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夹层椭圆形板的 1/ 3 亚谐解*

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摘要: 研究了夹层椭圆形板的非线性强迫振动问题· 在以 5 个位移分量表示的夹层椭圆板的运动方程的基础上, 导出了相应的非线性动力方程· 提出一类强非线性动力系统的叠加_叠代谐波平衡法· 将描述动力系统的二阶常微分方程, 化为基本解为未知函数的基本微分方程和派生解为未知函数的增量微分方程· 通过叠加_叠代谐波平衡法得出了椭圆板的 1/ 3 亚谐解· 同时, 对叠加_叠代谐波平衡法和数值积分法的精度进行了比较· 并且讨论了 1/ 3 亚谐解的渐近稳定性·

关 键 词: 夹层椭圆形板; 叠加_叠代谐波平衡法; 1/ 3 亚谐解; 分岔
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引 言

夹层板具有重量轻、刚度高等优良性能而成为航空、航天和海洋工程的重要结构元件· 近年来, 对夹层板壳的非线性问题的分析已引起了极大的研究兴趣, 尤其在夹层板壳的非线性振动方面已取得了一些成果[1,2]· 但是, 由于非线性数学的困难, 对于夹层板壳的分岔问题还无人进行探讨· 本文研究了夹层椭圆形板的 1/ 3 亚谐解·

目前, 新近兴起学科 ——混沌学为非线性系统的分析开拓了广阔前景[3~7]· 经典摄动法等难以求解强非线性问题, 主要局限在于不合理的常频率假设· 近 10 多年来, 虽然强非线性振动研究取得了一系列成果[8~9], 但对于亚(超) 谐解的定量研究仍然是悬而未决的问题, 不得不依赖于数值解法[10]·

本文提出了叠加_叠代谐波平衡法(SIHB), 把强非线性动力系统的稳态解问题转化为较少数量的非线性代数方程组问题· 利用 Maple 程序可以方便地求解这些非线性代数方程组, 得到稳态解的近似解析表达式· 通过叠加_叠代逐步求出亚(超) 谐解·

1 基 本 方 程

考虑在均匀横向载荷 $Q_0\cos\Omega_0 t$ 作用下的夹层椭圆形板, 如图 1 所示, a 和 b 分别是椭圆板

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的长半轴和短半轴, h_1 是表层厚度, h_0 是上、下表层间的距离• 坐标平面 xy 与夹芯中面一致• 假定上下表层的材料性质和厚度相同•

采用 Reissner 的假定: 1) 材料服从于 Hooke 定律; 2) 夹心横向不可压缩; 3) 夹心沿板面方向不能承受载荷; 4) 表层处于薄膜应力状态; 5) 夹心中面法线在变形后仍保持为直线•

在上述假设的基础上, 应用 Hamilton 原理, 导出以 5 个位移分量 u、v、w、ψ_x 和 ψ_y 表示的夹层椭圆形板非线性振动的运动方程[1, 2]:

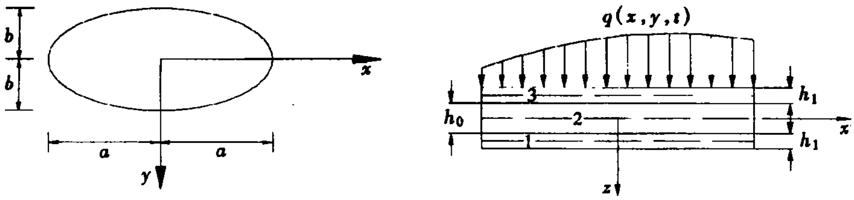


图 1 夹层椭圆形板的坐标和几何尺寸

$$2\frac{\partial^2 u}{\partial x^2}+(1-\nu)\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial w}{\partial x}\nabla^2 w\right)+(1+\nu)\left\{\frac{\partial^2 v}{\partial x\partial y}+\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\}=0, \tag{1a}$$

$$2\frac{\partial^2 v}{\partial y^2}+(1-\nu)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial w}{\partial y}\nabla^2 w\right)+(1+\nu)\left\{\frac{\partial^2 u}{\partial x\partial y}+\frac{1}{2}\frac{\partial}{\partial y}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\}=0, \tag{1b}$$

$$\begin{aligned}D\left(\frac{\partial^3\psi_x}{\partial x^3}+\frac{\partial^3\psi_x}{\partial x\partial y^2}+\frac{\partial^3\psi_y}{\partial x^2\partial y}+\frac{\partial^3\psi_y}{\partial y^3}+\nabla^2 w\right)&=\rho\frac{\partial^2 w}{\partial t^2}+\delta\frac{\partial w}{\partial t}-Q_0\cos\Omega t-\\
&\frac{2Eh_1}{1-\nu^2}\left[\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)\frac{\partial^2 w}{\partial x^2}+\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)\frac{\partial^2 w}{\partial y^2}+(1-\nu)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial^2 w}{\partial x\partial y}\right]+\\
&\frac{Eh_1}{1-\nu^2}\left\{\left[\left(\frac{\partial w}{\partial x}\right)^2+\nu\left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial^2 w}{\partial x^2}+\right.\\
&\left.\left[\left(\frac{\partial w}{\partial y}\right)^2+\nu\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial y^2}+2(1-\nu)\frac{\partial w}{\partial x}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right\},\end{aligned} \tag{1c}$$

