+1},\qquad 0<i<N_1, \tag{22a}$$

$$\begin{cases}\zeta_0^{n+1}=\zeta_{N_1}^{n+1}=0,\\ \zeta_i^{n+1}=\omega_{i,0}^{n+1},\qquad 0<i<N_1,\end{cases} \tag{22b}$$

此处

$$|\varepsilon_{3,i}^{n+1}|\leqslant M\left\{\left\|\frac{\partial^2 v}{\partial\tau_3^2}\right\|_{L^\infty(J^n,L^\infty)},\ \|v\|_{L^\infty(J^n,W^{4,\infty})},\ \|w\|_{L^\infty(J^n,W^{4,\infty})}\right\}\left\{h^2+\Delta t\right\},$$

$J^n=[t^n,t^{n+1}]$, $\tau_3=\tau_{3,i}^{n+1}$,特征方向在 (x_i^{n+1},t^{n+1}) 处. 函数 $\zeta^n(x)$ 是按节点值 $\left\{\zeta_i^n\right\}$ 的二次插值函数,于是式(22)可改写为

$$\hat{\phi}_{3,i}^{n+1}\frac{\zeta_i^{n+1}-\hat{\zeta}_i^n}{\Delta t}-\delta_x(K_3(x_i^{n+1},t^{n+1})\delta_x\zeta^{n+1})_i\leqslant$$
$$K_2(x_i^{n+1},z_{1/2}^{n+1},t^{n+1})\delta_z\omega_{i,0}^{n+1}+M\left\{|\zeta_i^n|^2+h^2+\Delta t\right\}-$$
$$\hat{\phi}_{3,i}^{n+1}\frac{(I-I_2)v^n(\hat{x}_{3,i}^n)}{\Delta t}+\varepsilon_{3,i}^{n+1},\qquad 0<i<N_1. \tag{23}$$

对式(23)乘以 $\zeta_i^{n+1}h_i^{n+1}$ 并求和,逐次分析诸项,最后可得

$$\frac{1}{2\Delta t}\left\{\|(\hat{\phi}_3^{n+1})^{1/2}\zeta^{n+1}\|_0^2-\|(\hat{\phi}_3^n)^{1/2}\zeta^n\|_0^2\right\}+\frac{K_*}{4}\left\{\|\delta_x\zeta^{n+1}\|_0^2+\|\delta_x\zeta^{n+1}\|_0^2\right\}\leqslant$$
$$\sum_{i=0}^{N_1}K_2(x_i^{n+1},z_{1/2}^{n+1},t^{n+1})\delta_z\omega_{i,0}^{n+1}\zeta_i^{n+1}h_i^{n+1}+M\left\{\|\zeta^{n+1}\|_0^2+\|\zeta^n\|_0^2+h^4+(\Delta t)^2\right\}. \tag{24}$$

最后讨论方程(9),由方程(9)和(1b) $(t=t^{n+1})$,可得下述关于 ω 的误差方程:

$$\phi_{2,ij}^{n+1}\frac{\omega_{ij}^{n+1}-\omega_{ij}^n}{\Delta t}-\delta_z(K_2(x_i^{n+1},z_j^{n+1},t^{n+1})\delta_z\omega^{n+1})_{ij}=\varepsilon_{2,ij}^{n+1},$$
$$0<i<N_1;\ 0<j<N_2. \tag{25a}$$

$$\begin{cases}\omega_{0,j}^{n+1}=\omega_{N_1,j}^{n+1}=0,\\ \omega_{i,0}^{n+1}=\zeta_i^{n+1},\quad \omega_{i,N_2}^{n+1}=\zeta_i^{n+1},\qquad 0<i<N_1,\end{cases} \tag{25b}$$

此处 $|\varepsilon_{2,ij}^{n+1}| \leqslant M\left\{\left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^\infty(J^n,L^\infty)},\ \|w\|_{L^\infty(J^n,W^{4,\infty})}\right\}\left\{k^2+\Delta t\right\}$.

