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具非线性边界条件的 Volterra 型泛函微分方程边值问题奇摄动*

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摘要: 首先利用微分不等式理论和一些分析技巧, 探讨了一类具非线性边界条件的二阶 Volterra 型泛函微分方程边值问题解的存在性问题• 然后通过对右端边界层函数和外部解的构造, 进一步研究了一类具小参数的二阶 Vottera 型非线性边值问题• 利用微分中值定理和上、下解方法得到了边值问题解的存在性, 并给出了解的关于小参数的一致有效渐近展开式•

关 键 词: 奇异摄动; 泛函微分方程; 边值问题; 一致有效渐近展开式

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引 言

近年来利用拓扑度理论和一些不动点原理研究泛函微分方程边值问题, 已有一些结果出现[1~6]• 但利用奇异摄动理论研究含小参数的泛函微分方程边值问题还不是很多[7~11]• 其原因是泛函微分方程的上、下解构造难度要大些• 作者曾在文[10]中研究一类奇摄动拟线性 Volterra 型时滞微分方程边值问题, 在文[11]中进一步研究了一类非线性 Volterra 型时滞微分方程边值问题•

$$\begin{cases} \varepsilon x''(t) = f(t, x(t), x(t-\tau(t)), [Tx](t), x'(x), \varepsilon), & t \in (0, 1), \\ x(t) = \varphi(t, \varepsilon), \quad t \in [-\tau, 0], & ax(1) + bx'(1) = A(\varepsilon)• \end{cases}$$

本文在文[11]的基础上继续研究具非线性边界条件下的泛函微分方程边值问题

$$\begin{cases} \varepsilon x'' = f(t, x(t), x_t, [Tx](t), x'(t), \varepsilon), & t \in (0, 1), \quad (1) \\ x(t) = \varphi(t, \varepsilon), \quad t \in [-\tau, 0], & \Phi(x(1), x'(1), \varepsilon) = A(\varepsilon), \quad (2) \end{cases}$$

其中 $\varepsilon > 0$ 为小参数,

$$[Tx](t) = \phi(t) + \int_0^t k(t, s) x(s) \mathrm{d}s,$$

$\phi(t)$ 和 $k(t, s) > 0$ 分别在[0, 1]和[0, 1] × [0, 1]上连续• 通过利用新的方法构造上、下解和微分不等式理论, 我们得到了边值问题(1) ~ (2)解的存在性新的结果, 还给出了边值问题(1) ~ (2)解的一致有效渐近展开式• 本文设 $C_\tau = C([-\tau, 0], R)$, 其中 $\tau \in (0, 1)$ 为常数, 其模

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定义为 $\|\varphi\| = \sup_{t\in[-\tau,0]} |\varphi(t)|$, $\forall \varphi \in C_\tau$. 则 C_τ 为 Banach 空间. 对于给定的 $x \in C([-\tau, 1], R)$, 令 $x_t(\theta) = x(t+\theta)$, $\forall \theta \in [-\tau, 0]$, 显然, $x_t \in C_\tau$, $\forall t \in [0, 1]$. 另外, 我们记 $\varphi_1 \geqslant \varphi_2 \Leftrightarrow \varphi_2(\theta)$, 对 $\forall \theta \in [-\tau, 0]$, 和 $\forall \varphi_1, \varphi_2 \in C_\tau$.

本文给出下列假设. 在第 2 节中, 它们将被用来研究边值问题(1)~(2).

[H1] $f(t, x, \varphi, y, w, \varepsilon)$ 为 $[0,1] \times \mathbf{R} \times C_\tau \times R^2 \times [0, \varepsilon_0]$ 上的连续可微泛函. $\varphi(t, \varepsilon)$ 在 $[-\tau, 0] \times [0, \varepsilon_0]$ 上连续且关于 ε 适当可微, $A(\varepsilon)$ 在 $[0, \varepsilon_0]$ 上连续, 其中 ε_0 为常数. $\Phi(x, y, \varepsilon)$ 在 $R^2 \times [0, \varepsilon_0]$ 上连续且关于 x 可微. 另外, $\Phi'_x(x, y, \varepsilon) \leqslant -l_0$, 且 $\Phi(x, y, \varepsilon)$ 关于 y 单调不增, l_0 为正常数.

[H2] 下列退化问题

$$\begin{cases} f(t, x(t), x_t, [Tx](t), x'(t), 0) = 0, \\ x(t) = \varphi(t, 0), \qquad t \in [-\tau, 0], \end{cases}$$

存在解 $\tilde{x}(t)$ 满足 $\tilde{x}(t) \in C^2_{[-\tau,1]}$.

