

具有延伸表面的驻点流动 和传热问题的级数解

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摘要: 研究了在延伸表面上不可压缩二维驻点流动的动量和热量传输问题. 通过一系列相似变换把轴对称和平面二维驻点流的控制方程组转化为常微分方程组, 利用同伦分析方法求得了速度分布和温度分布的级数解. 结果表明, 当主流流速大于平面延伸的速度时, 就形成了一个边界层, 而当主流流速小于平面延伸的速度时, 却形成一个反边界层. 通过图形和表分析各个物性参数对速度边界层和温度边界层的影响.

关键词: 边界层; 热传导; 驻点; 延伸表面; 同伦分析法

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引 言

具有延伸表面边界层流动问题和传热问题显见于科学研究的诸多领域, 且具有广阔的应用背景. 比如冶金、化工和工业制造工艺中热轧、拔丝、玻璃纤维和造纸、塑料薄膜拉伸、金属聚合物的挤出等领域. 对于二维延伸平面驻点流动, Crane^[1] 首先给出此模型的封闭方程组, 并得到问题的相似解. Brady 和 Acrivos^[2] 求出延伸圆管内层流流动的相似解, 而 Jacobi^[3] 给出了此问题的数值解. Gupta 等^[4], Hussaini 等^[5], McLeod 和 Rajagopal^[6], Chen 和 Char^[7], Riley 和 Weidman^[8], Mahapatra 和 Gupta^[9] 后来也对具有延伸表面的驻点流动问题进行了理论的、实验的和数值的相关研究. 最近 Khan^[10] 针对存在热源和热辐射项的具有延伸表面的粘弹性流体的热传递问题做了研究.

由于具有延伸表面驻点附近的流动和热传导在许多领域有重要的应用, 对该现象的深入了解是有必要的. 本文应用同伦分析法给出该问题的近似解析解, 并将该解析解与 Mahapatra 和 Gupta^[9] 的数值解进行比较, 验证其正确性. 同伦分析方法^[11-14] 是求解非线性问题一个强有力的方法. 一些学者已经利用同伦方法成功的求解了流体力学和传热问题中的一些模型^[15-18].

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1 数学模型

考虑稳定的二维不可压缩黏性流体滞止点附近的流动和传热。流体在远上方垂直流向平板, 流体在平板上冲击点附近形成驻点流。现用流动类型指标 k 来一起讨论平面流动和轴对称流动问题。该问题的控制方程组可写为

$$\frac{(x^k u)}{x} + \frac{(x^k v)}{y} = 0, \quad (1)$$

$$u \frac{u}{x} + v \frac{u}{y} = -\frac{1}{x} \frac{P}{x} + \left[\frac{2u}{x^2} + \frac{1}{x} \left(\frac{u^k}{x^k} \right) + \frac{2u}{y^2} \right], \quad (2)$$

$$u \frac{v}{x} + v \frac{v}{y} = -\frac{1}{y} \frac{P}{y} + \left[\frac{2v}{x^2} + \frac{k}{x} \frac{v}{x} + \frac{2v}{y^2} \right], \quad (3)$$

$$c_p \left[u \frac{T}{x} + v \frac{T}{y} \right] = \frac{2T}{y^2} + \left[\frac{u}{y} \right]^2 \quad (4)$$

相应的初始边界条件为

$$\begin{cases} u(x, 0) = cx, & v(x, 0) = 0, & u(x, \infty) = u_e = \alpha x, \\ T(x, 0) = T_w, & T(x, \infty) = T \end{cases} \quad (5)$$

其中 k 是流动类型指标, $k = 1$, 方程(1) ~ (5) 对应于轴对称流, 而 $k = 0$ 代表着平面流动。选取流体驻点为坐标系原点, x 轴是平板方向, 即 $y = 0$, y 轴垂直于平板。 u 和 v 分别为沿 x 轴和 y 轴的速度。对于轴对称流动, x 为圆柱坐标系中的径向坐标, y 为轴向坐标, 相应的 u 和 v 分别为径向速度和轴向速度。沿 x 轴有两个大小相等, 方向相反的力拉伸平板, 因此在驻点处切向速度为 $u_w = cx$, 其中 c 是正常数。 u_e 是边界层外的主流速度。 T_w 是壁面温度, T 是主流温度。 ρ 是流体密度, μ 是动力粘度, c_p 和 k 分别为定压比热容和热传导系数。

设黏性流体的流函数为

$$f = \frac{x^{k+1}}{k+1} F(y),$$

那么相应的速度分布为

$$u = \frac{1}{x^k} \frac{df}{dy} = \frac{x}{k+1} F'(y), \quad v = -\frac{1}{x^k} \frac{df}{dx} = -F(y) \quad (6)$$

把式(6)代入 x 方向的动量方程(2), 则得

$$\frac{1}{x} \frac{P}{x} = x \left[\frac{F}{k+1} + \frac{FF'}{k+1} - \frac{(F')^2}{(k+1)^2} \right] \quad (7)$$

由于 f 只是 y 的函数, 因此设黏性流体的压力公式为

$$P_0 - P = \frac{1}{2} \left[x^2 \left[\frac{F}{k+1} + \frac{FF'}{k+1} - \frac{(F')^2}{(k+1)^2} \right] + K(y) \right], \quad (8)$$

其中 P_0 是驻点处的压力。把式(6)和(8)同时代入 y 方向的动量方程, 得

$$F'' + FF' - \frac{(F')^2}{k+1} = -a^2(k+1), \quad (9)$$

相应的边界条件为

$$F(0) = 0, \quad F'(0) = (k+1)c, \quad F'(\infty) = a(k+1) \quad (10)$$

进一步, 引入下面的变换:

$$\begin{cases} f(\eta) = \frac{F(y)}{(c)^{1/2}}, & \eta = y \left[\frac{c}{x} \right]^{1/2}, & T = \frac{T - T_w}{T_w - T}, \\ T(x, y) = T_w + (T - T_w) \theta(\eta) \end{cases} \quad (11)$$