对式(25)乘以 $\omega_{ij}^{n+1}h_i^{n+1}k_j^{n+1}$ 作内积分求和可得

$$\sum_{\substack{0\leqslant i\leqslant N_1\\0\leqslant j\leqslant N_2}}\Phi_{2,ij}^{n+1}\left(\frac{\omega_{ij}^{n+1}-\omega_{ij}^n}{\Delta t}\right)\omega_{ij}^{n+1}h_i^{n+1}k_j^{n+1}-$$
$$\sum_{\substack{0\leqslant i\leqslant N_1\\0\leqslant j\leqslant N_2}}\delta_{\bar z}(K_2(x_i^{n+1},z_j^{n+1},t^{n+1})\delta_z\omega^{n+1})_{ij}\omega_{ij}^{n+1}h_i^{n+1}k_j^{n+1}=$$
$$\sum_{\substack{0\leqslant i\leqslant N_1\\0\leqslant j\leqslant N_2}}\varepsilon_{2,ij}^{n+1}\omega_{ij}^{n+1}h_i^{n+1}k_j^{n+1}. \tag{26}$$

注意到

$$\sum_{\substack{0\leqslant i\leqslant N_1\\0\leqslant j\leqslant N_2}}\Phi_{2,ij}^{n+1}\left(\frac{\omega_{ij}^{n+1}-\omega_{ij}^n}{\Delta t}\right)\omega_{ij}^{n+1}h_i^{n+1}k_j^{n+1}\geqslant$$
$$\frac{1}{2\Delta t}\left\{\sum_{ij}\Phi_{2,ij}^{n+1}(\omega_{ij}^{n+1})^2h_i^{n+1}k_j^{n+1}-\sum_{ij}\Phi_{2,ij}^{n+1}(\omega_{ij}^n)^2h_i^nk_j^n\right\}+$$
$$O\left\{\sum_{ij}\Phi_{2,ij}^n(\omega_{ij}^n)^2h_i^nk_j^n\right\}, \tag{27}$$

$$-\sum_{\substack{0\leqslant i\leqslant N_1\\0\leqslant j\leqslant N_2}}\delta_{\bar z}(K_2(x_i^{n+1},z_j^{n+1},t^{n+1})\delta_z\omega^{n+1})_{ij}\omega_{ij}^{n+1}h_i^{n+1}k_j^{n+1}=$$
$$-\sum_{0\leqslant i\leqslant N_1}h_i^{n+1}\left\{\sum_{0\leqslant j\leqslant N_2}\delta_{\bar z}(K_2(x_i^{n+1},z_j^{n+1},t^{n+1})\delta_z\omega^{n+1})_{ij}\omega_{ij}^{n+1}k_j^{n+1}\right\}=$$
$$\sum_i h_i^{n+1}\left\{\sum_j K_2(x_i^{n+1},z_j^{n+1},t^{n+1})\delta_z\omega_{ij}^{n+1}k_j^{n+1}-\right.$$
$$\left.\omega_{i,N_2}^{n+1}K_2(x_i^{n+1},z_{N_2-1/2}^{n+1},t^{n+1})\delta_{\bar z}\omega_{i,N_2}^{n+1}+\omega_{i,0}^{n+1}K_2(x_i^{n+1},z_{1/2}^{n+1},t^{n+1})\delta_z\omega_{i,0}^{n+1}\right\}=$$
$$\sum_{i,j}K_2(x_i^{n+1},z_j^{n+1},t^{n+1})(\delta_z\omega_{ij}^{n+1})^2h_i^{n+1}k_j^{n+1}-$$
$$\sum_i K_2(x_i^{n+1},z_{N_2-1/2}^{n+1},t^{n+1})\delta_{\bar z}\omega_{i,N_2}^{n+1}\xi_i^{n+1}h_i^{n+1}+$$
$$\sum_i K_2(x_i^{n+1},z_{1/2}^{n+1},t^{n+1})\delta_z\omega_{i,0}^{n+1}\zeta_i^{n+1}h_i^{n+1}, \tag{28}$$

可得

$$\frac{1}{2\Delta t}\left\{|||(\Phi_2^{n+1})^{1/2}\omega^{n+1}|||^2-|||(\Phi_2^n)^{1/2}\omega^n|||^2\right\}+$$
$$\frac{1}{2}\left\{\sum_{i,j}K_2(x_i^{n+1},z_j^{n+1},t^{n+1})(\delta_z\omega_{ij}^{n+1})h_i^{n+1}k_j^{n+1}+\right.$$
$$\left.K_2(x_i^{n+1},z_j^{n+1},t^{n+1})(\delta_{\bar z}\omega_{ij}^{n+1})^2h_i^{n+1}k_j^{n+1}\right\}\leqslant$$
$$\sum_{i=0}^{N_1}K_2(x_i^{n+1},z_{N_2-1/2}^{n+1},t^{n+1})\delta_{\bar z}\omega_{i,N_2}^{n+1}\xi_i^{n+1}h_i^{n+1}-$$
$$\sum_{i=0}^{N_1}K_2(x_i^{n+1},z_{1/2}^{n+1},t^{n+1})\delta_z\omega_{i,0}^{n+1}\zeta_i^{n+1}h_i^{n+1}+$$
$$M\left\{|||(\Phi_2^n)^{1/2}\omega^n|||^2+k^4+(\Delta t)^2\right\}, \tag{29}$$

此处记号 $|||\Phi^{1/2}\omega|||^2=\sum_{ij}\Phi_{ij}(\omega_{ij})^2h_ik_j$.