$$\frac{D}{G_2h_0}\left(\frac{\partial^2\psi_x}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2\psi_x}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2\psi_y}{\partial x\partial y}\right)-\left(\psi_x+\frac{\partial w}{\partial x}\right)=0, \tag{1d}$$

$$\frac{D}{G_2h_0}\left(\frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2}+\frac{\partial^2\psi_y}{\partial y^2}\right)-\left(\psi_x+\frac{\partial w}{\partial x}\right)=0, \tag{1e}$$

式中: u, v 是夹层板中面上点沿 x 和 y 方向的位移; w 是板的挠度; ψ_x, ψ_y 分别是夹心中面在 xz 和 yz 平面内的转角; G_2 是夹心的剪切模量, E、ν 分别是表层材料的弹性模量和 Poisson 比, $D = Eh_0^2h_1/(2(1-\nu^2))$ 是夹层板的抗变刚度, $\rho = \rho_c + 2\rho_f$ 是夹层板的面密度•

引进下列无量纲量

$$\left\{\begin{aligned}&\lambda=\frac{a}{b},\quad \xi=\frac{x}{a},\quad \eta=\frac{y}{b},\quad U=\frac{au}{h_0^2},\quad V=\frac{bv}{h_0^2},\quad W=\frac{w}{h_0},\\
&\Phi_x=\frac{a}{h_0}\psi_x,\quad \Phi_y=-\frac{a}{h_0}\psi_y,\quad k=\frac{D}{G_2h_0a^2},\quad \tau=\sqrt{\frac{D}{a^4\rho}}t,\\
&q_0=\frac{a^4}{Dh_0}Q_0,\quad \Omega=\sqrt{\frac{\rho a^4}{D}}\Omega_0,\quad 2n=\frac{\delta a^2}{h_0\sqrt{\rho D}}•\end{aligned}\right. \tag{2}$$

利用这些无量纲量, 方程(1) 在非线性强迫振动情况下的无纲量形式为

$$2\frac{\partial^2 U}{\partial\xi^2}+(1-\nu)\left[\lambda^2\frac{\partial^2 U}{\partial\eta^2}+\frac{\partial W}{\partial\xi}\left(\frac{\partial^2 W}{\partial^2\xi}+\lambda^2\frac{\partial^2 W}{\partial^2\xi\partial\eta}\right)\right]+$$
$$(1+\nu)\left\{\lambda^2\frac{\partial^2 V}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial}{\partial\xi}\left[\left(\frac{\partial W}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial W}{\partial\eta}\right)^2\right]\right\}=0, \tag{3a}$$

$$2\lambda^2\frac{\partial^2 V}{\partial\eta^2}+(1-\nu)\left[\frac{\partial^2 V}{\partial\xi^2}+\frac{\partial W}{\partial\eta}\left(\frac{\partial^2 W}{\partial\xi^2}+\lambda^2\frac{\partial^2 W}{\partial\eta^2}\right)\right]+$$
$$(1+\nu)\left\{\frac{\partial^2 U}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial}{\partial\eta}\left[\left(\frac{\partial W}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial W}{\partial\eta}\right)^2\right]\right\}=0, \tag{3b}$$

$$\frac{\partial^2 W}{\partial\tau^2}+2n\frac{\partial W}{\partial\tau}-\frac{\partial^3\Phi_x}{\partial\xi^3}-\lambda^2\frac{\partial^3\Phi_x}{\partial\xi\partial\eta^2}-\lambda\frac{\partial^3\Phi_y}{\partial\xi^2\partial\eta}-\lambda\frac{\partial^3\Phi_y}{\partial\eta^3}-$$
$$4\left[\left(\frac{\partial U}{\partial\xi}+\nu\lambda^2\frac{\partial V}{\partial\eta}\right)\frac{\partial^2 W}{\partial\xi^2}+\lambda^2\left(\lambda^2\frac{\partial V}{\partial\eta}+\nu\frac{\partial U}{\partial\xi}\right)\frac{\partial^2 W}{\partial\eta^2}+\right.$$
$$\left.(1-\nu)\lambda^2\left(\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\xi}\right)\frac{\partial^2 W}{\partial\xi\partial\eta}\right]-2\left[\left(\frac{\partial W}{\partial\xi}\right)^2+\nu\lambda^2\left(\frac{\partial W}{\partial\eta}\right)^2\right]\frac{\partial^2 W}{\partial\xi^2}-$$
$$2\lambda^2\left[\lambda^2\left(\frac{\partial W}{\partial\eta}\right)^2+\nu\left(\frac{\partial W}{\partial\xi}\right)^2\right]\frac{\partial^2 W}{\partial\eta^2}-$$
$$4(1-\nu)\lambda^2\frac{\partial W}{\partial\xi}\frac{\partial W}{\partial\eta}\frac{\partial^2 W}{\partial\xi\partial\eta}-q_0\cos\Omega\tau=0, \tag{3c}$$