由此方程(1)~(5)化简为

$$f''(\eta) + f(\eta)f'(\eta) - \eta f''(\eta)^2 + nd^2 = 0, \quad (12)$$

$$\begin{cases} f(0) + Prf'(0) = 0, \\ f''(0) + Pr[f'(\eta)f''(\eta) - 2f'(\eta)f''(\eta)] = -PrE(f'(\eta))^2, \end{cases} \quad (13)$$

$$\begin{cases} f(0) = 0, f'(0) = 1, f''(0) = d, \\ f'''(0) = 1, f^{(4)}(0) = 0, f^{(5)}(0) = 0, f^{(6)}(0) = 0, \end{cases} \quad (14)$$

其中, $n = \frac{1}{k+1}$, $d = \frac{a}{c}$, $Pr = \frac{c_p}{k}$ 是 Prandtl 数, $E = \frac{c_p(T_w - T_\infty)}{c_p(T_w - T_\infty)}$ 是 Eckert 数

2 同伦分析方法求解 $f(\eta)$, $\theta(\eta)$ 和 $\psi(\eta)$

2.1 零阶变形方程组

下面用同伦分析法求解非线性方程组(12)~(14) 表示一个函数需要一个完备的基函数序列来表达, 根据边界条件(14), 函数 $f(\eta)$, $\theta(\eta)$ 和 $\psi(\eta)$ 可以由以下的基函数

$$\left\{ \exp(-k\eta) \mid i = 0, k = 0 \right\}$$

表示为下列形式:

$$\begin{cases} f(\eta) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{i,k} \exp(-k\eta), \\ \theta(\eta) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} w_{i,k} \exp(-k\eta), \\ \psi(\eta) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} b_{i,k} \exp(-k\eta), \end{cases} \quad (15)$$

这里 $a_{i,k}$, $b_{i,k}$, $w_{i,k}$ ($i, k = 0, 1, 2, \dots$) 均是系数

根据边界条件(14), 以

$$\begin{cases} f_0(\eta) = 1 + d - d \exp(-\eta) - \exp(-\eta), \\ \theta_0(\eta) = \exp(-\eta), \quad \psi_0(\eta) = \exp(-\eta), \end{cases} \quad (16)$$

作为函数 $f(\eta)$, $\theta(\eta)$ 和 $\psi(\eta)$ 的初始近似, 并构造同伦的辅助线性算子

$$\mathcal{L}_f(f) = f + f, \quad \mathcal{L}_\theta(\theta) = \theta - \theta, \quad \mathcal{L}_\psi(\psi) = \psi + \psi, \quad (17)$$

且它们具有如下属性:

$$\begin{cases} \mathcal{L}_f[C_1 + C_2 + C_3 \exp(-\eta)] = 0, \\ \mathcal{L}_\theta[C_4 \exp(\eta) + C_5 \exp(-\eta)] = 0, \\ \mathcal{L}_\psi[C_6 + C_7 \exp(-\eta)] = 0, \end{cases} \quad (18)$$

其中 C_i ($i = 1, 2, 3, 4, 5, 6, 7$) 均是常数

下面构造零阶变形方程组:

$$\begin{cases} (1-q)\mathcal{L}_f[F(\eta, q) - f_0] = qh_f \mathcal{N}_f[F(\eta, q)], \\ F(0, q) = 0, F'(0, q) = 1, F''(\eta, q) = d, \end{cases} \quad (19)$$

$$\begin{cases} (1-q)\mathcal{L}_\theta[\theta^*(\eta, q) - \theta_0] = qh_\theta \mathcal{N}_\theta[\theta^*(\eta, q)], \\ \theta^*(0, q) = 1, \theta^*(\eta, q) = 0, \end{cases} \quad (20)$$

$$\begin{cases} (1-q)\mathcal{L}_\psi[\psi^*(\eta, q) - \psi_0] = qh_\psi \mathcal{N}_\psi[\psi^*(\eta, q)], \\ \psi^*(0, q) = 0, \psi^*(\eta, q) = 0, \end{cases} \quad (21)$$

其中非线性算子 \mathcal{N}_f , \mathcal{N}_θ 和 \mathcal{N}_ψ 分别定义为

$$\mathcal{N}_f [F] = \frac{3F}{3} + F \frac{2F}{2} - n \left[\frac{F}{2} \right]^2 + nd^2, \quad \mathcal{N} [*] = \frac{2 *}{2} + PrF \frac{*}{2}, \quad (22)$$

$$\mathcal{N} [*] = \frac{2 *}{2} + PrF \frac{*}{2} - 2Pr \frac{F}{2} * + PrE \left[\frac{2F}{2} \right]^2, \quad (23)$$

这里 $q \in [0, 1]$ 是一个嵌入系数, h_f, h 和 h 均为辅助非零参数. 当 q 从 0 连续的变到 1, 函数 $F(\cdot, q)$ 就从初始解 $f_0(\cdot)$ 连续变化到精确解 $f(\cdot)$, 相应的 $*(\cdot, q)$ 和 $*(\cdot, q)$ 也从初始解 $o(\cdot, q), o(\cdot)$ 分别连续变化到精确解 $(\cdot), (\cdot)$. 利用 Taylor 级数展开定理, 得

$$F(\cdot, q) = F(\cdot, 0) + \sum_{m=1}^{+\infty} f_m(\cdot) q^m, \quad f_m(\cdot) = \frac{1}{m!} \frac{d^m F(\cdot, q)}{dq^m} \Big|_{q=0}, \quad (24)$$

$$*(\cdot, q) = *(\cdot, 0) + \sum_{m=1}^{+\infty} m(\cdot) q^m, \quad m(\cdot) = \frac{1}{m!} \frac{d^m *(\cdot, q)}{dq^m} \Big|_{q=0}, \quad (25)$$

$$*(\cdot, q) = *(\cdot, 0) + \sum_{m=1}^{+\infty} m(\cdot) q^m, \quad m(\cdot) = \frac{1}{m!} \frac{d^m *(\cdot, q)}{dq^m} \Big|_{q=0} \quad (26)$$

