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关于AKNS族和一个耦合的MKdV族的规范等价可积系统与 r_矩阵*

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摘要: 得到了一个新的耦合的MKdV族• 通过规范变换, 首次从AKNS族中得到耦合的MKdV族的Lax表示、可积系与约束流; 利用Lax表示, 构造了耦合MKdV族的约束流的 r_矩阵, 同时也给出了该方程族约束流的第二组守恒积分与对合性•

关 键 词: 规范等价; r_矩阵; 可积系统

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引 言

在不同的特征值问题中存在着规范变换, 可用它来研究由不同的谱问题产生的无穷维可积约束流[1, 2]• 最近, 依赖于谱参数的经典的 r_矩阵已被广泛地用来研究有限维可积系统[3, 4], 然而, 用规范变换来研究它们之间的关系的文献还不多见• 本文的主要目的是介绍规范变换的一些新的应用, 诸如构造Hamilton系统、守恒积分• 因此, 可用规范变换将众所周知的约束流的Lax表示直接用来产生新的约束流的经典的Poisson结构与 r_矩阵, 等等• 为此, 我们给出下列结果•

对于AKNS特征值问题

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \boldsymbol{U}(\boldsymbol{w},\lambda)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad \boldsymbol{U}(\boldsymbol{w},\lambda) = \begin{pmatrix}-\lambda & q\\ r & \lambda\end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix}q\\ r\end{pmatrix}, \tag{1}$$

约束为

$$q = -\frac{1}{2}\langle \Psi_1, \Psi_1\rangle, \quad r = \frac{1}{2}\langle \Psi_2, \Psi_2\rangle, \tag{2}$$

对应的约束流为

$$\begin{cases}\Psi_{1x} = -\Lambda\Psi_1 - \dfrac{1}{2}\langle \Psi_1, \Psi_1\rangle\Psi_2 = \dfrac{\partial H}{\partial \Psi_2},\\[2mm] \Psi_{2x} = \dfrac{1}{2}\langle \Psi_2, \Psi_2\rangle\Psi_1 + \Lambda\Psi_2 = -\dfrac{\partial H}{\partial \Psi_1},\\[2mm] H = -\langle \Lambda\Psi_1, \Psi_2\rangle - \dfrac{1}{4}\langle \Psi_1, \Psi_1\rangle\langle \Psi_2, \Psi_2\rangle,\end{cases} \tag{3}$$

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这里 $\Psi_i = (\phi_{i1}, \phi_{i2}, \cdots, \phi_{iN})^T, (i = 1, 2)$; $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$ 且 $\langle \cdot, \cdot \rangle$ 表示 R^N 中的标准内积•

结论 1　系统(3)的第一组守恒积分如下

$$F_0 = -\langle \Psi_1, \Psi_2 \rangle$$

$$F_m = -\langle \Lambda^m \Psi_1, \Psi_2 \rangle - \frac{1}{4}\sum_{j=1}^{m}\begin{vmatrix} \langle \Lambda^{j-1}\Psi_1, \Psi_1\rangle & \langle \Lambda^{j-1}\Psi_1, \Psi_2\rangle \\ \langle \Lambda^{m-j}\Psi_2, \Psi_1\rangle & \langle \Lambda^{m-j}\Psi_2, \Psi_2\rangle \end{vmatrix} \quad (m = 1, 2, \cdots) \tag{4}$$

结论 2　约束流(3)的 Lax 表示为

$$\boldsymbol{M}_x(\lambda) = [\boldsymbol{U}(\lambda, \Psi_1, \Psi_2), \boldsymbol{M}(\lambda)], \tag{5}$$

$$\boldsymbol{U}(\lambda, \Psi_1, \Psi_2) = \begin{pmatrix} -\lambda & -\frac{1}{2}\langle \Psi_1, \Psi_1\rangle \\ \frac{1}{2}\langle \Psi_2, \Psi_2 \rangle & \lambda \end{pmatrix}, \tag{6}$$

$$\boldsymbol{M}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda)\end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j}\end{pmatrix}. \tag{7}$$