$$k\left(\frac{\partial^2\Phi_x}{\partial\xi^2}+\frac{1-\nu}{2}\lambda^2\frac{\partial^2\Phi_x}{\partial\eta^2}+\frac{1+\nu}{2}\lambda\frac{\partial^2\Phi_y}{\partial\xi\partial\eta}\right)-\left(\Phi_x+\frac{\partial W}{\partial\xi}\right)=0, \tag{3d}$$

$$k\left(\frac{1+\nu}{2}\lambda\frac{\partial^2\Phi_x}{\partial\xi\partial\eta}+\frac{1-\nu}{2}\frac{\partial^2\Phi_y}{\partial\xi^2}+\lambda^2\frac{\partial^2\Phi_y}{\partial\eta^2}\right)-\left(\Phi_y+\lambda\frac{\partial W}{\partial\eta}\right)=0. \tag{3e}$$

考虑夹层椭圆板的边缘为刚性夹紧固定, 因此, 求解方程(3) 的边界条件的无量纲形式为

$$\text{当 }\xi^2+\eta^2=1\text{ 时},\quad U=V=W=\Phi_x=\Phi_y=0. \tag{4}$$

采用伽辽金方法对具有固定边界的夹层椭圆板的非线性问题进行单模态分析. 选取满足边界条件(3) 的位移和转角函数具有如下分离形式

$$U=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}(\tau)(1-\xi^2-\eta^2)\xi^{2i+1}\eta^{2j}, \tag{5a}$$

$$V=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}(\tau)(1-\xi^2-\eta^2)\xi^{2i}\eta^{2j+1}, \tag{5b}$$

$$W=\varphi(\tau)\left[1-\frac{2+16k}{1+16k}(\xi^2+\eta^2)+\frac{1}{1+16k}(\xi^2+\eta^2)^2\right], \tag{5c}$$

$$\Phi_x=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}(\tau)(1-\xi^2-\eta^2)\xi^{2i+1}\eta^{2j}, \tag{5d}$$

$$\Phi_y=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}d_{ij}(\tau)(1-\xi^2-\eta^2)\xi^{2i}\eta^{2j+1}, \tag{5e}$$

其中 $\varphi(\tau)$ 是无量纲时间 τ 的函数, 其最大值为 $\varphi_{\max}=W_m=w_m/h_0$, 这里 w_m 是夹层椭圆板的中心挠度.

将式(5) 代入方程(3), 并利用 Galerkin 积分方程, 便得如下的关于时间函数 φ 的非线性常微分方程

$$\ddot{\varphi}+2n\dot{\varphi}+p^2\varphi+\mu\varphi^3=q\cos\Omega\tau, \tag{6}$$

式中 p 是夹层椭圆形板的线性振动固有频率,

$$p=\left(\frac{\alpha_1}{\alpha_3}\right)^{1/2},\quad \mu=\frac{\alpha_2}{\alpha_3},\quad q=\frac{\alpha_4}{\alpha_3}q_0, \tag{7}$$

其中

$$\alpha_1=-\int_{-1}^{1}\int_{-\sqrt{1-\eta^2}}^{\sqrt{1-\eta^2}}\left[\frac{\partial^3\Phi_x}{\partial\xi^3}+\lambda^2\frac{\partial^3\Phi_x}{\partial\xi\partial\eta^2}+\lambda\frac{\partial^3\Phi_y}{\partial\xi^2\partial\eta}+\lambda^3\frac{\partial^3\Phi_y}{\partial\eta^3}\right]\left[1-\frac{2+16k}{1+16k}(\xi^2+\eta^2)+\frac{1}{1+16k}(\xi^2+\eta^2)^2\right]\mathrm{d}\eta\mathrm{d}\xi, \tag{8a}$$

$$\begin{aligned}\alpha_2=-\int_{-1}^{1}\int_{-\sqrt{1-\eta^2}}^{\sqrt{1-\eta^2}}\Bigg\{&4\left[\left(\frac{\partial U}{\partial\xi}+\nu\lambda^2\frac{\partial V}{\partial\eta}\right)\frac{\partial^2 W}{\partial\xi^2}+\lambda^2\left(\lambda^2\frac{\partial V}{\partial\eta}+\nu\frac{\partial U}{\partial\xi}\right)\frac{\partial^2 W}{\partial\eta^2}+(1-\nu)\lambda^2\left(\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\xi}\right)\frac{\partial^2 W}{\partial\xi\partial\eta}\right]+\\&2\left[\left(\frac{\partial W}{\partial\xi}\right)^2+\nu\lambda^2\left(\frac{\partial W}{\partial\eta}\right)^2\right]\frac{\partial^2 W}{\partial\xi^2}+2\lambda^2\left[\lambda^2\left(\frac{\partial W}{\partial\eta}\right)^2+\nu\left(\frac{\partial W}{\partial\eta}\right)^2\right]\frac{\partial^2 W}{\partial\eta^2}+\\&4(1-\nu)\lambda^2\frac{\partial W}{\partial\xi}\frac{\partial W}{\partial\eta}\frac{\partial^2 W}{\partial\xi\partial\eta}\Bigg\}\left[1-\frac{2+16k}{1+16k}(\xi^2+\eta^2)+\frac{1}{1+16k}(\xi^2+\eta^2)^2\right]\mathrm{d}\eta\mathrm{d}\xi,\end{aligned} \tag{8b}$$