级数(24)~(26)的敛散性强烈的依赖于辅助参数 h_f, h 和 h . 如果参数 h_f, h 和 h 选取的足够合理, 使得级数(24)~(26)在 $q = 1$ 收敛, 那么就有

$$f(\cdot) = f_0(\cdot) + \sum_{k=1}^{+\infty} f_k(\cdot),$$

$$(G) = (o(G) + \sum_{k=1}^{+j} (k(G), H(G) = H_0(G) + \sum_{k=1}^{+j} H_k(G) I$$

2.2 高阶变形方程组

为了方便, 引入下面 3 个向量

$$\begin{aligned} f_n(G) &= \{f_0(G), f_1(G), f_2(G), \dots, f_n(G)\}, \\ (n(G) &= \{(0(G), (1(G), (2(G), \dots, (n(G)\}, \\ H_n(G) &= \{H_0(G), H_1(G), H_2(G), \dots, H_n(G)\} I \end{aligned}$$

在零阶变形方程组(19)~(21)的两边均对 q 求 k 阶导数, 然后令 $q = 0$, 最后在等式两边同除以 $k!$, 得到如下的 k 阶变形方程组:

$$\mathcal{L}[f_k(G) - V_k f_{k-1}(G)] = h_f H_f(G) R_k^f(f_{k-1}), \quad (27)$$

$$\mathcal{L}[(k(G) - V_k (k-1(G)] = h(H((G) R_k^H((k-1), \quad (28)$$

$$\mathcal{L}[H(G) - V_k H_{k-1}(G)] = h(H H(G) R_k^H(H_{k-1}) I \quad (29)$$

它们满足如下的边界条件:

$$f_k(0) = f_k^c(0) = f_k^c(+j) = H_k(0) = H_k(+j) = (k(0) = (k(+j) = 0, \quad (30)$$

其中

$$R_m^f((m-1) = (d_{m-1} + Pr \sum_{s=0}^{m-1} f_s^c(m-1-s), \quad (31)$$

$$R_m^H(f_{m-1}) = f_{m-1} + \sum_{s=0}^{m-1} f_s^c d_{m-1-s} - n \sum_{s=0}^{m-1} f_s^c f_{m-1-s}^c + n(1 - V_m) d^2, \quad (32)$$

$$R_m^H(H_{m-1}) = H_{m-1} + Pr \sum_{s=0}^{m-1} [f_s^c H_{m-1-s} - 2f_s^c H_{m-1-s} + E f_s^c d_{m-1-s}^c], \quad (33)$$

以及

$$V_m = \begin{cases} 0, & m > 1, \\ 1, & m = 1 \end{cases} \quad (34)$$

2.3 递推公式

因为方程(27)~(30)是线性方程组,利用数学软件知函数 $f_m(G)$, $H_m(G)$ 和 $({}_m(G)$ 可以表示为如下形式:

$$f_m(G) = a_{m,0}^0 + \prod_{k=1}^{m+1} \prod_{i=0}^{2m+2-k} a_{m,k}^i G^i \exp(-kG), \quad (35)$$

$$({}_m(G) = w_{m,0}^0 + \prod_{k=1}^{m+1} \prod_{i=0}^{2m+2-k} w_{m,k}^i \exp(-kG), \quad (36)$$

$$H_m(G) = b_{m,0}^0 + \prod_{k=1}^{m+1} \prod_{i=0}^{2m+2-k} b_{m,k}^i \exp(-kG)I \quad (37)$$

将上式(35)~(37)代入方程组(27)~(30),得到如下的递推公式1当 $m \setminus 1$ 时,有

$$a_{m,0}^0 = V_m a_{m-1,0}^0 - \prod_{u=0}^{2m-1} L_{1,1}^u (h_f r_{m-1,1}^u + D_{m,1}^u) - \prod_{k=2}^{m+1} k \prod_{u=1}^{2m+2-k} L_{k,1}^u (D_{m,k}^u - \#_{m,k}^u) +$$

$$\prod_{k=2}^m (k-1) \left[\prod_{u=1}^{2m-k} L_{k,0}^u (h_f r_{m-1,k}^u + \mathcal{S}_{m,k}^u) + \frac{1}{k^2(1-k)} (\mathcal{S}_{m,k}^0 + h_f r_{m-1,k}^0) \right] -$$

$$\prod_{k=2}^m \prod_{u=1}^{2m-k} L_{k,1}^u (h_f r_{m-1,k}^u + \mathcal{S}_{m,k}^u) + \prod_{k=2}^{m+1} (k-1) \prod_{u=0}^{2m+2-k} L_{k,0}^u (D_{m,k}^u - \#_{m,k}^u),$$

$$a_{m,1}^0 = V_m a_{m-1,1}^0 + \prod_{u=0}^{2m-1} L_{1,1}^u (h_f r_{m-1,1}^u + D_{m,1}^u) + \prod_{k=2}^{m+1} k \prod_{u=1}^{2m+2-k} L_{k,1}^u (D_{m,k}^u - \#_{m,k}^u) +$$