[H3] $f(t, x, \varphi, y, w, \varepsilon)$ 关于 φ, y 单调不增且

$$f'_w(t, x, \varphi, y, w, \varepsilon) \geqslant m > 0, \qquad 当\ (t, x, \varphi, y, w, \varepsilon) \in \Omega 时,$$

其中 m 为常数 $\Omega = \left\{(t, x, \varphi, y, w, \varepsilon) \,\middle|\, t \in [0,1], |x - \tilde{x}(t)| \leqslant d, \|\varphi - \tilde{x}_t\| \leqslant d, |y - [Tx](t)| \leqslant d, |w| < \infty, \varepsilon \in [0, \varepsilon_0]\right\}$, $d = |\varphi(0,0) - \tilde{x}(0)| + \delta$, $\delta > 0$ 为常数.

[H4] 对 $\forall r > 0$, 存在 $[0, \infty)$ 上的正值连续函数 $h(s)$ 满足

$$\int_0^{+\infty} \frac{s}{1 + h(s)} \mathrm{d}s = \infty$$

且 $$|f(t, x, \varphi, y, w, \varepsilon)| \leqslant h(|w|),$$

当 $t \in [0, 1]$, $|x| \leqslant r$, $\|\varphi\| \leqslant r$, $|y| \leqslant r$, $|w| < \infty$, $\varepsilon \in [0, \varepsilon_0]$ 时.

1 边值问题解的存在性

本节我们将给出一类一般的 Volterra 型泛函微分方程

$$\begin{cases} \varepsilon x''(t) = F(t, x(t), x_t, [Tx](t), x'(t)), \qquad t \in (0, 1), & (3) \\ x(t) = \varphi(t), \qquad t \in [-\tau, 0], \quad \Phi_0(x(1), x'(1)) = A, & (4) \end{cases}$$

边值问题解的存在性结果, 其中 $F(t, x, \varphi, y, w)$ 为 $[0, 1] \times \mathbf{R} \times C_\tau \times R^2$ 上的连续泛函, $\Phi_0(x, y)$ 在 R^2 上连续, $\varphi \in C_\tau$.

引理[12] 假设

1) $F_1(t, x, y)$ 在 $[0, 1] \times R^2$ 上连续, 对 $\forall r > 0$, 存在 $[0, \infty)$ 上的正值连续函数 $h_1(s)$ 使得 $\int_0^\infty \frac{s}{1 + h_1(s)} \mathrm{d}s = \infty$ 且 $|F_1(t, x, y)| \leqslant h_1(|y|)$, 当 $t \in [0, 1]$, $|x| \leqslant r$, $|y| < \infty$ 时.

2) 存在 $\alpha(t), \beta(t) \in C^2_{[0,1]}$ 使得

$$\begin{aligned} &\alpha(t) \leqslant \beta(t), && t \in [0, 1], \\ &\alpha''(t) \geqslant F_1(t, \alpha(t), \alpha'(t)), && t \in [0, 1], \\ &\beta''(t) \leqslant F_1(t, \beta(t), \beta'(t)), && t \in [0, 1]. \end{aligned}$$

则对 $\alpha(0) \leqslant A \leqslant \beta(0)$, $\alpha(1) \leqslant B \leqslant \beta(1)$, 边值问题

$$\begin{cases} x''(t) = F_1(t, x(t), x'(t)), & t \in (0, 1), \\ x(0) = A, x(1) = B, \end{cases}$$

存在解 $x(t)$ 使得

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \qquad t \in [0, 1].$$

定理 1 设

1) 对 $\forall r > 0$，存在 $[0, \infty)$ 上的连续函数 $h_2(s)$，使得 $\int_0^\infty \frac{s}{1 + h_2(s)} ds = \infty$，和 $|F(t, x, \varphi, y, w)| \leqslant h_2(|w|)$，当 $t \in [0, 1]$，$|x| \leqslant r$，$\|\varphi\| \leqslant r$，$|y| \leqslant r$，$|w| < \infty$ 时.

2) $F(t, x, \varphi, y, w)$ 关于 φ, y 单调不增.

