应用数学和力学编委会编
重 庆 出 版 社 出 版

文章编号: 1000_0887(2003) 12_1267_09

板弯曲问题的具两组高阶基本解序列的 MRM 方法*

丁方允[1], 丁 睿[2], 李炳杰[1]

(1. 兰州大学 数学系, 兰州 730000;
2. 苏州大学 数学科学学院, 苏州 215006)

(程昌钧推荐)

摘要: 讨论了双参数地基上薄板弯曲问题• 利用两组高阶基本解序列, 即调和及重调和基本解序列, 采用多重替换方法(MRM 方法) , 得到了板弯曲问题的 MRM 边界积分方程• 证明了该方程与边值问题的常规边界积分方程是一致的• 因此由常规边界积分方程的误差估计即可得到板弯曲问题 MRM 方法的收敛性分析• 此外该方法还可推广到具多组高阶基本解序列的情形•

关 键 词: 板弯曲问题; MRM 方法; 边界积分方程; 高阶基本解序列
中图分类号: O175. 8 文献标识码: A

引 言

众所周知[1,2] Helmholtz 方程

$$\Delta u(x) + k^2 u(x) = 0 \qquad (x \in \Omega:\ R^2 \text{ 中有界开区域})$$

的基本解 $u^*(x, y) = -iH_0^{(2)}(k|x-y|)/4$ 的形式较为复杂, 给数值计算带来不便• 如果采用形式简单的 Laplace 方程的基本解 $u_0^*(x, y) = -\ln|x-y|/(2\pi)$, 则方程在闭区域 Ω 上的解的表达式为:

$$c(y)u(y) + \int_\Gamma \left[u(x)\frac{\partial u_0^*(x, y)}{\partial n_x} - u_0^*(x, y)\frac{\partial u(x)}{\partial n} \right] \mathrm{d}s_x =$$
$$- k^2 \int_\Omega u(x)u_0^*(x, y)\,\mathrm{d}\Omega_x•$$

由于右端出现含未知函数 u 的区域积分, 因此无法充分发挥 BEM 所具有的降维优点• 为消除区域积分, 文[3] 给出了 Laplace 方程高阶基本解序列, 对区域积分中的基本解经多重替换可使其趋于零• 出于相同目的, 文[4]、[5] 引入重调和方程高阶基本解序列, 分别建立了结构声学的耦合问题和弹性薄板谐振问题的 MRM 方法• 而关于 MRM 方法收敛性分析的讨论至今未见•

本文将给出双参数地基上薄板弯曲边值问题的 MRM 方法• 由于问题的控制方程既含重

* 收稿日期: 2001_11_27; 修订日期: 2003_07_02
基金项目: 国家自然科学基金资助项目(10201026) ; 国家自然科学基金资助预研项目(T4107015)
作者简介: 丁方允(1941—) , 男, 浙江定海人, 教授, 已发表论文近 30 篇(E_mail: fyding@ hotmail. com. cn) .

调和算子又含 Laplace 算子, 因而需引入相应两组高阶基本解序列. 与[3~5]不同的是, 这里既建立了每组序列的替换公式, 还给出了两组序列间的替换公式. 反复使用替换公式便可导出解的 MRM 积分表达式和 MRM 边界积分方程. 本文将证明MRM 方法所导出的解在闭区域 Ω 上的表达式与常规边界积分方法得到的是一致的. 由此可见由常规边界积分方程的误差估计方法([6, 7])便可得到板弯曲问题MRM 方法的收敛性分析. 从而说明了本文方法是可靠的并且可将此法推广到具两组以上高阶基本解序列的情形.

以下总假设 Ω 是平面中的区域, 其边界 Γ 是逐段光滑的闭曲线. $\boldsymbol{n}$ 表示 Γ 上的单位外法向量.

1　常规边界积分方法

双参数地基上薄板弯曲问题的控制微分方程可写成如下形式

$$\Delta^2 w - s\Delta w + k^2 w = f \qquad (\Omega), \tag{1}$$

其中 $s = G_p/D$, $k^2 = K/D$, $f = q/D$, 式中 D 为板的弯曲刚度, q 为荷载, G_p、K 为表征土壤模型的两个弹性参数. 工程中一般采用的有 Filonenko_Berodich 模型, Pasternak 模型, Vazov 模型及 Reissner 模型. 模型不同双参数 (G_p, K) 的取值也不同. 对方程(1) 中的参数 s、k 要求满足条件: $4k^2 - s^2 > 0$. 为简便计, 这里讨论的边界条件为:

$$w\,|_\Gamma = w_0,\quad \left.\frac{\partial w}{\partial n}\right|_\Gamma = \theta_0, \tag{2}$$

其中 w_0、θ_0 分别是挠度和转角.