$$\alpha_3=\int_{-1}^{1}\int_{-\sqrt{1-\eta^2}}^{\sqrt{1-\eta^2}}\left[1-\frac{2+16k}{1+16k}(\xi^2+\eta^2)+\frac{1}{1+16k}(\xi^2+\eta^2)^2\right]^2\mathrm{d}\eta\mathrm{d}\xi, \tag{8c}$$

$$\alpha_4=\int_{-1}^{1}\int_{-\sqrt{1-\eta^2}}^{\sqrt{1-\eta^2}}\left[1-\frac{2+16k}{1+16k}(\xi^2+\eta^2)+\frac{1}{1+16k}(\xi^2+\eta^2)^2\right]\mathrm{d}\eta\mathrm{d}\xi, \tag{8d}$$

其中

$$U=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1-\xi^2-\eta^2)\xi^{2i+1}\eta^{2j},\quad V=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1-\xi^2-\eta^2)\xi^{2i}\eta^{2j+1},$$

$$W=1-\frac{2+16k}{1+16k}(\xi^2+\eta^2)+\frac{1}{1+16k}(\xi^2+\eta^2)^2,$$

$$\Phi_x=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1-\xi^2-\eta^2)\xi^{2i+1}\eta^{2j},\quad \Phi_y=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1-\xi^2-\eta^2)\xi^{2i}\eta^{2j+1},$$

方程(6)是硬弹簧型 Duffing 方程·

引入如下无量纲参数

$$x=\frac{\varphi}{p}\sqrt{\mu},\quad \tau=p\tau,\quad n=\frac{n}{p},\quad q=\frac{q\sqrt{\mu}}{p^3},\quad \Omega=\frac{\Omega}{p},$$

得到标准 Duffing 方程(仍将 τ, Ω, q 和 n 换写成 t, Ω, q 和 n)

$$\ddot{x}+2n\dot{x}+x+x^3=q\cos\Omega t\cdot \tag{9}$$

2 叠加_叠代谐波平衡法

代替特例(6),我们考虑一般系统的强非线性单自由度强迫振动问题:

$$\ddot{x}+2n\dot{x}+f(x)=q\cos\Omega t\cdot \tag{10}$$

叠加_叠代谐波平衡法求解周期解的过程分成两个主要步骤:

第一步 Newton 叠代过程

假设

$$x(t) + x_0(t) + y(t) \tag{11}$$

为方程(10)的所求周期解，其中 $x_0(t)$ 为基本解，$y(t)$ 为派生解，基本解满足基本方程：

$$\ddot{x}_0 + 2n\dot{x}_0 + f(x_0) = q\cos\Omega t; \tag{12}$$

派生解满足增量方程：

$$\ddot{y} + 2n\dot{y} + \sum_{k=1}^{\infty}\frac{1}{k!}f^{(k)}(x)\Big|_{x=x_0} y^k = 0. \tag{13}$$

第二步 W. Ritz 平均过程

由(12)采用 Ritz 平均法，求基谐解

$$x = a_0 + a_1\cos\Omega t + b_1\sin\Omega t. \tag{14}$$

由(13)采用 W. Ritz 平均法，逐步求分岔解 $y(t)$.

3 解析解

3.1 基谐解

假设方程(6)的基谐解为

$$\varphi(t) = a_1(t)\cos\Omega t + b_1(t)\sin\Omega t. \tag{15}$$

将假设解(15)代入方程(6)，用 Ritz 平均法，得到关于幅值参数的一个非线性微分方程组：

$$\ddot{a}_1 + 2n\dot{a}_1 + 2\Omega\dot{b}_1 + (p^2 - \Omega^2)a_1 + 2n\Omega b_1 + \frac{3}{4}\mu a_1(a_1^2 + b_1^2) - q = 0, \tag{16a}$$

$$\ddot{b}_1 - 2\Omega\dot{a}_1 + 2n\dot{b}_1 - 2n\Omega a_1 + (p^2 - \Omega^2)b_1 + \frac{3}{4}\mu b_1(a_1^2 + b_1^2) = 0. \tag{16b}$$