$$\prod_{k=2}^m (-k) \left[\prod_{u=1}^{2m-k} L_{k,0}^u (h_f r_{m-1,k}^u + \mathcal{S}_{m,k}^u) + \frac{1}{k^2(1-k)} (\mathcal{S}_{m,k}^0 + h_f r_{m-1,k}^0) \right] +$$

$$\prod_{k=2}^m \prod_{u=1}^{2m-k} L_{k,1}^u (h_f r_{m-1,k}^u + \mathcal{S}_{m,k}^u) - \prod_{k=2}^{m+1} k \prod_{u=0}^{2m+2-k} L_{k,0}^u (D_{m,k}^u - \#_{m,k}^u),$$

$$b_{m,0}^0 = V_m b_{m-1,0}^0 = 0,$$

$$b_{m,1}^0 = V_m b_{m-1,1}^0 - \prod_{k=2}^m \frac{1}{k(k-1)} (h_{m-1,k}^0 - \#_{m,k}^0 + \cdot_{m,k}^0) -$$

$$\prod_{k=2}^m \prod_{u=1}^{2m-k} M_{k,0}^u (h_{m-1,k}^u + \cdot_{m,k}^u + \#_{m,k}^u) + \prod_{k=2}^{m+1} \prod_{u=0}^{2m+2-k} M_{k,0}^u (4_{m,k}^u + O_{m,k}^u),$$

$$w_{m,1}^0 = V_m V_{2m} w_{m-1,1}^0 - \prod_{n=2}^{m+1} \prod_{q=0}^{2m+1-n} N_{m,n}^q \mathcal{P}_n^q, 0,$$

$$w_{m,0}^i = V_m V_{2m+1-i} w_{m-1,0}^i, \quad 0 \leq i \leq 2m+1,$$

$$w_{m,1}^i = V_m V_{2m-i} w_{m-1,1}^i + \prod_{q=i-1}^{2m} N_{m,1}^q \mathcal{P}_{1,k-1}^q, \quad 0 \leq i \leq 2m,$$

$$w_{m,k}^i = V_m V_{2m+1-i-k} w_{m-1,k}^i + \prod_{q=i}^{2m+1-k} N_{m,k}^q \mathcal{P}_{k,i}^q,$$

$$2 \leq k \leq m+1, 0 \leq i \leq 2m+1-k,$$

$$a_{m,1}^i = V_m a_{m-1,1}^i + \prod_{u=i-1}^{2m-1} L_{1,1}^u (h_f r_{m-1,1}^u + D_{m,1}^u), \quad 1 \leq i \leq 2m-1,$$

$$a_{m,1}^{2m} = L_{1,2m-1}^{2m-1} (h_f r_{m-1,1}^{2m-1} + D_{m,1}^{2m-1}), \quad a_{m,1}^{2m+1} = 0,$$

$$b_{m,1}^i = V_m b_{m-1,1}^i + \prod_{u=i-1}^{2m-1} M_{1,i}^u (h_{m-1,1}^u + O_{m,1}^u - 4_{m,1}^u), \quad 1 \leq i \leq 2m-1,$$

$$b_{m,1}^{2m} = M_{1,2m-1}^{2m-1} (h_{m-1,1}^{2m-1} + O_{m,1}^{2m-1} - 4_{m,1}^{2m-1}), \quad b_{m,1}^{2m+1} = 0$$

当 $2 \leq k \leq m$ 时, 有

$$\begin{aligned} a_{m,k}^0 &= V_m a_{m-1,k}^0 + \sum_{u=0}^{2m+2-k} L_{k,0}^u (D_{m,k}^u - \#_{m,k}^u) + \sum_{u=1}^{2m-k} L_{k,0}^u (h_f r_{m-1,k}^u + \mathcal{S}_{m,k}^u) + \\ &\quad \frac{1}{k^2(1-k)} (\mathcal{S}_{m,k}^0 + h_f r_{m-1,k}^0), \\ b_{m,k}^0 &= V_m b_{m-1,k}^0 - \sum_{u=0}^{2m+2-k} M_{k,0}^u (4_{m,k}^u + O_{m,k}^u + F_{m,k}^u) + \\ &\quad \frac{1}{k(1-k)} (\cdot_{m,k}^0 + h H_{m-1,k}^0 - \cdot_{+m,k}^0) + \sum_{u=1}^{2m-k} M_{k,0}^u (h H_{m-1,k}^u + \cdot_{m,k}^u - \cdot_{+m,k}^u) I \end{aligned}$$

当 $2 \leq k \leq m, 2 \leq i \leq 2m-k$ 时, 有

$$\begin{aligned} a_{m,k}^i &= V_m a_{m-1,k}^i + \sum_{u=i}^{2m-k} L_{k,i}^u (h_f r_{m-1,k}^u + \mathcal{S}_{m,k}^u) + \sum_{u=i}^{2m+2-k} L_{k,i}^u (D_{m,k}^u - \#_{m,k}^u), \\ b_{m,k}^i &= V_m b_{m-1,k}^i + \sum_{u=i}^{2m-k} M_{k,i}^u (h H_{m-1,k}^u + \cdot_{m,k}^u - \cdot_{+m,k}^u) + \\ &\quad \sum_{u=i}^{2m+2-k} M_{k,i}^u (4_{m,k}^u + O_{m,k}^u + F_{m,k}^u) I \end{aligned}$$

当 $2 \leq k \leq m, 2m+1-k \leq i \leq 2m+2-k$ 时, 有

$$\begin{aligned} a_{m,k}^i &= \sum_{u=i}^{2m+2-k} L_{k,i}^u (D_{m,k}^u - \#_{m,k}^u), \quad a_{m,m+1}^i = \sum_{u=i}^{m+1} L_{m+1,i}^u (D_{m,m+1}^u - \#_{m,m+1}^u), \\ b_{m,k}^i &= \sum_{u=i}^{2m+2-k} M_{k,i}^u (4_{m,k}^u + O_{m,k}^u + F_{m,k}^u), \\ b_{m,m+1}^i &= \sum_{u=i}^{m+1} M_{m+1,i}^u (4_{m,m+1}^u + O_{m,m+1}^u + F_{m,m+1}^u), \end{aligned}$$