线性偏微分方程(1) 的基本解是

$$w^*(x, y) = [\mathrm{H}_0^1(a_1 r) - \mathrm{H}_0^1(a_2 r)]/(4\sqrt{4k^2 - s^2}), \tag{3}$$

式中 $r = |x - y|$, $\mathrm{H}_0^1(z)$ 表示第一类零阶 Hankle 函数,

$$\begin{cases} a_1^2 = -\dfrac{s}{2} + \dfrac{1}{2}\sqrt{4k^2 - s^2}\,\mathrm{i}, \\[2ex] a_2^2 = -\dfrac{s}{2} - \dfrac{1}{2}\sqrt{4k^2 - s^2}\,\mathrm{i}, \end{cases}$$

i 是虚数单位. 第一类零阶 Hankle 函数有如下表达式

$$\mathrm{H}_0^1(z) = \mathrm{J}_0(z) + \mathrm{i}\mathrm{Y}_0(z),$$

其中 Bessel 函数 $\mathrm{J}_0(z)$ 和 Neumann 函数 $\mathrm{Y}_0(z)$ 可分别表示为

$$\mathrm{J}_0(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m},$$

$$\mathrm{Y}_0(z) = \frac{2}{\pi}\mathrm{J}_0(z)\ln\frac{z}{2} - \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m}\Psi(m+1);$$

式中　$$\Psi(m+1) = \sum_{k=1}^{m}\frac{1}{k} - c_E,$$

$c_E = \ln 2 = 0.577\,21\cdots$, 为 Euler 常数.

文[8]中先将 $\mathrm{J}_0(z)$ 和 $\mathrm{Y}_0(z)$ 代入基本解 w^* 的表达式, 然后作如下变换

$$a_1 = \rho_1\exp(\mathrm{i}\beta_1),\quad a_2 = \rho_2\exp(\mathrm{i}\beta_2).$$

经简单计算不难看出基本解 w^* 是实函数且具如下级数形式:

$$w^*(r) = \frac{1}{4\sqrt{4k^2 - s^2}}\left[\frac{2}{\pi}(\beta_2 - \beta_1)\sum_{m=0}^{\infty}\frac{(-1)^m k^m}{(m!)^2 4^m}r^{2m}\cos 2m\beta_1 - \right.$$

$$\frac{4}{\pi}\ln\frac{\sqrt{k}r}{2}\sum_{m=0}^{\infty}\frac{(-1)^m k^m}{(m!)^2 4^m}r^{2m}\sin 2m\beta_1+$$
$$\left.\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m k^m}{(m!)^2 4^m}\Psi(m+1)r^{2m}\sin 2m\beta_1\right] \cdot \tag{4}$$

接着按常规边界积分方法, 对方程(1) 的解 w 及基本解 w^* 使用第二 Green 公式得到解在闭区域 Ω 上的表达式

$$\begin{aligned} c(y)w(y) = {} & \iint_{\Omega} w^*(x,y)p(x)\,\mathrm{d}\Omega - \int_{\Gamma}\Big[w^*(x,y)\frac{\partial}{\partial n}\Delta w(x) - \\ & w(x)\frac{\partial}{\partial n_x}\Delta w^*(x,y) + \Delta w^*(x,y)\frac{\partial}{\partial n}w(x) - \\ & \Delta w(x)\frac{\partial}{\partial n_x}w^*(x,y)\Big]\mathrm{d}\Gamma_x + \\ & s\int_{\Gamma}\Big[w^*(x,y)\frac{\partial}{\partial n}w(x) - \frac{\partial}{\partial n}w^*(x,y)\Big]\mathrm{d}\Gamma_x \cdot \end{aligned} \tag{5}$$

式中当 $y\in\Omega$ 时, $c(y)=1$, 这里(5) 式成为问题(1) 的解 w 的积分表达式; 当 $y\in\Gamma$ 时, 若 y 是光滑边界点, $c(y)=1/2$, 若 y 是边界尖点, $c(y)=\theta/(2\pi)$, $\theta=\theta(y)$ 表示点 y 处左、右切线的外夹角, 此时(5) 式为问题(1) 的边界积分方程· 通常称(5) 式为问题(1) 在闭区域 Ω 上解的边界积分表达式·

2 具两组高阶基本解序列的 MRM 方法

这里先引入两类高阶基本解序列

$$\begin{cases} w_j^*(r) = \dfrac{\ln r - l(2j+1)}{2\pi 4^{2j+1}((2j+1)!)^2}r^{2(2j+1)} \\ m_j^*(r) = \dfrac{\ln r - l(2j)}{2\pi 4^{2j}((2j)!)^2}r^{4j} \end{cases} \qquad (j=0,1,2,\cdots), \tag{6}$$