3) 存在 $\alpha(t), \beta(t) \in C_{[-\tau, 1]} \cap C^2_{[0, 1]}$ 使得

$$\begin{aligned} &\alpha(t) \leqslant \beta(t), && t \in [-\tau, 1], \\ &\alpha''(t) \geqslant F(t, \alpha(t), \alpha_t, [T\alpha](t), \alpha'(t)), && t \in [0, 1], \\ &\beta''(t) \leqslant F(t, \beta(t), \beta_t, [T\beta](t), \beta'(t)), && t \in [0, 1]. \end{aligned}$$

则 $\forall \varphi(t) \in C_\tau$ 和 $A \in \mathbf{R}$ 满足

$$\alpha(t) \leqslant \varphi(t) \leqslant \beta(t), \qquad t \in [-\tau, 0]; \ \alpha(1) \leqslant A \leqslant \beta(1),$$

边值问题

$$\begin{cases} \varepsilon x''(t) = F(t, x(t), x_t, [Tx](t), x'(t)), & t \in (0, 1), & (5) \\ x(t) = \varphi(t), & t \in [-\tau, 0], \ x(1) = A & (6) \end{cases}$$

存在解 $x(t)$ 满足

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \qquad t \in [0, 1].$$

证明 令 $u_0(t) = \beta(t)$ 为初始迭代函数，考虑下列边值问题

$$\begin{cases} u''(t) = F_1(t, u(t), u'(t)), & t \in (0, 1), & (7) \\ u(0) = \varphi(0), \ u(1) = A, & & (8) \end{cases}$$

其中 $F_1(t, u(t), u'(t)) = F(t, u(t), \beta_t, [T\beta](t), u'(t))$，$t \in [0, 1]$. 由定理 1 的条件 2) 和 3)，易得 BVP(7)～(8) 存在解 $u_1(t)$ 满足 $\alpha(t) \leqslant u_1(t) \leqslant \beta(t)$，$t \in [0, 1]$. 令

$$\tilde{u}_1(t) = \begin{cases} u_1(t), & t \in [0, 1], \\ \varphi(t), & t \in [-\tau, 0]. \end{cases}$$

显然，$\tilde{u}_1(t) \in C_{[-\tau, 1]} \cap C^2_{[0, 1]}$. 利用迭代方法，我们可得函数列 $\{\tilde{u}_i\}_1^\infty$ 满足

$$\begin{cases} \tilde{u}_i''(t) = F(t, \tilde{u}_i(t), (\tilde{u}_{i-1})_t, [T\tilde{u}_{i-1}](t), \tilde{u}_i'(t)), & t \in (0, 1), & (9) \\ \tilde{u}_i(0) = \varphi(0), \ \tilde{u}_i(1) = A, & & (10) \end{cases}$$

$i = 1, 2, \cdots, \infty$. 如果对给出的正整数 k，我们有 $\alpha(t) \leqslant \tilde{u}_k(t) \leqslant \tilde{u}_{k-1}(t) \leqslant \tilde{u}_1(t) \leqslant u_0 = \beta(t)$，$t \in (0, 1)$，则考虑

$$\begin{cases} u''(t) = F(t, u(t), \tilde{u}_{kt}, [T\tilde{u}_k](t), u'(t)), & t \in (0, 1), & (11) \\ u(0) = \varphi(0), \ u(1) = A. & & (12) \end{cases}$$

由于

$$\begin{aligned} \alpha''(t) &\geqslant F(t, \alpha(t), \alpha_t, [T\alpha](t), \alpha'(t)) \geqslant \\ &\quad F(t, \alpha(t), (\tilde{u}_k)_t, [T\tilde{u}_k](t), \alpha'(t)), \qquad t \in (0, 1), \\ \tilde{u}_k''(t) &= F(t, \tilde{u}_k(t), (\tilde{u}_{k-1})_t, [T\tilde{u}_{k-1}](t), \tilde{u}_k'(t)) \leqslant \\ &\quad F(t, \tilde{u}_k(t), \tilde{u}_{kt}, [T\tilde{u}_k](t), \tilde{u}_k'(t)), \qquad t \in (0, 1) \end{aligned}$$

和 $\alpha(0)\leqslant\varphi(0)\leqslant\tilde{u}_k(0)$, $\alpha(1)\leqslant A\leqslant\tilde{u}_k(1)$, 则由引理得边值问题(11)～(12)存在解 $u_{k+1}(t)\leqslant\tilde{u}_k(t)$, $t\in[0,1]$• 令

$$\tilde{u}_{k+1}(t)=\begin{cases}u_{k+1}(t), & t\in[0,1],\\ \varphi(t), & t\in[-\tau,0],\end{cases}$$

易见

$$\alpha(t)\leqslant\tilde{u}_{k+1}(t)\leqslant\tilde{u}_k(t)\leqslant\cdots\leqslant\tilde{u}_1(t)\leqslant\beta(t),\qquad t\in[-\tau,1]• \tag{13}$$