结论 3　由(7)给出的 $A(\lambda)$, $B(\lambda)$ 和 $C(\lambda)$ 关于标准的 Poisson 括号满足下列关系式•

$$\begin{cases} \{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0, \\ \{A(\lambda), B(\mu)\} = \dfrac{1}{\mu - \lambda}(B(\mu) - B(\lambda)), \\ \{A(\lambda), C(\mu)\} = \dfrac{1}{\mu - \lambda}(C(\lambda) - C(\mu)), \\ \{B(\lambda), C(\mu)\} = \dfrac{2}{\mu - \lambda}(A(\mu) - A(\lambda)). \end{cases} \tag{8}$$

结论 4　约束流(3)满足下面经典 Poisson 结构[5]

$$\{\boldsymbol{M}^{(1)}(\alpha_1) \overset{\leftarrow}{,} \boldsymbol{M}^{(2)}(\alpha_2)\} = [\boldsymbol{r}^{(12)}(\alpha_1, \alpha_2), \boldsymbol{M}^{(1)}(\alpha_1)] - [\boldsymbol{r}^{(21)}(\alpha_2, \alpha_1), \boldsymbol{M}^{(2)}(\alpha_2)], \tag{9}$$

其中 r_矩阵为

$$\boldsymbol{r}^{(ij)}(\alpha_i, \alpha_j) = \frac{1}{\alpha_j - \alpha_i}\boldsymbol{p}^{(ij)}, \tag{10}$$

这里 $\boldsymbol{M}^{(1)}(\alpha_1) = \boldsymbol{M}(\alpha_1) \otimes \boldsymbol{I}$, $\boldsymbol{M}^{(2)}(\alpha_2) = \boldsymbol{I} \otimes \boldsymbol{M}(\alpha_2)$, $\boldsymbol{I}$ 为 2×2 单位矩阵, $\{\boldsymbol{M}^{(1)}(\alpha_1) \overset{\leftarrow}{,} \boldsymbol{M}^{(2)}(\alpha_2)\}$ 表示 4×4 矩阵 $\boldsymbol{M}^{(1)}(\alpha_1)$ 和 $\boldsymbol{M}^{(2)}(\alpha_2)$ 的积, 但两个矩阵元素的积是用它们的 Poisson 括号来代替的, 而且矩阵 $\boldsymbol{p}^{(ij)} = \frac{1}{2}\sum_{n=0}^{3}\sigma_n^{(i)} \otimes \sigma_n^{(j)}$, $\sigma_n^{(i)}$ 是标准 Pauli 矩阵•

1　耦合的 MKdV 族的规范变换, Lax 表示和约束流

考虑如下特征值问题

$$\begin{cases} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_x = \boldsymbol{U}(\boldsymbol{w}, \lambda)\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \\ \boldsymbol{U}(\boldsymbol{w}, \lambda) = \begin{pmatrix} -u & \lambda \\ \lambda + 2v & u \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} u \\ v \end{pmatrix}. \end{cases} \tag{11}$$

其伴随表示为

$$V_x = [U(w,\lambda), V] = U(w,\lambda)V - VU(w,\lambda), \tag{12}$$

这里 V 有 Laurent 级数表示

$$V = V(w,\lambda) = \sum_{m=0}^{\infty}\begin{pmatrix} a_m(w) & b_m(w) \\ c_m(w) & -a_m(w)\end{pmatrix}\lambda^{-m}. \tag{13}$$

方程(12)和(13)导出下列递推关系式

$$\begin{cases} a_0 = 0,\ b_0 = 1,\ c_0 = 1, \\ a_1 = -u,\ b_1 = -v,\ c_1 = v,\ \cdots \\ a_{jx} = c_{j+1} - b_{j+1} - 2vb_j \qquad (j = 1, 2, \cdots). \\ b_{jx} = -2a_{j+1} - 2ub_j \\ c_{jx} = 2u + 2a_{j+1} + 4va_j, \end{cases} \tag{14}$$

递推关系式(14)也可表为

$$\begin{cases} \begin{pmatrix} a_{j+1} \\ b_{j+1}\end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\partial - u \\ -\frac{1}{2}\partial + \partial^{-1}u\partial & -v - \partial^{-1}v\partial \end{pmatrix}\begin{pmatrix} a_j \\ b_j\end{pmatrix}, \\ c_{j+1} = a_{jx} + b_{j+1} + 2vb_j \qquad (j = 1, 2, \cdots) \end{cases} \tag{15}$$

其中　$\partial = \dfrac{\partial}{\partial x}, \partial\partial^{-1} = \partial^{-1}\partial = 1$.