式中

$$r=|x-y|,\quad l(0)=0,\quad l(m)=\sum_{k=1}^{m}\frac{1}{k}\cdot$$

容易验证高阶基本解序列 w_j^* 和 m_j^* 具如下性质·

引理 1 w_0^* 和 m_0^* 分别是重调和算子和 Laplace 算子的基本解并分别满足方程:

$$\Delta^2 w_0^*(x,y)=\delta(y-x),\quad \Delta m_0^*(x,y)=\delta(y-x)\cdot$$

高阶基本解序列 w_j^* 和 m_j^* 满足下列替换公式:

$$\Delta^2 w_{j+1}^* = w_j^*,\quad \Delta^2 m_{j+1}^* = m_j^*,\quad \Delta w_j^* = m_j^* \qquad (j=0,1,2,3,\cdots)\cdot \tag{7}$$

在描述 MRM 方法时要用到下列两类记号: 设 $u_l^*=u_l^*(x,y)$, 记

$$I(u_l^*)(y)=\int_{\Gamma}\Big[\Delta w\frac{\partial u_l^*}{\partial n_x}-u_l^*\frac{\partial\Delta w}{\partial n}+w\frac{\partial\Delta u_l^*}{\partial n_x}-\Delta u_l^*\frac{\partial w}{\partial n}\Big]\mathrm{d}s_x,$$

$$J(u_l^*)(y)=\int_{\Gamma}\Big[u_l^*\frac{\partial w}{\partial n}-w\frac{\partial u_l^*}{\partial n_x}\Big]\mathrm{d}s_x,\quad P(u_l^*)(y)=\iint_{\Omega}u_l^* p\,\mathrm{d}x,$$

$$W_l(y)=\iint_{\Omega}w_l^* w\,\mathrm{d}x,\quad M_l(y)=\iint_{\Omega}m_l^* w\,\mathrm{d}x\cdot$$

以上 $l=0,1,2,\cdots$·

另一类记号是由 $A_0=1$, $B_0=0$ 按下列递推公式产生的:

$$\begin{cases} A_l = (s^2 - k^2)A_{l-1} - sk^2 B_{l-1}, \\ B_l = sA_{l-1} - k^2 B_{l-1}. \end{cases}$$

并记

$$q_l^*(r) = A_l w_l^* + B_l m_l^*, \quad R_l(y) = B_{l+1} M_l - k^2 A_l W_l.$$

以上 $l = 1, 2, \cdots$.

现在将控制方程(1)左端第2、第3项移至右端, 利用第二 Green 公式及重调和算子的基本解 w_0^* 可将问题(1)在闭区域 Ω 上的解 w 表示成

$$\begin{aligned} c(y)w(y) = & \iint_\Omega (w\Delta^2 w_0^* - w_0^* \Delta^2 w)\mathrm{d}x + \iint_\Omega w_0^* p\,\mathrm{d}x + \\ & s\iint_\Omega (w_0^* \Delta w - w\Delta w_0^*)\mathrm{d}x + \iint_\Omega w\Delta w_0^*\,\mathrm{d}x - k^2\iint_\Omega w w_0^*\,\mathrm{d}x = \\ & I(q_0^*) + sJ(q_0^*) + P(q_0^*) + [(sA_0 - k^2 B_0)M_0 - k^2 A_0 W_0] = \\ & \int_\Gamma \left[\Delta w \frac{\partial w_0^*}{\partial n_x} - w_0^* \frac{\partial \Delta w}{\partial n} + w\frac{\partial \Delta w_0^*}{\partial n_x} - \Delta w_0^* \frac{\partial w}{\partial n}\right]\mathrm{d}s_x + \\ & S\int_\Gamma \left[w_0^* \frac{\partial w}{\partial n} - w\frac{\partial w_0^*}{\partial n_x}\right]\mathrm{d}s_x + \iint_\Omega w_0^* p\,\mathrm{d}x + R_0, \end{aligned} \tag{8}$$

式中 $c(y)$ 的取值与式(5)中的相同. 上面第2个等号的推导中, 用到了替换公式 $\Delta w_0^* = m_0^*$ 及引理1.