利用数学归纳法可得(13)式对一切正整数均成立• 设

$$r=\max\left\{\max_{t\in[-\tau,1]}|\alpha(t)|,\ \max_{t\in[-\tau,1]}|\beta(t)|\right\}•$$

从定理1的条件1)不难发现存在 $R_0>0$ 使得

$$\int_{2r}^{R_0}\frac{s\mathrm{d}s}{1+h_2(s)}>2r, \tag{14}$$

和

$$|F(t,x,\varphi,y,w)|\leqslant h_2(|w|),$$
$$\text{当 } t\in[0,1],\ |x|\leqslant r,\ \|\varphi\|\leqslant r,\ |z|\leqslant r,\ |w|<\infty\text{ 时}• \tag{15}$$

另一方面,由微分中值定理知,存在 $\xi\in(0,1)$ 使得 $\tilde{u}_i'(\xi)=\tilde{u}_i(1)-\tilde{u}_i(0)$• 并结合 r 的定义知, $|\tilde{u}_i'(\xi)|\leqslant 2r$• 如果存在 $t_0\in(0,1)$ 使得 $\tilde{u}_i'(\tau_1)=2r$, $\tilde{u}_i'(\tau_2)=R_0$ 和 $2r<\tilde{u}_i'(t)<R_0$ 当 $t\in(\tau_1,\tau_2)$• 则由(13)和(15)得

$$\int_{2r}^{R_0}\frac{s}{1+h_2(s)}\mathrm{d}s=\int_{\tau_1}^{\tau_2}\frac{\tilde{u}_i'(t)\tilde{u}_i''(t)}{1+h_2(\tilde{u}_i'(t))}\mathrm{d}t=$$
$$\int_{\tau_1}^{\tau_2}\frac{\tilde{u}_i'(t)F(t,\tilde{u}_i(t),(\tilde{u}_i)_t,[T\tilde{u}_i](t),\tilde{u}_i'(t))}{1+h_2(\tilde{u}_i'(t))}\mathrm{d}t\leqslant$$
$$\int_{\tau_1}^{\tau_2}\tilde{u}_i'(t)\frac{h(\tilde{u}_i'(t))}{1+h_2(\tilde{u}_i'(t))}\mathrm{d}t<\int_{\tau_1}^{\tau_2}\tilde{u}_i'(t)\mathrm{d}t=$$
$$\tilde{u}_i(\tau_2)-\tilde{u}_i(\tau_1)•$$

于是 $|\tilde{u}_i(\tau_2)|\leqslant r$, $|\tilde{u}_i(\tau_1)|\leqslant r$• 因而 $\int_{2r}^{R_0}\frac{s}{1+h_2(s)}\mathrm{d}s\leqslant 2r$, 这与(14)矛盾• 故 $\tilde{u}_i'(t)<R_0$, 当 $t\in[0,1]$ 时• 类似可得, $\tilde{u}_i'(t)>-R_0$, 当 $t\in[0,1]$ 时• 由此可见, 对一切正整数 i, $|\tilde{u}_i'(t)|<R_0$, $t\in[0,1]$, 再结合(13)易得 $\left\{\tilde{u}_i(t)\right\}_1^\infty$ 一致收敛于 $\bar{x}_0(t)$, $t\in[0,1]$• 利用 Lebesgue 控制收敛定理, 易得 $\bar{x}_0(t)$ 为边值问题(5)～(6)的解•

定理2　设

1) 定理1的假设1)～2)满足•

2) 存在 $\alpha(t),\beta(t)\in C_{[-\tau,1]}\cap C^2_{[0,1]}$ 使得定理1的假设3)成立, 且 $\alpha(1)<\beta(1)$•

3) $\Phi_0(x,y)$ 关于 y 单调不增, $\Phi_0(\alpha(1),\alpha'(1))\geqslant A$, $\Phi_0(\beta(1),\beta'(1))\leqslant A$•

则对 $\forall\varphi(t)\in C_\tau$ 和 $A\in\mathbf{R}$ 满足

$$\alpha(t)\leqslant\varphi(t)\leqslant\beta(t),\qquad t\in[-\tau,0],\ \alpha(1)\leqslant A\leqslant\beta(1),$$