其中

$$\begin{aligned} M_{k,i}^u &= \frac{u!}{i!}, \quad L_{k,i}^u = \frac{i!(u-i+2)}{u!}, \quad k=1, 0 \leq i \leq u+1, \\ M_{k,i}^u &= \frac{u!}{i!(k-1)^{u-i+1}} \left\{ 1 - \left(\frac{1}{n} \right)^{u-i+1} \right\}, \quad k \geq 2, 0 \leq i \leq u, \\ \mathcal{P}_{1,i}^u &= \frac{u!}{(i+1)! 2^{u-i+1}}, \quad u \geq 0, 0 \leq i \leq u, \\ \mathcal{P}_{k,i}^u &= \frac{u!}{i!(k-1)^{u-i+1}} \left\{ 1 - \left(\frac{1}{n} \right)^{u-i+1} \right\}, \quad k \geq 2, 0 \leq i \leq u, \\ L_{k,i}^u &= \sum_{j=i}^u \frac{u!}{i!(k+1)^{u-j+1} (k-1)^{j-i+1}}, \quad k \geq 2, 0 \leq i \leq u, u \geq 0 \end{aligned}$$

且有系数 $D_{m,k}^i, O_{m,k}^i, 4_{m,k}^i, \mathcal{S}_{m,k}^i, \cdot_{m,k}^i, \#_{m,k}^i, F_{m,k}^i, \mathcal{I}_{m,k}^i$ 和 $N_{m,k}^i$ 分别为

$$\begin{aligned} D_{m,k}^i &= \sum_{s=0}^i \sum_{r=\max\{1, k+s-m\}}^6 \sum_{t=\max\{0, i+2s+k-r-2m\}}^6 h_f a_{m-1-s, k-r}^i d_{s,r}^t, \\ &\quad 2 \leq k \leq m+1, \\ D_{m,1}^i &= \sum_{t=0}^{m-1} h_f a_{m-t-1, 0}^i d_{t,1}^0, \quad O_{m,1}^i = \sum_{t=0}^{m-1} Pr h H_{m-t-1, 0}^i s_{t,1}^0, \\ O_{m,k}^i &= \sum_{s=0}^i \sum_{r=\max\{1, k+s-m\}}^6 \sum_{t=\max\{0, i+2s+k-r-2m\}}^6 Pr h H_{m-1-s, k-r}^i s_{s,r}^t, \\ &\quad 2 \leq k \leq m+1, \end{aligned}$$

$$\begin{aligned}
 i_{m,k} &= \begin{matrix} m-1 \\ s=0 \end{matrix} \min \left(s+1, k-1 \right) \begin{matrix} \min \left(2s+2-r, i \right) \\ t = \max \left(0, i+2s+k-r-2m \right) \end{matrix} 2PrhH_{m-1-s, k-r}^i c_{s,r}^i, \\
 & 2 [k [m+1, \\
 4_{m,k}^i &= \begin{matrix} m-k \\ t=0 \end{matrix} 2PrhH_{m-t-k, 0}^0 c_{t+k-1, k}^0, \quad k = 1, \\
 \mathcal{S}_{m,k}^0 &= \begin{matrix} m-k \\ t=0 \end{matrix} h_{f} a_{m-t-k, 0}^0 d_{t+k-1, k}^0, \quad \mathcal{S}_{m,k}^i = \begin{matrix} m-k \\ t = [(i-k+1)/2] \end{matrix} h_{f} a_{m-t-k, 0}^0 d_{t+k-1, k}^i, \quad 1 [i, \\
 \cdot_{m,k}^0 &= \begin{matrix} m-k \\ t=0 \end{matrix} 2PrhH_{m-t-k, 0}^0 s_{t+k-1, k}^0, \quad \cdot_{m,k}^i = \begin{matrix} m-k \\ t = [(i-k+1)/2] \end{matrix} 2PrhH_{m-t-k, 0}^0 s_{t+k-1, k}^i, \\
 & 1 [i, \\
 \#_{m,k}^i &= \begin{matrix} m-1 \\ s=0 \end{matrix} \min \left(s+1, k-1 \right) \begin{matrix} \min \left(2s+2-r, i \right) \\ t = \max \left(0, i+2s+k-r-2m \right) \end{matrix} h_{f} n c_{m-1-s, k-r}^i c_{s,r}^i, \\
 F_{m,k}^i &= \begin{matrix} 6 \\ s=0 \end{matrix} \begin{matrix} r = \max \left(1, k+s-m \right) \\ m-1 \end{matrix} \min \left(s+1, k-1 \right) \begin{matrix} t = \max \left(0, i+2s+k-r-2m \right) \\ \min \left(2s+2-r, i \right) \end{matrix} PrEh_{f} d_{m-1-s, k-r}^i d_{s,r}^i, \\
 \mathcal{I}_{m,k}^i &= \begin{matrix} 6 \\ s=0 \end{matrix} \begin{matrix} r = \max \left(1, k+s-m \right) \\ m-1 \end{matrix} \min \left(s+1, k-1 \right) \begin{matrix} t = \max \left(0, i+2s+k-r-2m \right) \\ \min \left(2s+2-r, i \right) \end{matrix} PrEh (a_{m-1-s, k-r}^i g_{s,r}^i), \\
 \dot{N}_{m,k} &= h((q_{m-1, k}^i + Pr \mathcal{I}_{m,k}^i) I
 \end{aligned}$$