组合式(21)、(24)和(29),乘以 $2\Delta t$ 并对 t 求和, $0 \leqslant n \leqslant L-1$, 注意到 $\xi^0 = \zeta^0 = \omega^0 = 0$, 可得下述估计式:

$$\|\xi^L\|_0^2 + \|\zeta^L\|_0^2 + |||\omega^L|||_0^2 + \sum_{n=1}^{L}\Big\{\|\delta_x\xi^n\|^2 + \|\delta_{\bar{x}}\xi^n\|^2 + \|\delta_x\zeta^n\|^2 + \|\delta_{\bar{x}}\zeta^n\|^2 + |||\delta_z\omega^n|||^2 + |||\delta_{\bar{z}}\omega^n|||^2\Big\}\Delta t \leqslant M\Big\{\sum_{n=1}^{L}\big[\|\xi^n\|_0^2 + \|\zeta^n\|_0^2 + |||\omega^n|||_0^2\big]\Delta t + h^4 + k^4 + (\Delta t)^2\Big\}. \tag{30}$$

应用 Gronwall 引理可得

$$\|\xi^L\|_0^2 + \|\zeta^L\|_0^2 + |||\omega^L|||_0^2 + \sum_{n=1}^{L}\Big\{\|\delta_x\xi^n\|_0^2 + \|\delta_{\bar{x}}\xi^n\|_0^2 + \|\delta_x\zeta^n\|_0^2 + \|\delta_{\bar{x}}\zeta^n\|_0^2 + |||\delta_z\omega^n|||_0^2 + |||\delta_{\bar{z}}\omega^n|||_0^2\Big\}\Delta t \leqslant M\Big\{h^4 + (\Delta t)^2\Big\}. \tag{31}$$

定理 假定问题(1)~(4)的精确解满足光滑性条件(R),采用特征差分格式(7)~(9)逐层计算,则下述误差估计成立:

$$\|u-U\|_{l^\infty([0,T],l^2)} + \|v-V\|_{l^\infty([0,T],l^2)} + \|w-W\|_{l^\infty([0,T],l^2)} + \|u-U\|_{l^2([0,T],h^1)} + \|v-V\|_{l^2([0,T],h^2)} + \|w-W\|_{l^2([0,T],h^1)} \leqslant M^*\Big\{\Delta t + h^2 + k^2\Big\}, \tag{32}$$

此处

$$\|f\|_{l^\infty(J,x)} = \sup_{n\Delta t\leqslant T}\|f^n\|_x,\quad \|g\|_{l^2(J,x)} = \sup_{N\Delta t\leqslant T}\Big\{\sum_{n=0}^{N}\|g^n\|_x^2\Delta t\Big\}^{1/2},$$

M^* 依赖于 u、v、w 及其导数.

3 二维问题的拓广

本文所提出的动边值问题的特征差分方法,可以拓广到二维多层对流扩散耦合系统的初边值问题:

$$\phi_1(x,y,t)\frac{\partial u}{\partial t} + \boldsymbol{a}(x,y,t)\cdot\nabla u - \nabla\cdot(K_1(x,y,t)\nabla u) + K_2(x,y,z,t)\frac{\partial w}{\partial z}\Big|_{z=\mu(t)} = Q_1(x,y,t,u),$$
$$(x,y)^{\mathrm T}\in\Omega(t);\ t\in J=(0,T], \tag{33a}$$

$$\phi_2(x,y,z,t)\frac{\partial w}{\partial t} = \frac{\partial}{\partial z}\left(K_2(x,y,z,t)\frac{\partial w}{\partial z}\right),$$
$$(x,y,z)^{\mathrm T}\in\Theta(t);\ t\in J, \tag{33b}$$

$$\phi_3(x,y,t)\frac{\partial v}{\partial t} + \boldsymbol{b}(x,y,t)\cdot\nabla v - \nabla\cdot(K_3(x,y,z)\nabla v) - K_2(x,y,z,t)\frac{\partial w}{\partial z}\Big|_{z=0} = Q_3(x,y,t,v),\qquad (x,y)^{\mathrm T}\in\Omega(t);\ t\in J. \tag{33c}$$