对 R_0 使用替换公式(7)得

$$\begin{aligned} R_0 = & B_1 \iint_\Omega w\Delta^2 m_1^*\,\mathrm{d}x - k^2 A_0 W_0 = \\ & B_1(I(m_1^*) + P(m_1^*) + sJ(m_1^*)) + \\ & sB_1\iint_\Omega m_1^* \Delta^2 w\,\mathrm{d}x - k^2 B_1 M_1 - k^2 A_0 W_0 = \\ & B_1(I(m_1^*) + P(m_1^*) + sJ(m_1^*)) - k^2 B_1 M_1 + \\ & sB_1\iint_\Omega w\Delta^2 w_1^*\,\mathrm{d}x - k^2 A_0\iint_\Omega w\Delta^2 w_1^*\,\mathrm{d}x = \\ & B_1(I(m_1^*) + sJ(m_1^*) + P(m_1^*)) - k^2 B_1 M_1 + \\ & A_1\left(I(w_1^*) + sJ(w_1^*) + P(w_1^*) + s\iint_\Omega w\Delta^2 w_1^*\,\mathrm{d}x - k^2 W_1\right) = \\ & B_1(I(m_1^*) + sJ(m_1^*) + P(m_1^*)) + A_1(I(w_1^*) + \\ & sJ(w_1^*) + P(w_1^*)) - k^2 B_1 M_1 + A_1(sM_1 - k^2 W_1) = \\ & B_1(I(m_1^*) + sJ(m_1^*) + P(m_1^*)) + \\ & A_1(I(w_1^*) + sJ(w_1^*) + P(w_1^*)) + R_1 = \\ & I(q_1^*) + sJ(q_1^*) + P(q_1^*) + R_1 = \\ & \int_\Gamma \left[\Delta w \frac{\partial q_1^*}{\partial n_x} - q_1^* \frac{\partial \Delta w}{\partial n} + w\frac{\partial \Delta q_1^*}{\partial n_x} - \Delta q_1^* \frac{\partial w}{\partial n}\right]\mathrm{d}s_x + \\ & s\int_\Gamma \left[q_1^* \frac{\partial w}{\partial n} - w\frac{\partial q_1^*}{\partial n_x}\right]\mathrm{d}s_x + \iint_\Omega q_1^* p\,\mathrm{d}x + R_1. \end{aligned} \tag{9}$$

将 R_0 代入(8)式得

$$c(y)w(y)=\sum_{l=0}^{1}\left[I(q_l^*)+sJ(q_l^*)+P(q_l^*)\right]+R_1=$$
$$\sum_{l=0}^{1}\left\{\int_\Gamma\left[\Delta w\frac{\partial q_l^*}{\partial n_x}-q_l^*\frac{\partial\Delta w}{\partial n}+w\frac{\partial\Delta q_1^*}{\partial n_x}-\Delta q_l^*\frac{\partial w}{\partial n}\right]\mathrm{d}s_x+\right.$$
$$\left.s\int_\Gamma\left[q_l^*\frac{\partial w}{\partial n}-w\frac{\partial q_l^*}{\partial n_x}\right]\mathrm{d}s_x+\iint_\Omega q_l^*p\,\mathrm{d}x\right\}+R_1.$$

与(9)式的推导类似,反复利用替换公式(7)可归纳地得到

引理 2 递推公式

$$R_l=I(q_l^*)+sJ(q_l^*)+P(q_l^*)+R_{l+1}\tag{10}$$

成立, $l=0,1,2,\cdots$.

对(8)式利用引理 2 经 n 次递推得

$$c(y)w(y)=\sum_{l=0}^{n}\left[I(q_l^*)+sJ(q_l^*)+P(q_l^*)\right]+R_n$$
$$(y\in\Omega\ (n=0,1,2,\cdots)).$$

定理 1 设 Ω是平面中的有界区域, Γ是光滑边界曲线, 方程(1)中的参数 s、k 满足 $4k^2-s^2>0$, 则在闭区域 Ω 上, (1)的解 w 可表为

$$c(y)w(y)=\sum_{l=0}^{\infty}\left\{\int_\Gamma\left[\Delta w\frac{\partial q_l^*}{\partial n_x}-q_l^*\frac{\partial\Delta w}{\partial n}+w\frac{\partial\Delta q_l^*}{\partial n_x}-\Delta q_l^*\frac{\partial w}{\partial n}\right]\mathrm{d}s_x+\right.$$
$$\left.s\int_\Gamma\left[q_l^*\frac{\partial w}{\partial n}-w\frac{\partial q_l^*}{\partial n_x}\right]\mathrm{d}s_x+\iint_\Omega q_l^*p\,\mathrm{d}x\right\}\qquad(y\in\Omega).\tag{11}$$

证明 记 $p=\max\left\{|s^2-k^2|,sk^2,s,k^2\right\}$, 则有 $|A_n|,|B_n|<(2p)^2$, $n=1,2,\cdots$. 于是

$$|R_n|\leqslant(2p)^{n+1}(|M_n|+|W_n|).$$

容易看出, 由于 Ω有界, 故$(2p)^{n+1}m_n^*(r)$ 和$(2p)^{n+1}w_n^*(r)\to0$, $(n\to\infty)$, 所以 $R_n\to0$. 进而(11)式成立. 证毕.

称(11)式边值问题(1)~(3)在闭区域 Ω 上解的MRM 积分表达式.

引理 3 若边界 Γ 光滑, 且 $\varphi\in C^1(\Gamma)$, 则积分

$$u(y)=\int_\Gamma\varphi(x)\frac{\partial}{\partial n_x}\Delta w_0^*(x,y)\mathrm{d}s_x\qquad(y\in\Gamma)$$

的偏导数可表示为如下弱奇异积分:

$$\frac{\partial u(y)}{\partial y_i}=\int_\Gamma\frac{\partial}{\partial y_j}w_0^*(x,y)\frac{\partial w}{\partial\tau_x}\mathrm{d}s_x\qquad((i,j=1,2;i\neq j)(y\in\Gamma)).\tag{12}$$

式中 τ_x 表示边界点x 处的单位切向量.