稳态基谐解为

$$\varphi(t) = a_1\cos\Omega t + b_1\sin\Omega t, \tag{17}$$

它的系数是方程组(16)的奇点，可通过求解下列非线性代数方程组得到

$$(p^2 - \Omega^2)a_1 + 2n\Omega b_1 + \frac{3}{4}\mu a_1(a_1^2 + b_1^2) - q = 0, \tag{18a}$$

$$-2n\Omega a + (p^2 - \Omega^2)b_1 + \frac{3}{4}\mu b_1(a_1^2 + b_1^2) = 0. \tag{18b}$$

稳态基谐解的稳定性由微分方程组(16)的线性化系数矩阵

$$\boldsymbol{A}_1 = \begin{bmatrix} \dfrac{3a_1b_1}{4\Omega} - n & \dfrac{1-\Omega^2}{2\Omega} + \dfrac{3(a_1^2 + 3b_1^2)}{8\Omega} \\ -\dfrac{1-\Omega^2}{2\Omega} - \dfrac{3(3a_1^2 + b_1^2)}{8\Omega} & -n - \dfrac{3}{4\Omega}a_1b_1 \end{bmatrix} \tag{19}$$

的特征根进行判定 $(p = 1, \mu = 1)$.

3.2 1/3 亚谐解

假设方程(6)的 1/3 亚谐解为：

$$\varphi(t) = \varphi_1(t) + \varphi_{1/3}(t); \tag{20}$$

基本解

$$\varphi_1(t) = a_1(t)\cos\Omega t + b_1(t)\sin\Omega t; \tag{21a}$$

派生解

$$\varphi_{1/3}(t) = a_{1/3}(t)\cos\frac{\Omega t}{3} + b_{1/3}(t)\sin\frac{\Omega t}{3}. \tag{21b}$$

将(20)代入(6)得到派生解满足增量方程:

$$\ddot{\varphi}_{1/3} + 2n\dot{\varphi}_{1/3} + p^2\varphi_{1/3} + 3\mu\varphi_1^2\varphi_{1/3} + 3\mu\varphi_1\varphi_{1/3}^2 + \mu\varphi_{1/3}^3 = 0. \tag{22}$$

将(20)代入(6),用 Ritz 平均法,得到关于幅值参数的一个非线性微分方程组:

$$\begin{aligned}&\ddot{a}_1 + 2n\dot{a}_1 + 2\Omega\dot{b}_1 + (p^2 - \Omega^2)a_1 + 2n\Omega b_1 - q + \\ &\quad \mu\left\{\frac{3}{4}a_1(a_1^2 + b_1^2 + 2a_{1/3}^2 + 2b_{1/3}^2) + \frac{1}{4}a_{1/3}(a_{1/3}^2 - 3b_{1/3}^2)\right\} = 0,\end{aligned} \tag{23a}$$

$$\begin{aligned}&\ddot{b}_1 + 2n\dot{b}_1 - 2\Omega\dot{a}_1 + (p^2 - \Omega^2)b_1 + 2n\Omega a_1 + \\ &\quad \mu\left\{\frac{3}{4}b_1(a_1^2 + b_1^2 + 2a_{1/3}^2 + 2b_{1/3}^2) + \frac{1}{4}b_{1/3}(3a_{1/3}^2 - b_{1/3}^2)\right\} = 0,\end{aligned} \tag{23b}$$

$$\begin{aligned}&\ddot{a}_{1/3} + 2n\dot{a}_{1/3} + \frac{2}{3}\Omega\dot{b}_{1/3} + \left(p^2 - \frac{1}{9}\Omega^2\right)a_{1/3} + \frac{2}{3}n\Omega b_{1/3} + \\ &\quad \mu\left\{\frac{3}{4}a_{1/3}(a_{1/3}^2 + b_{1/3}^2 + 2a_1^2 + 2b_1^2) + \right. \\ &\quad \left. \frac{3}{4}a_1(a_{1/3}^2 - b_{1/3}^2) + \frac{3}{2}b_1a_{1/3}b_{1/3}\right\} = 0,\end{aligned} \tag{23c}$$

$$\begin{aligned}&\ddot{b}_{1/3} + 2n\dot{b}_{1/3} - \frac{2}{3}\Omega\dot{a}_{1/3} + \left(p^2 - \frac{1}{9}\Omega^2\right)b_{1/3} - \frac{2}{3}n\Omega a_{1/3} + \\ &\quad \mu\left\{\frac{3}{4}b_{1/3}(a_{1/3}^2 + b_{1/3}^2 + 2a_1^2 + 2b_1^2) - \frac{3}{2}a_1a_{1/3}b_{1/3} + \right. \\ &\quad \left. \frac{3}{4}b_1(a_{1/3}^2 - b_{1/3}^2)\right\} = 0.\end{aligned} \tag{23d}$$

稳态 1/3 亚谐解为

$$\varphi(t) = \varphi_1(t) + \varphi_{1/3}(t); \tag{24}$$

基本解

$$\varphi_1(t) = a_1\cos\Omega t + b_1\sin\Omega t, \tag{25a}$$

派生解

$$\varphi_{1/3}(t) = a_{1/3}\cos\frac{\Omega t}{3} + b_{1/3}\sin\frac{\Omega t}{3} \tag{25b}$$