这里还定义系数 $c_{m,k}^i, d_{m,k}^i, r_{m,k}^i, s_{m,k}^i, t_{m,k}^i, g_{m,k}^i$ 和 $q_{m,k}^i$ 它们分别为

$$\begin{aligned}
 c_{m,k}^i &= (i+1) a_{m,k}^{i+1} - k a_{m,k}^i, \quad 0 [i [2m, \quad c_{m,k}^{2m+1} = -k a_{m,k}^{2m+1}, \\
 d_{m,k}^i &= (i+1)(i+2) a_{m,k}^{i+2} - 2k(i+1) a_{m,k}^{i+1} + k^2 a_{m,k}^i, \quad 0 [i [2m-1, \\
 d_{m,k}^{2m} &= -2k(2m+1) a_{m,k}^{2m+1} + k^2 a_{m,k}^{2m}, \quad d_{m,k}^{2m+1} = k^2 a_{m,k}^{2m+1}, \\
 r_{m,k}^i &= (i+1) d_{m,k}^{i+1} - k d_{m,k}^i, \quad 0 [i [2m, \quad r_{m,k}^{2m+1} = -k d_{m,k}^{2m+1}, \\
 s_{m,k}^i &= (i+1) b_{m,k}^{i+1} - k b_{m,k}^i, \quad 0 [i [2m, \quad s_{m,k}^{2m+1} = -k b_{m,k}^{2m+1}, \\
 t_{m,k}^i &= (i+1)(i+2) b_{m,k}^{i+2} - 2k(i+1) b_{m,k}^{i+1} + k^2 b_{m,k}^i, \quad 0 [i [2m-1, \\
 t_{m,k}^{2m} &= -2k(2m+1) b_{m,k}^{2m+1} + k^2 b_{m,k}^{2m}, \quad t_{m,k}^{2m+1} = k^2 b_{m,k}^{2m+1}, \\
 g_{m,k}^i &= (i+1) w_{m,k}^{i+1} - k w_{m,k}^i, \quad 0 [i [2m, \quad g_{m,k}^{2m+1} = -k w_{m,k}^{2m+1}, \\
 q_{m,k}^i &= -(i+1) g_{m,k}^{i+1} - k g_{m,k}^i, \quad 0 [i [2m, \quad q_{m,k}^{2m+1} = -k g_{m,k}^{2m+1}
 \end{aligned}$$

当 $m = 0$ 时, 由初始近似解 (16) 得

$$a_{0,0}^0 = 1, a_{0,0}^1 = d, a_{0,1}^0 = -1, a_{0,1}^1 = -d, b_{0,1}^0 = 1, w_{0,1}^0 = 11$$

应用前面的递推公式, 可以依次求出 $m \setminus 1$ 时的各个系数 $a_{m,k}^i, b_{m,k}^i, w_{m,k}^i$ 显然, 当 $N \rightarrow \infty$ 时, 就得到了方程组 (12) ~ (14) 的级数解

$$f(G) = \lim_{N \rightarrow \infty} \sum_{m=0}^N f_m(G) = \lim_{N \rightarrow \infty} \left[\sum_{m=0}^N a_{m,0}^0 + \sum_{k=1}^{N+1} e^{-kG} \left[\sum_{m=k-1}^{2N-2m+1-k} \sum_{i=0}^6 a_{m,k}^i \right] \right], \quad (38)$$

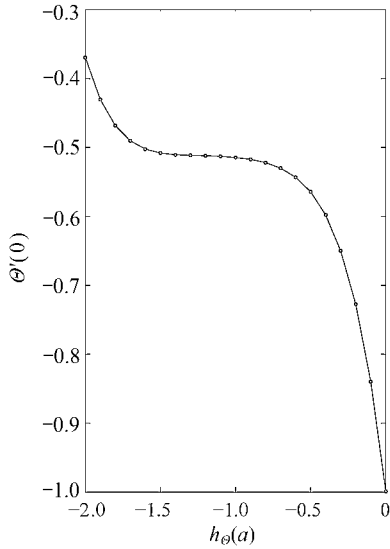
$$((G) = \lim_{N \rightarrow \infty} \sum_{m=0}^N ((G) = \lim_{N \rightarrow \infty} \left[\sum_{m=0}^N w_{m,0}^0 + \sum_{k=1}^{N+1} e^{-kG} \left[\sum_{m=k-1}^{2N-2m+1-k} \sum_{i=0}^6 w_{m,k}^i \right] \right], \quad (39)$$

$$H(G) = \lim_{N \rightarrow \infty} \sum_{m=0}^N H_m(G) = \lim_{N \rightarrow \infty} \left[\sum_{m=0}^N b_{m,0}^0 + \sum_{k=1}^{N+1} e^{-kG} \left[\sum_{m=k-1}^{2N-2m+1-k} \sum_{i=0}^6 b_{m,k}^i \right] \right] I \quad (40)$$

3 结果分析

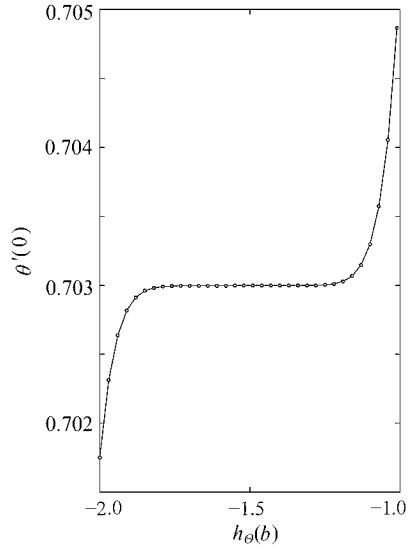
正如 Liao^[11] 所说, 同伦分析法的敛散性和收敛速度强烈的依赖于辅助参数 h_f, h_H 和 h_c 的