证明 不失一般, 仅对 $i=1$ 证明即可. 注意到$\partial r/\partial y_i=-\partial r/\partial x_i$ 及

$$\frac{\partial^2m_0^*}{\partial x_1\partial n_x}=-\frac{\partial^2m_0^*}{\partial x_2\partial\tau_x},$$

则有

$$\frac{\partial u(y)}{\partial y_1}=\int_\Gamma\varphi(x)\frac{\partial}{\partial y_1}\left(\frac{\partial m_0^*}{\partial n_x}\right)\mathrm{d}s_x=\int_\Gamma\varphi(x)\frac{\partial}{\partial\tau_x}\left(\frac{\partial m_0^*}{\partial x_2}\right)\mathrm{d}s_x=$$
$$\int_\Gamma\varphi(x)\left[\frac{\partial}{\partial x_1}\left(\frac{\partial m_0^*}{\partial x_2}\right)\tau_x^1+\frac{\partial}{\partial x_2}\left(\frac{\partial m_0^*}{\partial x_2}\right)\tau_x^2\right]\mathrm{d}s_x=$$

$$\int_\Gamma\left[\frac{\partial}{\partial x_1}\left(\varphi\frac{\partial m_0^*}{\partial x_2}\right)\mathrm{d}x_1+\frac{\partial}{\partial x_2}\left(\varphi\frac{\partial m_0^*}{\partial x_2}\right)\mathrm{d}x_2\right]-\int_\Gamma\frac{\partial m_0^*}{\partial x_2}(\because\varphi\cdot\tau)\mathrm{d}s_x=$$

$$-\int_\Gamma\frac{\partial m_0^*}{\partial x_2}\frac{\partial\varphi}{\partial\tau_x}\mathrm{d}s_x=\int_\Gamma\frac{\partial m_0^*}{\partial y_2}\frac{\partial\varphi}{\partial\tau_x}\mathrm{d}s_x\cdot$$

证毕·

定理 2 条件同定理 1, 则边值问题(1)、(2) 在区域 Ω 上的解可表示为:

$$w(y)=\sum_{l=0}^{\infty}\left\{\int_\Gamma\left[\Delta w\frac{\partial q_l^*}{\partial n_x}-q_l^*\frac{\partial\Delta w}{\partial n}+w\frac{\partial\Delta q_l^*}{\partial n_x}-\Delta q_l^*\frac{\partial w}{\partial n}\right]\mathrm{d}s_x+\right.$$
$$\left.s\int_\Gamma\left[q_l^*\frac{\partial w}{\partial n}-w\frac{\partial q_l^*}{\partial n_x}\right]\mathrm{d}s_x+\iint_\Omega q_l^* p\,\mathrm{d}x\right\}\quad(y\in\Omega)\cdot\tag{13}$$

边值问题(1)、(2) 的未知边界量 $\partial w/\partial\tau|_\Gamma$, $\Delta w|_\Gamma$, $\partial\Delta w/\partial n_x|_\Gamma$ 满足下列 3 个边界积分方程:

$$c(y)w(y)=\sum_{l=0}^{\infty}\left\{\int_\Gamma\left[\Delta w\frac{\partial q_l^*}{\partial n_x}-q_l^*\frac{\partial\Delta w}{\partial n}+w\frac{\partial\Delta q_l^*}{\partial n_x}-\Delta q_l^*\frac{\partial w}{\partial n}\right]\mathrm{d}s_x+\right.$$
$$\left.s\int_\Gamma\left[q_l^*\frac{\partial w}{\partial n}-w\frac{\partial q_l^*}{\partial n_x}\right]\mathrm{d}s_x+\iint_\Omega q_l^* p\,\mathrm{d}x\right\}\quad(y\in\Gamma),\tag{14}$$