的系数就是微分方程组(23)的奇点,可通过求解下列非线性代数方程组得到

$$\begin{aligned}&(p^2 - \Omega^2)a_1 + 2n\Omega b_1 - q + \mu\left\{\frac{3}{4}a_1(a_1^2 + b_1^2 + 2a_{1/3}^2 + 2b_{1/3}^2) + \right. \\ &\quad \left. \frac{1}{4}a_{1/3}(a_{1/3}^2 - 3b_{1/3}^2)\right\} = 0,\end{aligned} \tag{26a}$$

$$\begin{aligned}&(p^2 - \Omega^2)b_1 - 2n\Omega a_1 + \mu\left\{\frac{3}{4}b_1(a_1^2 + b_1^2 + 2a_{1/3}^2 + 2b_{1/3}^2) + \right. \\ &\quad \left. \frac{1}{4}b_{1/3}(3a_{1/3}^2 - b_{1/3}^2)\right\} = 0,\end{aligned} \tag{26b}$$

$$\begin{aligned}&\left(p^2 - \frac{1}{9}\Omega^2\right)a_{1/3} + \frac{2}{3}n\Omega b_{1/3} + \mu\left\{\frac{3}{4}a_{1/3}(a_{1/3}^2 + b_{1/3}^2 + 2a_1^2 + 2b_1^2) + \right. \\ &\quad \left. \frac{3}{4}a_1(a_{1/3}^2 - b_{1/3}^2) + \frac{3}{2}b_1a_{1/3}b_{1/3}\right\} = 0,\end{aligned} \tag{26c}$$

$$\begin{aligned}&\left(p^2 - \frac{1}{9}\Omega^2\right)b_{1/3} - \frac{2}{3}n\Omega a_{1/3} + \mu\left\{\frac{3}{4}b_{1/3}(a_{1/3}^2 + b_{1/3}^2 + 2a_1^2 + 2b_1^2) - \right. \\ &\quad \left. \frac{3}{2}a_1a_{1/3}b_{1/3} + \frac{3}{4}b_1(a_{1/3}^2 - b_{1/3}^2)\right\} = 0.\end{aligned} \tag{26d}$$

稳态 1/3 亚谐解(24) 的稳定性通过微分方程组(23) 的线性化系数矩阵的特征根进行判定• 利用 Maple 程序通过叠代求解, 并计算特征根判别其稳定性, 其叠代步骤如下:

第一步: 求解方程组(18) 得到初值 $a_1^{(0)}$, $b_1^{(0)}$, $a_{1/3}^{(0)} = 0$, $b_{1/3}^{(0)} = 0$;

第二步: 取 $a_1 = a_1^{(0)}$, $b_1 = b_1^{(0)}$, 求解(26c)、(26d), 得到 $a_{1/3}^{(1)}$、$b_{1/3}^{(1)}$;

第三步: 取 $a_{1/3} = a_{1/3}^{(1)}$, $b_{1/3} = b_{1/3}^{(1)}$, 求解(26a)、(26b), 得到 $a_1^{(1)}$、$b_1^{(1)}$;

…………

第四步: 取 $a_1 = a_1^{(k)}$, $b_1 = b_1^{(k)}$, 求解(26c)、(26d), 得到 $a_{1/3}^{(k+1)}$、$b_{1/3}^{(k+1)}$;

第五步: 取 $a_{1/3} = a_{1/3}^{(k+1)}$, $b_{1/3} = b_{1/3}^{(k+1)}$, 求解(26a)、(26b), 得到 $a_1^{(k+1)}$、$b_1^{(k+1)}$, 若 $| a_1^{(k+1)} - a_1^{(k)} | < \varepsilon = 10^{-6}$, 进行下一步, 否则, 转入第四步;

第六步: 最后得到 1/3 亚谐解(24) 的系数 $a_1 = a_1^{(k+1)}$、$b_1 = b_1^{(k+1)}$、$a_{1/3} = a_{1/3}^{(k+1)}$、$b_{1/3} = b_{1/3}^{(k+1)}$, 并判断稳定性(结束)•

4 数值计算结果

考虑方程(9), $n = 0.05$, $\Omega = 6.3$, $q = 40$ 的情况• 利用 Maple 程序求解, 并计算特征根判别其稳定性• 由方程(18) 可以求得基谐解(17) 的系数 A、B、C, 如图 2(a) 所示• 其中 B 是鞍点, 由 A 分岔出 7 个 1/3 亚谐解; 由 C 分岔出超谐解(另文讨论)•

$$A: \begin{cases} a_1 = -1.056\,434\,372 \\ b_1 = 0.017\,582\,713\,07 \end{cases} (\text{焦点}), \quad B: \begin{cases} a_1 = -6.563\,392\,791 \\ b_1 = 0.685\,889\,977\,4 \end{cases} (\text{鞍点}),$$

$$C: \begin{cases} a_1 = 7.593\,367\,164 \\ b_1 = 0.921\,507\,309\,5 \end{cases} (\text{焦点})•$$