边值问题(3)～(4)存在解 $x(t)$ 使得

$$\alpha(t)\leqslant x(t)\leqslant\beta(t),\qquad t\in[0,1]•$$

证明　$\forall c\in[\alpha(1),\beta(1)]$, 由定理1知, 下列边值问题

$$\begin{cases} x''(t) = F(t, x(t), x_t, [Tx](t), x'(t)), & t \in (0, 1), & (16) \\ x(t) = \varphi(t), & t \in [-\tau, 0],\ x(1) = c, & (17) \end{cases}$$

存在解 $x_c(t)$ 满足 $\alpha(t) \leqslant x_c(t) \leqslant \beta(t)$, $t \in [-\tau, 1]$•

如果 $c = \alpha(1)$, 则 $x_c'(1) \leqslant \alpha'(1)$• 因而

$$\Phi_0(x_c(1), x_c'(1)) = \Phi_0(\alpha(1), x_c'(1)) \geqslant \Phi_0(\alpha(1), \alpha'(1)) \geqslant A• \tag{18}$$

如果 $c = \beta(1)$,同样可得

$$\Phi_0(x_c(1), x_c'(1)) \leqslant \Phi_0(\beta(1), \beta'(1)) \leqslant A• \tag{19}$$

令

$$\Omega_1 = \left\{ c \mid c \in [\alpha(1), \beta(1)],\ \Phi_0(x_c(1), x_c'(1)) < A \right\},$$
$$\Omega_2 = \left\{ c \mid c \in [\alpha(1), \beta(1)],\ \Phi_0(x_c(1), x_c'(1)) > A \right\}•$$

如果定理 2 的结论不真, 则

$$\Omega_1 \cup \Omega_2 = [\alpha(1), \beta(1)],\ 且\ \Omega_1 \cap \Omega_2 = \not\subset• \tag{20}$$

由(18)和(19),不难得到 Ω_1 和 Ω_2 均非空且可证明 Ω_1 是闭集• 事实上,设 $c_n \in \Omega_1$, 则 $\lim_{n\to\infty} c_n = c_0$• 为方便起见,我们设 $x_n(t) = x_{c_n}(t)$• 显然, $\Phi_0(x_n(1), x_n'(1)) < A$• 由定理 2 的条件 1) 并通过计算可得,函数列 $\left\{x_n(t)\right\}_1^\infty$ 是等度连续的• 因而存在 $\left\{x_n(t)\right\}_1^\infty$ 中的子列在 $[-\tau, 1]$ 上收敛于 $x_0(t)$,其中 $x_0(t)$ 满足

$$\begin{cases} x_0''(t) = F(t, x_0(t), (x_0)_t, [Tx_0](t), x_0'(t)), & t \in (0, 1), \\ x_0(t) = \varphi(t), \quad t \in [-\tau, 0],\ x_0(1) = c_0 \end{cases}$$

和

$$\Phi_0(x_0(1), x_0'(1)) \leqslant A• \tag{21}$$

由假设知(21)不可能取等号• 因而 $\Phi_0(x_0(1), x_0'(1)) < A$,即 $c_0 \in \Omega_1$• 由此得 Ω_1 为闭集• 同样可证 Ω_2 也是闭集• 这样与(20)相矛盾• 这矛盾说明了定理 2 的结论成立•

2 奇摄动边值问题

令
$$\varphi(t, \varepsilon) = \varphi_0(t) + \varphi_1(t)\varepsilon + \cdots + \varphi_N(t)\varepsilon^N + O(\varepsilon^N), \tag{22}$$

$$\varphi_0(t) = \varphi(t, 0),\ \varphi_i(t) = \frac{\partial^i \varphi(t, 0)}{i!\partial^i \varepsilon} \tag{23a}$$

和

$$x(t, \varepsilon) = x_0(t) + x_1(t)\varepsilon + \cdots + x_N(t, \varepsilon)\varepsilon^N + \cdots + O(\varepsilon^N) \tag{23b}$$

为边值问题(1)~(2)的外部解• 将(22)和(23)分别代入(1)和(2),将其展开成关于 ε 的幂级数,并令同次幂的系数相等得

$$\begin{cases} f(t, x_0(t), x_{0t}, [Tx_0](t), x_0'(t)) = 0, & (24)_0 \\ x_0(t) = \varphi_0(t), \quad t \in [-\tau, 0], & (25)_0 \end{cases}$$

和

$$\begin{cases} x_{i-1}''(t) = f_1(t)x_i(t) + f_2(t)x_{it} + f_3(t)x_i'(t) + \\ \qquad f_4(t)\int_0^t k(t, s)x_i(s)\,\mathrm{d}s + P_i(t), & (24)_i \\ x_i(t) = \varphi_i(t), \quad t \in [-\tau, 0], & (25)_i \end{cases}$$