合适选取1 图1给出了确定参数 h_H 和 h_l 的 h 曲线图1 参数 h_H 和 h_l 的取值范围分别为 $-1.15 < h_l < -0.5$ 和 $-1.8 < h_H < -1.21$ 本文选取参数 $h_f = 0.13$, $h_H = -1.4$ 和 $h_l = -1.01$ 由表1可以看到,利用 $h_f = 0.13$,同伦级数解与数值解以及文献[9]中解吻合的较好1



$Pr = 0.7, d = 0.2$

图 1(a) 9阶同伦近似 $\theta(0)$ 的 h 曲线图



$Pr = 0.5, d = 0.2, E = 5.0$

图 1(b) 9阶同伦近似 $\theta(0)$ 的 h 曲线图

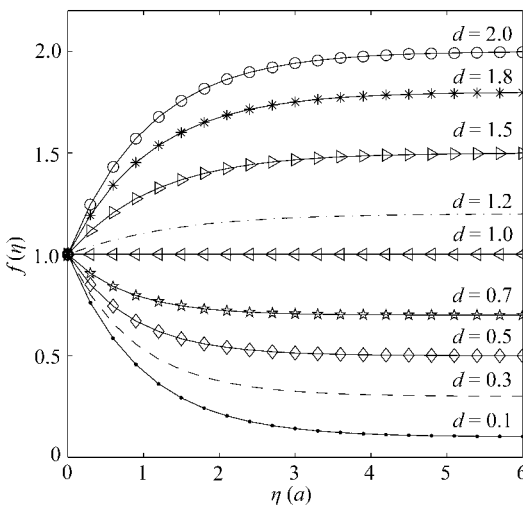


图 2(a) 当 d 取不同值时,平面流动的切向速度分布图

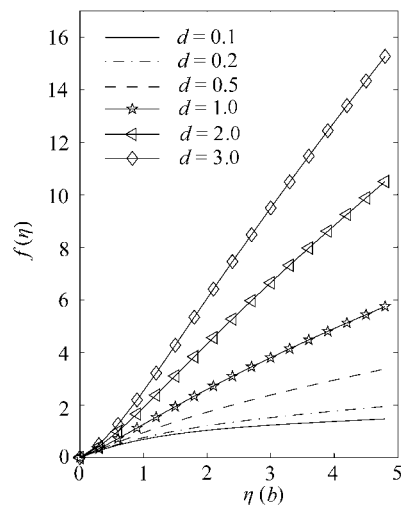
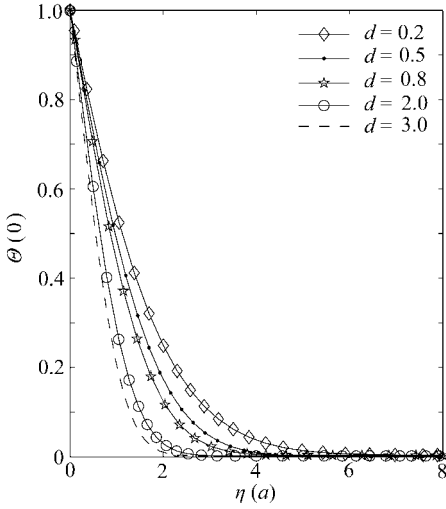


图 2(b) 当 d 取不同值时,平面流动的纵向速度 $f(G)$ 分布图

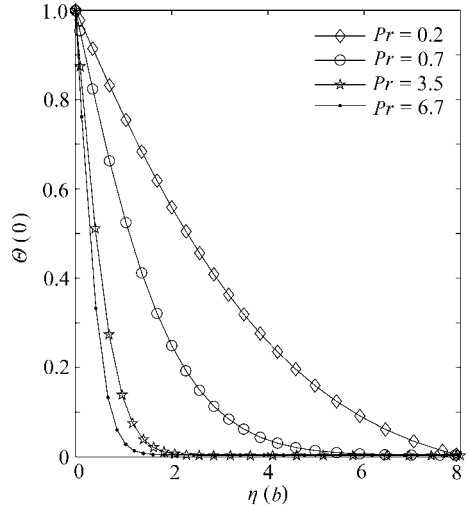
下面,主要讨论参数 $d = a/c$, Prandtl 数 Pr 和 Eckert 数 E 以及流动类型指标 k 对动量和能量边界层的影响1 图2描绘的是 d 取不同值时无量纲切向速度 $f^c(G)$ 与纵向速度 $f(G)$ 的分布曲线1 由图2(a)可以看出,无量纲切向速度 $f^c(G)$ 与延伸速度与主流速度之比有关1 当主流速度大于延伸速度时,形成一个边界层1 而且当 a/c 减少时,无量纲速度边界层变厚1 然而当 $a/c < 1$ 时,形成一个反边界层1 对于每一个确定的大于1的数 a/c , $f^c(G)$ 是 G 的增函数,而对于每一个确定的小于1的数 a/c , $f^c(G)$ 是 G 的减函数1 从图2(b)可以看出,纵向速