$$c(y)\frac{\partial w(y)}{\partial n_y}=\int_\Gamma\left[\Delta w\frac{\partial^2 w_0^*}{\partial n_y\partial n_x}-\frac{\partial w_0^*}{\partial n_y}\frac{\partial\Delta w}{\partial n}+\left(\frac{\partial m_0^*}{\partial y_2}n_y^1+\frac{\partial m_0^*}{\partial y_1}n_y^2\right)\frac{\partial w}{\partial\tau}-\right.$$
$$\left.\left(\frac{\partial m_0^*}{\partial n_y}-s\frac{\partial w_0^*}{\partial n_y}\right)\frac{\partial w}{\partial n}-sw\frac{\partial^2 w_0^*}{\partial n_y\partial n_x}\right]\mathrm{d}s_x+\iint_\Omega\frac{\partial q_l^*}{\partial n_y}p\,\mathrm{d}x+$$
$$\sum_{l=1}^{\infty}\left\{\int_\Gamma\left[\Delta w\frac{\partial^2 q_l^*}{\partial n_y\partial n_x}-\frac{\partial q_l^*}{\partial n_y}\frac{\partial\Delta w}{\partial n}+w\frac{\partial^2\Delta q_l^*}{\partial n_y\partial n_x}-\frac{\partial\Delta q_l^*}{\partial n_y}\frac{\partial w}{\partial n}\right]\mathrm{d}s_x+\right.$$
$$\left.s\int_\Gamma\left[\frac{\partial q_l^*}{\partial n_y}\frac{\partial w}{\partial n}-w\frac{\partial^2 q_l^*}{\partial n_y\partial n_x}\right]\mathrm{d}s_x+\iint_\Omega\frac{\partial q_l^*}{\partial n_y}p\,\mathrm{d}x\right\}\quad(y\in\Gamma),\tag{15}$$

$$c(y)\frac{\partial w(y)}{\partial\tau_y}=\int_\Gamma\left[\Delta w\frac{\partial^2 w_0^*}{\partial\tau_y\partial n_x}-\frac{\partial w_0^*}{\partial\tau_y}\cdot\frac{\partial w}{\partial n}+\left(\frac{\partial m_0^*}{\partial y_2}\tau_y^1+\frac{\partial m_0^*}{\partial y_1}\tau_y^2\right)\frac{\partial w}{\partial\tau}-\right.$$
$$\left.\left(\frac{\partial m_0^*}{\partial n_y}-s\frac{\partial w_0^*}{\partial n_y}\right)\frac{\partial w}{\partial n}-sw\frac{\partial^2 w_0^*}{\partial n_y\partial n_x}\right]\mathrm{d}s_x+\iint_\Omega\frac{\partial q_l^*}{\partial n_y}p\,\mathrm{d}x+$$
$$\sum_{l=1}^{\infty}\left\{\int_\Gamma\left[\Delta w\frac{\partial^2 q_l^*}{\partial\tau_y\partial n_x}-\frac{\partial q_l^*}{\partial\tau_y}\frac{\partial\Delta w}{\partial n}+w\frac{\partial^2\Delta q_l^*}{\partial\tau_y\partial n_x}-\frac{\partial\Delta q_l^*}{\partial\tau_y}\frac{\partial w}{\partial n}\right]\mathrm{d}s_x+\right.$$
$$\left.s\int_\Gamma\left[\frac{\partial q_l^*}{\partial\tau_y}\frac{\partial w}{\partial n}-w\frac{\partial^2 q_l^*}{\partial\tau_y\partial n_x}\right]\mathrm{d}s_x+\iint_\Omega\frac{\partial q_l^*}{\partial\tau_y}p\,\mathrm{d}x\right\}\quad(y\in\Gamma)\cdot\tag{16}$$

证明 (13) 及(14) 式分别是(11) 式中 $y\in\Omega$ 和 $y\in\Gamma$ 的两种情形· 根据引理 3 得

$$\frac{\partial I(q_0^*)}{\partial y_1}=\frac{\partial I(w_0^*)}{\partial y_1}=\int_\Gamma\left[\Delta w\frac{\partial^2 w_0^*}{\partial y_1\partial n_x}-\frac{\partial w_0^*}{\partial y_1}\frac{\partial w}{\partial n}+w\frac{\partial^2 m_0^*}{\partial y_1\partial n_x}-\frac{\partial m_0^*}{\partial y_1}\frac{\partial w}{\partial n}\right]\mathrm{d}s_x=$$
$$\int_\Gamma\left[\Delta w\frac{\partial^2 w_0^*}{\partial y_1\partial n_x}-\frac{\partial w_0^*}{\partial y_1}\frac{\partial w}{\partial n}+\frac{\partial m_0^*}{\partial y_2}\frac{\partial w}{\partial\tau}-\frac{\partial m_0^*}{\partial y_1}\frac{\partial w}{\partial n}\right]\mathrm{d}s_x,$$

同理

$$\frac{\partial I(q_0^*)}{\partial y_2}=\int_\Gamma\left[\Delta w\frac{\partial^2 w_0^*}{\partial y_2\partial n_x}-\frac{\partial w_0^*}{\partial y_2}\frac{\partial w}{\partial n}+\frac{\partial m_0^*}{\partial y_1}\frac{\partial w}{\partial\tau}-\frac{\partial m_0^*}{\partial y_2}\frac{\partial w}{\partial n}\right]\mathrm{d}s_x,$$