稳态 1/3 亚谐解(24) 的系数由方程组(26) 求得, 基本解(25a) 的系数为 A_0, A_1, A_2 如图 2(b) 所示; 派生解(25b) 的系数为①~⑦如图 2(c) 所示•

$$A_0: \begin{cases} a_1 = -1.056\,434\,372 \\ b_1 = 0.017\,582\,713\,07 \end{cases} (\text{焦点}), \quad A_1: \begin{cases} a_1 = -1.203\,942\,210\,0 \\ b_1 = 0.029\,939\,240\,54 \end{cases} (\text{焦点}),$$

$$A_2: \begin{cases} a_1 = -1.109\,564\,257 \\ b_1 = 0.021\,158\,117\,17 \end{cases} (\text{鞍点})• \quad ①\begin{cases} a_{1/3} = 0 \\ b_{1/3} = 0 \end{cases} (\text{焦点}),$$

$$②\begin{cases} a_{1/3} = 2.012\,720\,533\,00 \\ b_{1/3} = 0.060\,974\,947\,77 \end{cases} (\text{焦点}), \quad ③\begin{cases} a_{1/3} = -1.059\,166\,12 \\ b_{1/3} = 1.712\,579\,638 \end{cases} (\text{焦点}),$$

$$④\begin{cases} a_{1/3} = -0.953\,554\,412\,7 \\ b_{1/3} = -1.773\,554\,586\,0 \end{cases} (\text{焦点}), \quad ⑤\begin{cases} a_{1/3} = 0.578\,487\,054\,3 \\ b_{1/3} = 0.819\,441\,975\,5 \end{cases} (\text{鞍点}),$$

$$⑥\begin{cases} a_{1/3} = -0.998\,901\,094\,9 \\ b_{1/3} = 0.091\,263\,497\,09 \end{cases} (\text{鞍点}), \quad ⑦\begin{cases} a_{1/3} = 0.420\,414\,040\,50 \\ b_{1/3} = -0.910\,705\,472\,6 \end{cases} (\text{鞍点})•$$

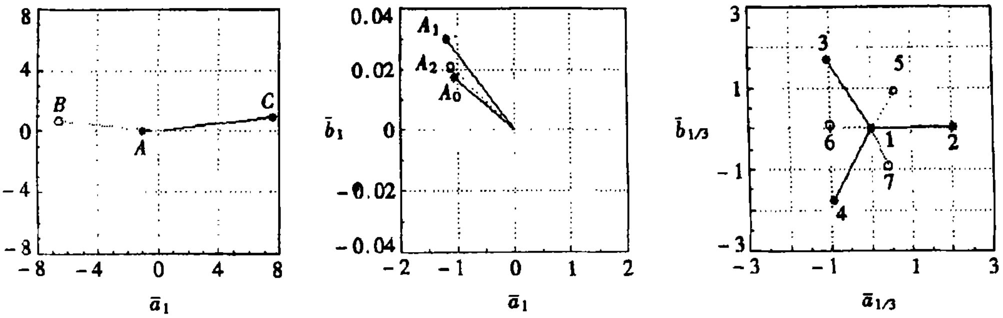
代入(24) 得到 7 个 1/3 亚谐解(27), 其中(27b) 如图 3 所示•

$$x = 1.056\,6\cos(6.3t - 3.125\,0) \quad (\text{稳定}), \tag{27a}$$

$$x = 1.204\,3\cos(6.3t - 3.116\,7) + 2.013\,6\cos(2.1t - 0.030\,286) \quad (\text{稳定}), \tag{27b}$$

$$x = 1.204\,3\cos(6.3t - 3.116\,7) + 2.013\,6\cos(2.1t - 2.124\,7) \quad (\text{稳定}), \tag{27c}$$

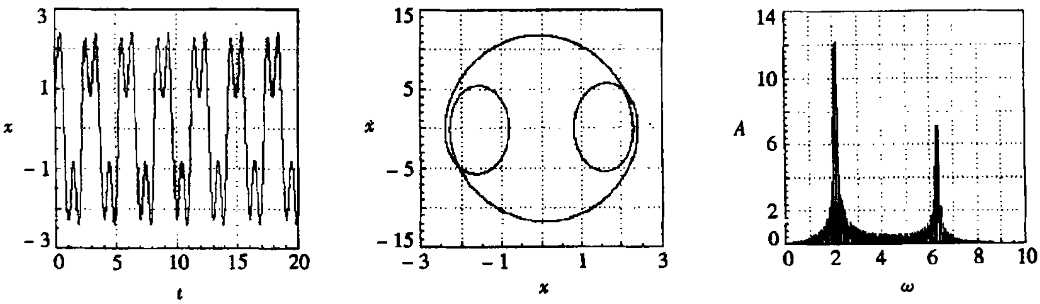
$$x = 1.204\,3\cos(6.3t - 3.116\,7) + 2.013\,6\cos(2.1t - 4.219\,1) \quad (\text{稳定}), \tag{27d}$$