此处

$$\Omega(t) = \Big\{X=(x,y)\mid \chi_1(t)\leqslant x\leqslant\chi_2(t),\ \sigma_1(t)\leqslant y\leqslant\sigma_2(t),\ t\in J\Big\},$$
$$\Theta(t) = \Big\{(X,z)\mid \chi_1(t)\leqslant x\leqslant\chi_2(t),\ \sigma_1(t)\leqslant y\leqslant\sigma_2(t),\ 0\leqslant z\leqslant\mu(t),\ t\in J\Big\},$$
$$\nabla = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)^{\mathrm T}.$$

初始条件

$$u(x,y,0)=\psi_1(x,y),\qquad (x,y)^{\mathrm T}\in\Omega(0),\tag{34a}$$

$$w(x,y,z,0)=\psi_2(x,y,z),\qquad (x,y,z)^{\mathrm T}\in\Theta(0),\tag{34b}$$

$$v(x,y,0)=\psi_3(x,y),\qquad (x,y)^{\mathrm T}\in\Omega(0).\tag{34c}$$

边界条件

$$\begin{cases}u(x,y,t)\mid_{\partial\Omega(t)}=0,\ w(x,y,z,t)\mid_{(x,y)^{\mathrm T}\in\partial\Omega(t);z\in(0,\mu(t))}=0,\\ v(x,y,t)\mid_{\partial\Omega(t)}=0,\\ w(x,y,z,t)\mid_{z=\mu(t)}=u(x,y,t),\ w(x,y,z,t)\mid_{z=0}=v(x,y,t),\\ \qquad (x,y)^{\mathrm T}\in\Omega(t);t\in J.\end{cases}\tag{35}$$

类似地我们可以构造特征差分格式, 对方程(33a) 我们有:

$$\hat{\phi}_1(x_i^{n+1},y_j^{n+1},t^{n+1})\frac{U_{ij}^{n+1}-\hat{U}_{ij}^n}{\Delta t}-\left\{\delta_{\bar x}(K_1\delta_x U)_{ij}^{n+1}+\delta_{\bar y}(K_1\delta_y U)_{ij}^{n+1}\right\}=$$
$$-K_2(x_i^{n+1},y_j^{n+1},z_{N_3-1/2}^{n+1},t^{n+1})\delta_{\bar z}W_{ij,N_3}^{n+1}+Q_1(x_i^{n+1},y_j^{n+1},\hat{U}_{ij}^n),$$
$$0<i<N_1;0<j<N_2.\tag{36a}$$

边界条件

$$U_{0,j}^{n+1}=U_{N_1,j}^{n+1}=0,\quad U_{i,0}^{n+1}=U_{i,N_2}^{n+1}=0.\tag{36b}$$

此处设$\left\{U_{ij}^n\right\}$是差分解的网点值, $U^n(x)$ 是网点值的双二次插值函数, $\hat{U}_{ij}^n=U^n(\hat{X}_{1,ij}^n)$, $\hat{X}_{1,ij}^n=X_{ij}^n-\boldsymbol{a}(X_{ij}^n,t^n)\Delta t/(\hat{\phi}_1(X_{ij}^n,t^n))$.

对方程(33c) 我们有

$$\hat{\phi}_3(x_i^{n+1},y_j^{n+1},t^{n+1})\frac{V_{ij}^{n+1}-\hat{V}_{ij}^n}{\Delta t}-\left\{\delta_{\bar x}(K_3\delta_x V)_{ij}^{n+1}+\delta_{\bar y}(K_3\delta_y V)_{ij}^{n+1}\right\}=$$
$$K_2(x_i^{n+1},y_j^{n+1},z_{1/2}^{n+1},t^{n+1})\delta_z W_{ij,0}^{n+1}+Q_3(x_i^{n+1},y_j^{n+1},\hat{V}_{ij}^n),$$
$$0<i<N_1;0<j<N_2.\tag{37a}$$

边界条件

$$V_{0,j}^{n+1}=V_{N_1,j}^{n+1}=0,\quad V_{i,0}^{n+1}=V_{i,N_2}^{n+1}=0,\tag{37b}$$