由此可得

$$\frac{\partial I(q_0^*)}{\partial n_y}=\int_\Gamma\left[\Delta w\frac{\partial^2 w_0^*}{\partial n_y\partial n_x}-\frac{\partial w_0^*}{\partial n_y}\frac{\partial w}{\partial n}+\right.$$

$$\left(\frac{\partial m_0^*}{\partial y_2}n_y^1+\frac{\partial m_0^*}{\partial y_1}n_y^2\right)\frac{\partial w}{\partial \tau}-\frac{\partial m_0^*}{\partial n_y}\frac{\partial w}{\partial n}\Bigg]\mathrm{d}s_x,$$

$$\frac{\partial I(q_0^*)}{\partial \tau_y}=\int_\Gamma\Bigg[\Delta w\frac{\partial^2 w_0^*}{\partial \tau_y\partial n_x}-\frac{\partial w_0^*}{\partial \tau_y}\frac{\partial w}{\partial n}+\left(\frac{\partial m_0^*}{\partial y_2}\tau_y^1+\frac{\partial m_0^*}{\partial y_1}\tau_y^2\right)\frac{\partial w}{\partial \tau}-\frac{\partial m_0^*}{\partial \tau_y}\frac{\partial w}{\partial n}\Bigg]\mathrm{d}s_x\cdot$$

在(11)式中除 $I(q_0^*)$ 经求法向或切向导数后积分中含有强奇异核外,其它各项可逐项求导,因为它们满足级数逐项求导的条件• 对(14)式分别求法向和切向导数,并用刚得到的结果立即推得(15)、(16)成立• 证毕•

称(13)式为边值问题(1)~(3)在区域 Ω 内解的 MRM 积分表达式;称(14)~(16)为 MRM 边界积分方程•

定理 3 对边值问题(1)~(3),采用常规积分方法得到的闭区域 Ω 上的积分表达式(5)与闭区域 Ω 上 MRM 积分表达式(11)是相同的•

证明 由于 $\Psi(m+1)=l(m)-\ln 2$,方程(1)的基本解(4)可写成

$$w^*(r)=\frac{1}{2\pi\sqrt{4k^2-s^2}}(\beta_2-\beta_1)\sum_{m=0}^\infty\frac{(-1)^m k^m}{(m!)^2 4^m}r^{2m}\cos 2m\beta_1-\frac{4}{\pi\sqrt{4k^2-s^2}}\ln\sqrt{k}\sum_{m=0}^\infty\frac{(-k)^m r^{2m}}{(m!)^2 4^m}\sin 2m\beta_1+\frac{4}{\pi\sqrt{4k^2-s^2}}\sum_{m=0}^\infty\frac{(-k)^m r^{2m}}{(m!)^2 4^m}[l(m)-\ln r]\sin 2m\beta_1\cdot\tag{17}$$

分别用 $u_1^*(r)$, $u_2^*(r)$, $u_3^*(r)$ 表示(17)式中的 3 项• 容易推得

$$\Delta u_1^*=\frac{s}{2}u_1^*+\frac{1}{4\pi}(\beta_2-\beta_1)\sum_{m=0}^\infty\frac{(-k)^m r^{2m}}{(m!)^2 4^m}\sin 2m\beta_1;$$

$$\Delta^2 u_1^*=\left(\frac{s^2}{2}-k^2\right)u_1^*+\frac{s}{4\pi}(\beta_2-\beta_1)\sum_{m=0}^\infty\frac{(-k)^m r^{2m}}{(m!)^2 4^m}\sin 2m\beta_1\cdot$$

对 $u_2^*(r)$ 也有类似结果,从而不难得到:

$$\iint_\Omega(w\Delta^2 u_1^*-u_1^*\Delta^2 w)\mathrm{d}x+\iint_\Omega u_1^* p\,\mathrm{d}x+s\iint_\Omega(u_1^*\Delta w-w\Delta u_1^*)\mathrm{d}x=0,$$

$$\iint_\Omega(w\Delta^2 u_2^*-u_2^*\Delta^2 w)\mathrm{d}x+\iint_\Omega u_2^* p\,\mathrm{d}x+s\iint_\Omega(u_2^*\Delta w-w\Delta u_2^*)\mathrm{d}x=0\cdot$$

因此方程(1)在区域 Ω 上的解可写成

$$w(y)=\iint_\Omega(w\Delta^2 w^*-w^*\Delta^2 w)\mathrm{d}\Omega_x+s\iint_\Omega(w^*\Delta w-w\Delta w^*)\mathrm{d}\Omega_x+\iint_\Omega w^* p\,\mathrm{d}\Omega_x=\iint_\Omega(w\Delta^2 u_3^*-u_3^*\Delta^2 w)\mathrm{d}\Omega_x+s\iint_\Omega(u_3^*\Delta w-w\Delta u_3^*)\mathrm{d}\Omega_x+\iint_\Omega u_3^* p\,\mathrm{d}\Omega_x=\int_\Gamma\left[\Delta w\frac{\partial u_3^*}{\partial n_x}-u_3^*\frac{\partial\Delta w}{\partial n}+w\frac{\partial\Delta u_3^*}{\partial n}-\Delta u_3^*\frac{\partial w}{\partial n}\right]\mathrm{d}s_x+s\int_\Gamma\left(u_3^*\frac{\partial w}{\partial n}-w\frac{\partial u_3^*}{\partial n_x}\right)\mathrm{d}s_x+\iint_\Omega u_3^* p\,\mathrm{d}\Omega_x\cdot\tag{18}$$