$i = 1, 2, \cdots, N$• 其中

$$f_1(t) = f'_x(t, x_0(t), x_{0t}, [Tx_0](t), x'_0(t), 0),$$
$$f_2(t) = f'_\varphi(t, x_0(t), x_{0t}, [Tx_0](t), x'_0(t), 0),$$

$(f'_\varphi(t, x, \varphi, z, w, \varepsilon)$ 为泛函 $f(t, x, \varphi, z, w, \varepsilon)$ 关于 φ 的 Frechet 导数)•

$$f_3(t) = f'_y(t, x_0(t), x_{0t}, [Tx_0](t), x'_0(t), 0),$$
$$f_4(t) = f'_w(t, x_0(t), x_{0t}, [Tx_0](t), x'_0(t), 0),$$

$P_i(t)$ 为关于 $x_0(t), x_1(t), \cdots, x_{i-1}(t), x_{0t}, x_{1t}, \cdots, x_{i-1\,t}, x'_0(t), x'_1(t), \cdots, x'_{i-1}(t)$ 和

$$\int_0^t k(t, s)x_0(s)\mathrm{d}s,\ \int_0^t k(t, s)x_1(s)\mathrm{d}s,\ \cdots,\ \int_0^t k(t, s)x_{i-1}(s)\mathrm{d}s$$

的已知函数• 由假设[H_2]知(24)$_0$~(25)$_0$ 存在唯一解

$$x_0(t) = \tilde{x}(t), \qquad t \in [-\varepsilon_0, 1]•$$

然后, 再利用泛函微分方程基本理论[13], 可得初值问题 (24)$_i$ ~ (25)$_i$ 在 $[-\tau, 1]$ 上存在解 $x_i(t)\ (i = 1, 2, \cdots, N)$•

令 $X_N(t, \varepsilon) = \sum_{i=0}^{N} x_i(t)\varepsilon^i$• 显然, $X_N(t, \varepsilon) \in C^2_{[0,1]} \cap C_{[-\tau,1]}$ 和

$$|\varepsilon X''_N(t, \varepsilon) - f(t, X_N(t, \varepsilon), X_{Nt}, [TX_N](t), X'_N(t, \varepsilon), \varepsilon)| \leqslant M\varepsilon^{N+1}, \qquad t \in [0, 1], \tag{26}$$

其中 M 为与 ε 无关的常数• 作

$$\Omega_1 = \Big\{(t, x, \varphi, y) \mid t \in [0, 1], |x - x_0(t)| \leqslant d, \|\varphi - x_{0t}\| \leqslant d, |y - [Tx_0](t)| \leqslant d\Big\},$$

并取 l 满足 $l > |f'_x(t, x, \varphi, y, x'_0(t), 0)|$, $l > K|f'_y(t, x, \varphi, y, x'_0(t), 0)|$, $l > |f'_\varphi(t, x, \varphi, y, x'_0(t), 0)|$, 当 $(t, x, \varphi, y) \in \Omega_1$ 时, 其中 $K = \max_{(t,s)\in[0,1]\times[0,1]} k(t, s)$•

设

$$\Phi(x, X'_N(1, \varepsilon), \varepsilon) = A(\varepsilon)• \tag{27}$$

由 $\Phi'_x(x, y, \varepsilon) \leqslant -l_0 < 0$ 知, 存在唯一函数 $w(\varepsilon)$ 使得

$$\Phi(w(\varepsilon), X'_N(1, \varepsilon), \varepsilon) = A(\varepsilon)•$$

令边界层函数为

$$w(t, \varepsilon) = |w(\varepsilon) - X_N(1, \varepsilon)|\, \mathrm{e}^{-\lambda_1(1-t)}, \qquad t \in [-\tau, 1], \tag{28}$$

余项为

$$\Gamma(t, \varepsilon) = \frac{\varepsilon^{N+1} r}{l}(2\mathrm{e}^{\lambda_2 t} - 1), \qquad t \in [-\tau, 1], \tag{29}$$

其中 $r > 0$ 为常数(待定), λ_1、λ_2 分别为方程 $\varepsilon\lambda^3 - m\lambda^2 + 2l\lambda + l = 0$ 在 $(m/\varepsilon - k, m/\varepsilon)$ 和 $(l/m, 2l/m)$ 内的两根, k 为常数满足 $k > 2l/m$• 根据泛函微分方程基本理论[13], 易见 $w(t, \varepsilon)$ 和 $\Gamma(t, \varepsilon)$ 满足

$$\varepsilon w''(t, \varepsilon) - m w'(t, \varepsilon) + 2l w(t, \varepsilon) + l\int_0^t w(s, \varepsilon)\mathrm{d}s = 0, \qquad t \in [0, 1] \tag{30}$$