此处 $V^n(x)$ 是网点值的双二次插值函数, 记

$$\hat{V}_{ij}^n=V^n(\hat{X}_{3,ij}^n),\ \hat{X}_{3,ij}^n=X_{ij}^n-\boldsymbol{b}(X_{ij}^n,t^n)/(\hat{\phi}_3(X_{ij}^n,t^n)).$$

对方程(33b) 我们有

$$\phi_2(x_i^{n+1},y_j^{n+1},z_k^{n+1},t^{n+1})\frac{W_{ijk}^{n+1}-W_{ijk}^n}{\Delta t}=\delta_{\bar z}(K_2(x,y,z,t)\delta_z W)_{ijk}^{n+1},$$
$$0<i<N_1;0<j<N_2;0<k<N_3.\tag{38a}$$

边界条件

$$\begin{cases}W_{0,jk}^{n+1}=W_{N_1,jk}^{n+1}=W_{i,0,k}^{n+1}=W_{i,N_2,k}^{n+1}=0,\\ W_{ij,N_3}^{n+1}=U_{ij}^{n+1},\ W_{ij,0}^{n+1}=V_{ij}^{n+1}.\end{cases}\tag{38b}$$

特征差分格式(36) ~ (38) 的计算程序: 在实际计算时, 式(36a) 中的 $\delta_{\bar z}W_{ij,N_3}^{n+1}$ 近似地取为 $\delta_{\bar z}W_{ij,N_3}^n$, 式(37a) 中的 $\delta_z W_{ij,0}^{n+1}$ 近似地取为 $\delta_z W_{ij,0}^n$. 若已知 $t=t^n$ 时刻的差分解$\left\{U_{ij}^n,W_{ijk}^n,V_{ij}^n\right\}$时, 寻求下一时刻 $t=t^{n+1}$ 的$\left\{U_{ij}^n,W_{ijk}^n,V_{ij}^n\right\}$. 首先由式(36) 计算出$\left\{U_{ij}^{n+1}\right\}$, 同时可并行的

由式(37)计算出$\{V_{ij}^{n+1}\}$，最后由式(38)用追赶法计算出$\{W_{ijk}^{n+1}\}$．由于问题是正定的，此差分解存在且唯一．

4 应　　用

动边值问题的特征差分方法，已应用到多层油资源运移聚集的软件系统和胜利油田油资源评估中①．问题的数学模型为

$$\nabla\cdot\left(K_1\frac{k_{ro}}{\mu_o}\nabla\psi_o\right)+B_oq-\left(K_2\frac{k_{ro}}{\mu_o}\frac{\partial\psi_o}{\partial z}\right)_{z=\mu(t)}=-\phi_1 s\left(\frac{\partial\psi_o}{\partial t}-\frac{\partial\psi_w}{\partial t}\right),$$
$$X=(x,y)^{\mathrm T}\in\Omega(t);\ t\in J=(0,T], \tag{39a}$$

$$\nabla\cdot\left(K_1\frac{k_{rw}}{\mu_w}\nabla\psi_w\right)+B_wq-\left(K_2\frac{k_{rw}}{\mu_w}\frac{\partial\psi_w}{\partial z}\right)_{z=\mu(t)}=\phi_1 s\left(\frac{\partial\psi_w}{\partial t}-\frac{\partial\psi_w}{\partial t}\right),$$
$$X\in\Omega(t);\ t\in J, \tag{39b}$$

$$\frac{\partial}{\partial z}\left(K_2\frac{k_{ro}}{\mu_o}\frac{\partial\psi_o}{\partial z}\right)=-\phi_2 s\left(\frac{\partial\psi_o}{\partial t}-\frac{\partial\psi_w}{\partial t}\right),$$
$$X=(x,y,z)^{\mathrm T}\in\Theta(t);\ t\in J, \tag{40a}$$

$$\frac{\partial}{\partial z}\left(K_2\frac{k_{rw}}{\mu_w}\frac{\partial\psi_w}{\partial z}\right)=\phi_2 s\left(\frac{\partial\psi_o}{\partial t}-\frac{\partial\psi_w}{\partial t}\right),$$
$$X=\Theta(t);\ t\in J, \tag{40b}$$

$$\nabla\cdot\left(K_3\frac{k_{ro}}{\mu_o}\nabla\psi_o\right)+B_oq+\left(K_2\frac{k_{ro}}{\mu_o}\frac{\partial\psi_o}{\partial z}\right)_{z=0}=-\phi_3 s\left(\frac{\partial\psi_o}{\partial t}-\frac{\partial\psi_w}{\partial t}\right),$$
$$X=(x,y)^{\mathrm T}\in\Omega(t);\ t\in J, \tag{41a}$$

$$\nabla\cdot\left(K_3\frac{k_{rw}}{\mu_w}\nabla\psi_w\right)+B_wq+\left(K_2\frac{k_{rw}}{\mu_w}\frac{\partial\psi_w}{\partial z}\right)_{z=0}=\phi_3 s\left(\frac{\partial\psi_o}{\partial t}-\frac{\partial\psi_w}{\partial t}\right),$$
$$X\in\Omega(t);\ t\in J. \tag{41b}$$