(a) 基谐解 (b) 基本解 (c) 1/3 谐波解

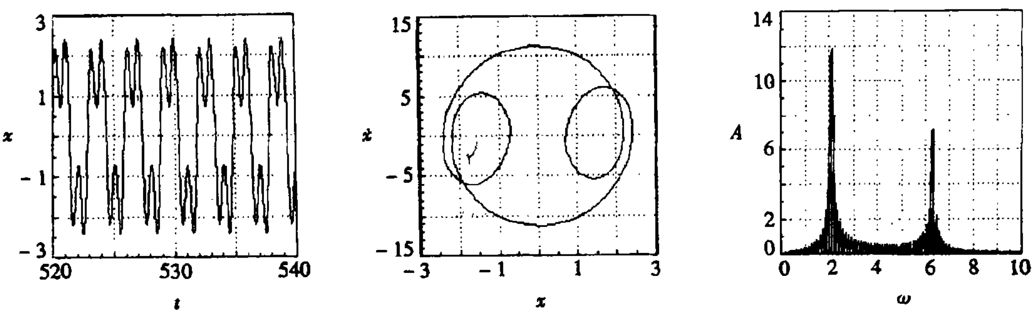
$n = 0.05$, $\Omega = 6.3$, $q = 40$, ●— 焦点, ○— 鞍点

图 2 Van der Pol 平面上 1/3 亚谐解的幅值分布



(a) 时程曲线 (b) 相图 (c) 频谱

图 3 1/3 亚谐分析解(27b)(SIHB 方法)



(a) 时程曲线 (b) 相图 (c) 频谱

图 4 1/3 亚谐数值解(RK 方法)

$$x = 1.1098\cos(6.3t - 3.1225) + 1.0031\cos(2.1t - 0.95609) \quad (\text{不稳定}), \tag{27e}$$

$$x = 1.1098\cos(6.3t - 3.1225) + 1.0031\cos(2.1t - 3.0505) \quad (\text{不稳定}), \tag{27f}$$

$$x = 1.1098\cos(6.3t - 3.1225) + 1.0031\cos(2.1t - 5.1449) \quad (\text{不稳定})\cdot \tag{27g}$$

采用双精度四阶 Runge_Kutta(RK) 法取时间步长 $\Delta t = T/35$, $T = 2\pi/\Omega$, 得到 $x_0 = 3$, $\dot{x}_0 = 1$ 时的 1/3 亚谐解见图 4; 振幅分岔如图 5 所示•

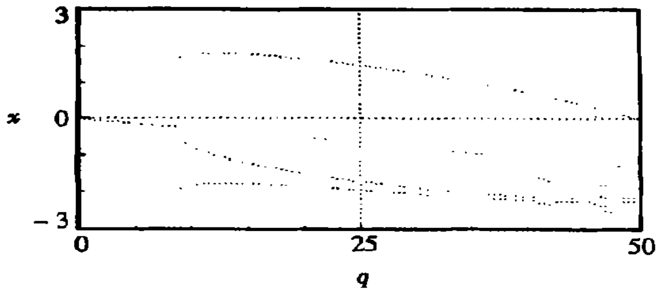


图 5　强迫振幅分岔图 (n = 0.05, Ω = 6.3)

5 结　　论

1) 夹层椭圆形板的 1/3 亚谐振动可以用假设解(24)描述,在亚谐频域内,由叠加_叠代谐波平衡法(SIHB)和数值积分法所得的稳态 1/3 亚谐解吻合得相当好•

2) 采用SIHB 法得到了 1/3 亚谐解的个数、解的稳定性、解的解析表达式和解的分岔情况•

3) 在 4 个稳定的 1/3 亚谐解中,其中一个亚谐解仍然为 1/1 基谐解,另外3 个亚谐解的基本解相同、1/3 亚谐成分具有对称性并且相位差是 2π/3•　由图 2(c)可以看出:焦点 1 为零解;3 个鞍点(5,6,7)构成正三角形;3 个焦点(2,3,4)亦构成正三角形•

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1/3 Subharmonic Solution of Elliptical Sandwich Plates

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Abstract: The problem of nonlinear forced oscillations for elliptical sandwich plates is dealt with. Based on the governing equations expressed in terms of five displacement components, the nonlinear dynamic equation of an elliptical sandwich plate under a harmonic force is derived. A superpositive_iterative harmonic balance (SIHB) method is presented for the steady_state analysis of strongly nonlinear oscillators. In a periodic oscillation, the periodic solutions can be expressed in the form of basic harmonics and bifurcate harmonics. Thus, an oscillation system which is described as a second order ordinary differential equation, can be expressed as fundamental differential equation with fundamental harmonics and incremental differential equation with derived harmonics. The 1/3 subharmonic solution of an elliptical sandwich plate is investigated by using the methods of SIHB. The SIHB method is compared with the numerical integration method. Finally, asymptotical stability of the 1/3 subharmonic oscillations is inspected.

Key words: elliptical sandwich plate; superpositive_ iterative harmonic balance (SIHB) method; 1/3 subharmonic solution; bifurcation