和

$$\varepsilon\Gamma''(t, \varepsilon) - m\Gamma(t, \varepsilon) + 2l\Gamma(t, \varepsilon) + l\int_0^t (s, \varepsilon)\mathrm{d}s = -2r\varepsilon^{N+1}, \qquad t \in [0, 1]• \tag{31}$$

定理 3　如果条件[H_1]~[H_4]满足, 则当 $\varepsilon > 0$ 充分小时, 边值问题(1)~(2)存在解 $x(t,$

$\varepsilon)$ 满足

$$|x(t,\varepsilon)-X_N(t,\varepsilon)|\leqslant w(t,\varepsilon)+M_1\varepsilon^{N+1},\qquad t\in[0,1],$$

其中 M_1 为与 ε 无关的常数•

证明 设

$$\alpha(t,\varepsilon)=X_N(t,\varepsilon)-w(t,\varepsilon)-\Gamma(t,\varepsilon),\qquad t\in[-\tau,1],$$
$$\beta(t,\varepsilon)=X_N(t,\varepsilon)+w(t,\varepsilon)+\Gamma(t,\varepsilon),\qquad t\in[-\tau,1]•$$

显然, $\alpha(t,\varepsilon),\ \beta(t,\varepsilon)\in C_{[-\tau,1]}\cap C^2_{[0,1]}$ 和

$$\alpha(t,\varepsilon)\leqslant\beta(t,\varepsilon),\qquad t\in[-\tau,1],\tag{32}$$
$$\alpha(1,\varepsilon)\leqslant A(\varepsilon)\leqslant\beta(1,\varepsilon)•\tag{33}$$

另外, 我们取

$$\sigma=\max_{\substack{\varepsilon\in[0,\varepsilon_0]\\ t\in[-\tau,0]}}\left|\frac{\partial^i\varphi(t,\varepsilon)}{(N+1)!\partial^i\varepsilon}\right|,$$

则

$$\left|\varphi(t,\varepsilon)-\sum_{i=0}^{N}\varphi_i(t)\varepsilon^i\right|\leqslant\sigma\varepsilon^{N+1}•$$

如果取, $r>l\sigma$, 则

$$\alpha(t,\varepsilon)\leqslant\varphi(t,\varepsilon)\leqslant\beta(t,\varepsilon),\qquad t\in[-\tau,1]•\tag{34}$$

另外, 还有

$$\begin{aligned}&\Phi(\beta(1,\varepsilon),\beta'(1,\varepsilon),\varepsilon)-A(\varepsilon)\leqslant\\&\quad\Phi(\beta(1,\varepsilon),X_N'(1,\varepsilon),\varepsilon)-A(\varepsilon)=\\&\quad\Phi(X_N(1,\varepsilon)+|w(\varepsilon)-X_N(1,\varepsilon)|+\Gamma(1,\varepsilon),X_N'(1,\varepsilon),\varepsilon)-\\&\quad\Phi(w(\varepsilon),X_N'(1,\varepsilon),\varepsilon)+\Phi(w(\varepsilon),X_N'(1,\varepsilon),\varepsilon)-A(\varepsilon)=\\&\quad\int_0^1\Phi_x'(w(\varepsilon)+\theta(X_N(1,\varepsilon)+|w(\varepsilon)-X_N(1,\varepsilon)|+\Gamma(1,\varepsilon)-\\&\quad w(\varepsilon)),X_N'(1,\varepsilon),\varepsilon)\mathrm{d}\theta\cdot[X_N(1,\varepsilon)+|w(\varepsilon)-\\&\quad X_N(1,\varepsilon)|+\Gamma(1,\varepsilon)-w(\varepsilon)]\leqslant\\&\quad-l_0\Gamma(1,\varepsilon)<0•\end{aligned}\tag{35}$$

同理可得,

$$\Phi(\alpha(1,\varepsilon),\alpha'(1,\varepsilon),\varepsilon)-A(\varepsilon)>0•\tag{36}$$