应用本文的计算方法，对胜利油田阳信洼陷多层油资源运移聚集的实际问题进行了数值模拟，结果符合油水运移聚集规律，可清晰地看到油在下层运移聚集的情况，并由中间层进一步运移到上层，最后形成油藏的全过程，其成藏位置基本上和实际油田的位置一致．

阳信洼陷的工区范围(20 516 000，4 128 000)，(50 576 000，4 178 000)，x方向长度为60 km，y方向长度为50 km，实际模拟面积为3 000 km^2．

模拟目标选择为：沙四Ⅰ段、沙四Ⅱ段、沙四Ⅲ段、沙四Ⅳ段4个层位．含油气盆地的油气生成量决定后期形成的油气藏的规模．依据阳信洼陷油气成因特点、储层类型及油气成藏规律，根据数值模拟结果表明：

(ⅰ) 阳信洼陷下步油气勘探要以沙四上亚段为主要的目标方向，实行立体勘探，以期有所突破．

(ⅱ) 具有5个油藏带：南部鼻状构造和中央隆起带、东部火成岩–断层遮挡油气藏带、西部潜山带、深洼区的岩性油藏带．

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[参　考　文　献]

[1]　Ewing R E. The Mathematics of Reservoir Simulation [M]. Philadelphia: SIAM, 1983.

[2] Bredehoeft J D, Pinder G F. Digital analysis of area flow in multiaquifer ground water systems: A quasi- three- dimensional model[J]. Water Resources Research, 1970, **6**(3): 883- 888.

[3] Don W, Emil O F. An iterative quasi- three- dimensional finite element model for heterogeneous multiaquifer systems[J]. Water Resources Research, 1978, **14**(5): 943- 952.

[4] Ungerer P, Dolyiez B, Chenet P Y, et al. A 2- D model of basin petroleum by two- phase fluid flows, application to some case studies[A]. In: Doligez B, Ed. Migration of Hydrocarbon in Sedimentary Basins[C]. Paris: Editions Techniq, 1987, 414- 455.

[5] Ungerer P. Fluid flow, hydrocarbon generation and migration[J]. AAPG Bull, 1990, **74**(3): 309- 335.

[6] Douglas Jr J, Russell T F. Numerical method for convection- dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures[J]. SIAM J Numer Anal, 1982, **19**(5): 871- 885.

[7] Douglas Jr J. Finite difference methods for two- phase incompressible flow in porous media[J]. SIAM J Numer Anal, 1983, **20**(4): 681- 696.

[8] 袁益让. 可压缩两相驱动问题的分数步长特征差分格式[J]. 中国科学, A辑, 1998, **28**(10): 893- 902.

[9] 萨马尔斯基A A, 安德烈耶夫B B. 椭圆型方程差分方法[M]. 武汉大学计算数学教研室 译. 北京: 科学出版, 1984.

[10] 袁益让. 多层渗流方程组合系统的迎风分数步长差分方法和应用[J]. 中国科学, A辑, 2001, **31**(9): 791- 806.

[11] 袁益让. 三维非线性多层渗流方程耦合系统的差分方法[J]. 中国科学, A辑, 2005, **35**(12): 1397- 1423.

Characteristic Finite Difference Method and Application for Moving Boundary Value Problem of the Coupled System

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Abstract: The coupled system of multilayer dynamics of fluids in porous media is to describe the history of oil- gas transport and accumulation in basin evolution. It is of great value in rational evaluation of prospecting and exploiting oil- gas resources. The mathematical model can be described as a coupled system of nonlinear partial differential equations with moving boundary values. A kind of characteristic finite difference schemes was put forward, from which optimal order estimates in norm was derived for the error in the approximate solutions. The research is important both theoretically and practically for the model analysis in the field, for model numerical method and for software development.

Key words: multilayer dynamics of fluids; moving boundary values; characteristic finite difference; error estimates; numerical simulation of oil deposit