下面，置

$$V^{(m)} = V^{(m)}(w,\lambda) = \sum_{j=0}^{m}\begin{pmatrix} a_j(w) & b_j(w) \\ c_j(w) & -a_j(w)\end{pmatrix}\lambda^{m-j} + \begin{pmatrix} 0 & -b_m(w) \\ -b_m(w) & 0\end{pmatrix}. \tag{16}$$

利用(16)，定义特征函数的第 m 阶流为

$$\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}_{t_m} = V^{(m)}(w,\lambda)\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}. \tag{17}$$

(11)和(17)的相容性条件产生一个零曲率表示

$$U_{t_m} - (V^{(m)})_x + [U, V^{(n)}] = 0, \qquad (m = 1, 2, \cdots), \tag{18}$$

$$\begin{pmatrix} u \\ v\end{pmatrix}_{t_m} = \begin{pmatrix} -\partial & 0 \\ 0 & -\partial\end{pmatrix}\begin{pmatrix} a_m \\ b_m\end{pmatrix}. \tag{19}$$

在 $\lambda_{t_m} = 0$ 的条件下，(11)和(17)的相容性条件(18)正好等价于发展方程(19).

例1

$$\begin{pmatrix} u \\ v\end{pmatrix}_{t_3} = \begin{pmatrix} -\partial & 0 \\ 0 & -\partial\end{pmatrix}\begin{pmatrix} a_3 \\ b_3\end{pmatrix} = \begin{pmatrix} \frac{1}{4}u_{xxx} - \frac{3}{2}u^2u_x + \frac{3}{2}(uv^2)_x + \frac{3}{4}(v^2)_{xx} \\ \frac{1}{4}v_{xxx} - \frac{1}{2}uv_{xx} + \frac{3}{4}v^2v_x - \frac{3}{2}u^2v_x - 3uvu_x + \frac{3}{2}u_xv_x + \frac{3}{2}vu_{xx}\end{pmatrix}. \tag{20}$$

在方程(20)中，令 $v = 0$，则(20)成为 MKdV 方程[6]

$$u_{t_3} = \frac{1}{4}u_{xxx} - \frac{3}{2}u^2u_x.$$

因此,方程(19)称为耦合的 m 阶 MKdV 方程族•

容易验证如下定理•

定理 1.1　AKNS 特征值问题(1)和特征值问题(11)是规范等价的,其规范变换为

$$\begin{cases}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = T\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \quad T = \dfrac{1}{\sqrt{2}}\begin{pmatrix}\beta & -\beta\\ \beta^{-1} & \beta^{-1}\end{pmatrix},\\ q = \beta^2(-u-v),\quad r = \beta^{-2}(-u+v),\quad \beta = \exp\left[\displaystyle\int_{x_0}^{x} v\,\mathrm{d}x\right]\bullet\end{cases} \tag{21}$$

定义　如果存在规范变换将一个约束流变成另一个约束流,则称这两个约束流是规范等价的•

定理 1.2　AKNS 族的约束(2)和约束流(3)规范等价于耦合的 MKdV 族的约束

$$u = -\langle \Phi_1, \Phi_2\rangle,\quad v = \langle \Phi_1, \Phi_1\rangle \tag{22}$$