对任意的 $t\in[0,1]$, 利用中值定理可得

$$\begin{aligned}&f((t,\alpha(t,\varepsilon),\alpha_t,\varepsilon),[T\alpha](t),\alpha'(t,\varepsilon),\varepsilon)-\varepsilon\alpha''(t,\varepsilon)=\\&\quad f_w'(t,\alpha(t,\varepsilon),\alpha_t,[T\alpha](t),X_N'(t,\varepsilon)+\theta_1(\alpha'(t,\varepsilon)-\\&\quad X_N'(t,\varepsilon)),\varepsilon)(\alpha'(t,\varepsilon)-X_N'(t,\varepsilon))+f_y'(t,\alpha(t,\varepsilon),\alpha_t,[TX_N](t)+\\&\quad\theta_2([T\alpha](t)-[TX_N](t)),X_N'(t,\varepsilon),\varepsilon)([T\alpha](t)-[TX_N](t))+\\&\quad f_\varphi'(t,\alpha(t,\varepsilon),X_{Nt}+\theta_3(\alpha_t-X_{Nt}),[TX_N](t),X_N'(t,\varepsilon),\varepsilon)(\alpha_t-X_{Nt})+\\&\quad f_x'(t,X_N(t,\varepsilon)+\theta_4(\alpha(t,\varepsilon)-X_N(t,\varepsilon)),\\&\quad X_{Nt},[TX_N](t),X_N'(t,\varepsilon),\varepsilon)(\alpha(t,\varepsilon)-X_N(t,\varepsilon))+\\&\quad M\varepsilon^{N+1}+\varepsilon\Gamma''(t,\varepsilon)+\varepsilon w''(t,\varepsilon)\leqslant\qquad(0<\theta_i<1,\ i=1,2,3,4)\end{aligned}$$

$$\varepsilon w''(t,\varepsilon)-mw'(t,\varepsilon)+lw(t,\varepsilon)+lw_t+l\int_0^t w(s,\varepsilon)\mathrm{d}s+$$
$$\varepsilon\Gamma''(t,\varepsilon)-m\Gamma'(t,\varepsilon)+l\Gamma(t,\varepsilon)+l\Gamma_t+l\int_0^t\Gamma(s,\varepsilon)\mathrm{d}s+M\varepsilon^{N+1}\leqslant$$
$$\varepsilon w''(t,\varepsilon)-mw'(t,\varepsilon)+2lw(t,\varepsilon)+l\int_0^t w(s,\varepsilon)\mathrm{d}s+$$
$$\varepsilon\Gamma''(t,\varepsilon)-m\Gamma'(t,\varepsilon)+2l\Gamma(t,\varepsilon)+l\int_0^t\Gamma(s,\varepsilon)\mathrm{d}s+M\varepsilon^{N+1}\leqslant$$
$$-2\varepsilon^{N+1}r+M\varepsilon^{N+1}.$$

如果取 $r>\max\{l\sigma,M/2\}$，则由上式得到

$$\varepsilon\alpha''(t,\varepsilon)\geqslant f(t,\alpha(t,\varepsilon),\alpha_t,[T\alpha](t),\alpha'(t,\varepsilon),\varepsilon),\qquad t\in[0,1].\tag{37}$$

同理可证

$$\varepsilon\beta''(t,\varepsilon)\leqslant f(t,\beta(t,\varepsilon),\beta_t,[T\beta](t),\beta'(t,\varepsilon),\varepsilon),\qquad t\in[0,1].\tag{38}$$

因而,由(32)、(34)、(35)、(36)、(37)和(38),并结合定理1易得边值问题(1)~(2)存在解 $x(t,\varepsilon)$ 满足

$$\alpha(t,\varepsilon)\leqslant x(t,\varepsilon)\leqslant\beta(t,\varepsilon),\qquad t\in[0,1],$$

即

$$|x(t,\varepsilon)-X_N(t,\varepsilon)|\leqslant w(t,\varepsilon)+M_1\varepsilon^{N+1},\qquad t\in[0,1],$$

其中 $M_1=(r/l)(2\mathrm{e}^{2l(1+r)/m}-1)$ 为与 ε 无关的常数.

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Singularly Perturbed Nonlinear Boundary Value Problem for a Kind of Volterra Type Functional Differential Equation

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Abstract: By employing the theory of differential inequality and some analysis methods, a nonlinear boundary value problem subject to a general kind of second order Volterra functional differential equation was considered first. Then, by constructing the right_side layer function and the outer solution, a nonlinear boundary value problem subject to a kind of second order Volterra functional differential equation with a small parameter was studied further. By using the differential mean value theorem and the technique of upper and lower solution, a new result on the existence of the solutions to the boundary value problem is obtained, and a uniformly valid asymptotic expansions of the solution is given as well.

Key words: singular perturbation; functional differential equation; boundary value problem; uniformly valid asymptotic expansion