度 $f(G)$ 是 a/c 的增函数, 同时, 对于每一个固定的 a/c , 函数 $f(G)$ 也是 G 的增函数¹



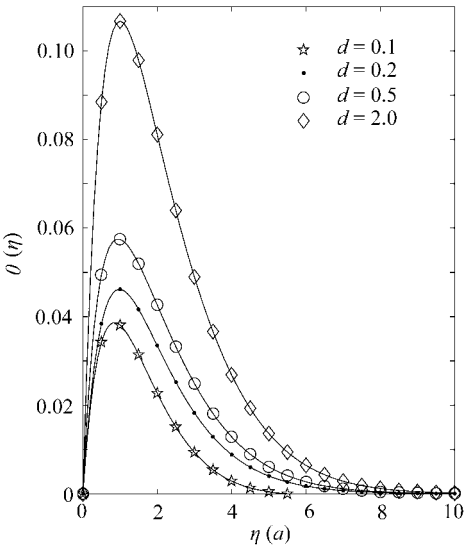
$Pr = 0.7, n = 1$

图 3(a) 变量 $\theta(G)$ 对于不同 d 的曲线分布图



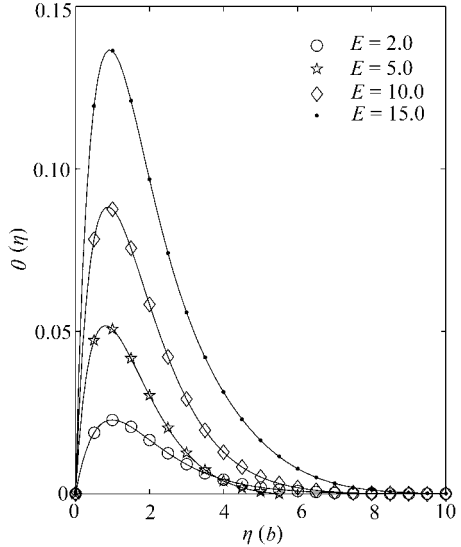
$d = 0.2, n = 1$

图 3(b) 变量 $\theta(G)$ 对于不同 Pr 的曲线分布图



$Pr = 0.05, E = 2.0, n = 1$

图 4(a) 变量 $H(G)$ 对于不同 d 的曲线分布图



$Pr = 0.05, d = 2.0, n = 1$

图 4(b) 变量 $H(G)$ 对于不同 E 的曲线分布图

图 3 和图 4 描绘的是相对于不同参数 Pr 和 E 的无量纲温度 $\theta(G)$ 和 $H(G)$ 的分布曲线¹ 从图 3 可以看出, 对于固定的 a/c , 忽略耗散项的无量纲温度 $\theta(G)$ 随着 Prandtl 数 Pr 的增加而减少, 这表明当 Prandtl 数 Pr 增大时, 温度边界层变薄; 而当 Prandtl 数 Pr 减少时, 温度边界层变厚¹ 对于固定的 Prandtl 数 Pr , $\theta(G)$ 也随着 d 的增加而减少¹ 这表面延伸表面能增强换热¹ 当考虑能量耗散项时, 无量纲温度 $H(G)$ 随着 Eckert 数 E 的增加而增加¹

最后, 还计算了壁面剪切力¹ 从表 1 中还可以看到, 当 $a/c > 1$ 时, 壁面剪切力 $f^d(0)$ 是

a/c 的增函数, 而当 $a/c > 1$ 时, 壁面剪切力 $f^d(0)$ 是 $a/c > 1$ 的减函数

4 结 论

本文研究了在延伸表面上不可压缩二维驻点流的动量和热量传输问题。其中表面延伸的速度为 cx , 主流的速度为 ax 。当主流速度大于延伸速度时, 形成一个边界层。反过来, 形成一个反边界层。而且随着比值 $d = a/c$ 的减少, 无量纲速度边界层变厚。忽略能量耗散项时, 无量纲温度 (G) 随着 Prandtl 数 Pr 数的增加而减少, 表明当 Prandtl 数 Pr 数增大时, 温度边界层变薄; 而当 Prandtl 数 Pr 数减少时, 温度边界层变厚。而对于固定的 Pr 数, 无量纲温度 (G) 也随着 a/c 的增加而减少。当考虑能量耗散项时, 无量纲温度 $H(G)$ 随着 Eckert 数 E 的增加而增加。最后计算了壁面热流和壁面剪切力。当主流速度固定时, 如果 $a/c > 1$, 壁面剪切力 $f^d(0)$ 随着壁面延伸速度的增加而增大, 而当 $a/c < 1$ 时, 壁面剪切力 $f^d(0)$ 随着壁面延伸速度的增加而减少。

表 1 壁面剪切力 $f^d(0)$

d	HAM 解 $n = 1$	Guptal ⁹⁾ $n = 1$	HAM 解 $n = 1/2$	数值解 $n = 1/2$
0.1	- 0.969 5	- 0.969 4	- 0.806 3	- 0.793 3
0.2	- 0.918 4	- 0.918 1	- 0.750 2	- 0.746 8
0.5	- 0.667 4	- 0.667 3	- 0.542 0	- 0.533 4
0.8	- 0.293 9		- 0.223 9	- 0.235 5
2.0	2.017 8	2.017 5	1.580 6	1.561 3
3.0	4.729 5	4.729 3	3.652 6	3.633 4
5.0	11.751 4		8.9721	8.960 8

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A n a l y t i c S o l u t i o n o f S t a g n a t i o n - P o i n t F l o w a n d
H e a t T r a n s f e r O v e r a S t r e t c h i n g S h e e t b y
M e a n s o f H o m o t o p y A n a l y s i s M e t h o d

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Abstract: The steady two-dimensional stagnation-point flow of an incompressible viscous fluid towards a stretching sheet whose velocity is proportional to the distance from the slit is concerned. The governing system of partial differential equations was first transformed into a system of dimensionless ordinary differential equations. The analytical solutions for the velocity distribution and dimensionless temperature profiles were obtained for the various values of the ratio of free stream velocity and stretching velocity, Prandtl number, Eckert number and dimensionality index in the series forms with the help of homotopy analysis method(HAM). It is shown that a boundary layer is formed when the free stream velocity exceeds the stretching velocity and an inverted boundary layer is formed when the free stream velocity is less than the stretching velocity. Graphs are plotted to discuss the effects of different parameters.

Key words: boundary-layer; heat transfer; stagnation point; stretching sheet; homotopy analysis method