对上式第一个等号利用 Green 公式即得(5)式,换言之(18)式即是(5)式•

另一方面令

$$\begin{cases} A_l = \dfrac{2(-k)^{2l+1}}{\sqrt{4k^2-s^2}}\sin(2(2l+1)\beta_1) \\ B_l = \dfrac{2(-k)^{2l}}{\sqrt{4k^2-s^2}}\sin(2(2l)\beta_1) \end{cases} \qquad (l = 0, 1, 2, \cdots),$$

则

$$u_3^* = \sum_{l=0}^{\infty}(A_l w_l^* + B_l m_l^*)•$$

注意到 $\sin 2\beta_1 = \sqrt{4k^2 - s^2}/(2k)$, 显然有 $A_0 = 1, B_0 = 0$ 及递推关系

$$A_l = (-k^2)A_{l-1} + sB_l,\ B_l = sA_{l-1} + (-k^2)B_{l-1} \qquad (l = 1, 2, \cdots)•$$

因此 $A_l = A_l,\ B_l = B_l,\ l = 0, 1, 2, \cdots$• 于是

$$u_3^* = \sum_{l=0}^{\infty}(A_l w_l^* + B_l m_l^*) = \sum_{l=0}^{\infty} g_l^*•$$

将其代入(18)式, 即得区域 Ω 上的(11)式• 至此证明了(5)式与(11)式在区域 Ω 上是相同的• 进而在边界上也是相同的• 证毕•

定理 3 说明 MRM 方法得到的解恰好与常规边界积分方法得到的解是一致的• 因此由常规边界积分方程的误差估计可得到板弯曲问题 MRM 方法的收敛性分析• 从而保证了 MRM 方法数值求解的可靠性• 此外该方法还可类似推广到具多组高阶基本解序列的情形•

[参 考 文 献]

[1] 丁方允, 吕涛涛. 二维 Helmholtz 方程非线性边值问题的边界元分析[J]. 兰州大学学报, 1994, **30**(2): 25—30.

[2] 丁方允. 三维 Helmholtz 方程 Dirichlet 问题的边界元法及其收敛性分析[J]. 兰州大学学报, 1995, **31**(3): 30—38.

[3] Kamiya N, Andon E. A note on multiple reciprocity method integral formulation for the Helmholtz equation[J]. Comm Numer Methods Engrg, 1993, **9**(1): 9—13.

[4] Sladek V, Sladek J, Tanaka M. Boundary element solution of some structure_acoustic coupling problems using the multiple reciprocity method[J]. Comm Mumer Methods Engrg, 1994, **10**(2): 237—248.

[5] Sladek V, Sladek J, Tanaka M. Multiple reciprocity method for harmonic vibration of thin elastic plates[J]. Applied Mathemaics and Model, 1993, **17**(4): 468—476.

[6] 丁睿, 朱正佑, 程昌钧. 粘弹性薄板动力响应的边界元方法(Ⅱ)——理论分析[J]. 应用数学和力学, 1998, **19**(2): 95—103.

[7] 丁睿, 丁方允, 张颖. 屈曲特征值问题的边界元方法及收敛性分析[J]. 应用数学和力学, 2002, **23**(2): 144—156.

[8] 李正良, 邓安福. 双参数地基上板弯曲问题的边界积分方程[J]. 应用数学和力学, 1992, **13**(7): 633—644.

Multiple Reciprocity Method With Two Series of Sequences of High_Order Fundamental Solution for Thin Plate Bending

DING Fang_yun[1], DING Rui[2], LI Bing_jie
(1. Department of Mathematics, Lanzhou University,
Lanzhou 730000, P. R. China;
2. School of Mathematical Sciences, Suzhou University,
Suzhou 215006, P. R. China)

Abstract: The boundary value problem of plate bending problem on two_parameter foundation was discussed. Using two series of the high_order fundamental solution sequences, namely the fundamental solution sequences for the multi_harmonic operator and Laplace operator, applying the multiple reciprocity method(MRM), the MRM boundary integral equation for plate bending problem was constructed. It proves that the boundary integral equation derived from MRM is essentially identical to the conventional boundary integral equation. Hence the convergence analysis of MRM for plate bending problem can be obtained by the error estimation for the conventional boundary integral equation. In addition this method can extend to the case of more series of the high_order fundamental solution sequences.

Key words: plate bending problem; multiple reciprocity method; boundary integral equation; high_order fundamental solution sequence