和约束流

$$\begin{cases}\Phi_{1x} = \langle \Phi_1, \Phi_2\rangle \Phi_1 + \Lambda\Phi_2 = \dfrac{\partial H}{\partial \Phi_2},\\ \Phi_{2x} = \Lambda\Phi_1 + 2\langle \Phi_1, \Phi_1\rangle \Phi_1 - \langle \Phi_1, \Phi_2\rangle \Phi_2 = -\dfrac{\partial H}{\partial \Phi_1},\\ H = \dfrac{1}{2}\langle \Lambda\Phi_2, \Phi_2\rangle - \dfrac{1}{2}\langle \Lambda\Phi_1, \Phi_1\rangle + \dfrac{1}{2}\langle \Phi_1, \Phi_2\rangle^2 - \dfrac{1}{2}\langle \Phi_1, \Phi_1\rangle^2,\end{cases} \tag{23}$$

其中 $\boldsymbol{w}$ 的边界条件[2] 为当 $|x| \to \infty$ 时, $\boldsymbol{w} \to 0$, $\Phi_i = (\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{iN})^{\mathrm{T}}$　$(i = 1, 2)$•

证明　利用(2)和(21)有

$$\begin{cases}u = -\langle \Phi_1, \Phi_2\rangle,\\ v = \dfrac{1}{2}(\langle \Phi_1, \Phi_1\rangle + \langle \Phi_2, \Phi_2\rangle)\bullet\end{cases} \tag{24}$$

应用(24)于(11),则 $\langle \Phi_1, \Phi_1\rangle = \langle \Phi_2, \Phi_2\rangle$• 代入(24),有

$$u = -\langle \Phi_1, \Phi_2\rangle,\quad v = \langle \Phi_1, \Phi_1\rangle\bullet \tag{25}$$

由(21),有

$$\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\end{pmatrix} = T^{-1}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix},\quad T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}\beta^{-1} & \beta\\ -\beta^{-1} & \beta\end{pmatrix}\qquad (j = 1, 2, \cdots, N)\bullet \tag{26}$$

借助于(25),(26)和 $\langle \Phi_1, \Phi_1\rangle = \langle \Phi_2, \Phi_2\rangle$, 得到

$$q = \beta^2(-u-v) = -\frac{1}{2}\langle \Psi_1, \Psi_1\rangle,$$

$$r = \beta^{-2}(-u+v) = \frac{1}{2}\langle \Psi_2, \Psi_2\rangle\bullet$$

下面,从系统(3)导出系统(23)• 为简便起见,假设系统(23)取形式

$$\Phi_x = U(\lambda, \Phi_1, \Phi_2)\Phi,\quad \Phi = (\Phi_1, \Phi_2)^{\mathrm{T}},,$$

系统(3)取形式

$$\Psi_x = U(\lambda, \Psi_1, \Psi_2)\Psi,\quad \Psi = (\Psi_1, \Psi_2)^{\mathrm{T}}\bullet$$

由规范变换(21),有

$$U(\lambda, \Phi_1, \Phi_2) = T^{-1}U(\lambda, \Psi_1, \Psi_2)T - T^{-1}T_x = \begin{pmatrix}\langle \Phi_1, \Phi_2\rangle & \Lambda\\ \Lambda + 2\langle \Phi_1, \Phi_1\rangle & -\langle \Phi_1, \Phi_2\rangle\end{pmatrix}\bullet$$

将(26)代入(3),并利用 $\langle \Phi_1, \Phi_1\rangle = \langle \Phi_2, \Phi_2\rangle$, 得到

$$H = H|_{(26)} = -\frac{1}{2}\langle \Lambda\beta(\Phi_1-\Phi_2), \beta^{-1}(\Phi_1+\Phi_2)\rangle - \frac{1}{4}\langle\beta^2(\Phi_1-\Phi_2), \Phi_1-\Phi_2\rangle\langle\beta^{-2}(\Phi_1+\Phi_2), \Phi_1+\Phi_2\rangle = \frac{1}{2}\langle\Lambda\Phi_2, \Phi_2\rangle - \frac{1}{2}\langle\Lambda\Phi_1, \Phi_1\rangle + \frac{1}{2}\langle\Phi_1, \Phi_2\rangle^2 - \frac{1}{2}\langle\Phi_1, \Phi_1\rangle^2.$$

所以系统(23)可从(3)得到. 反之, 利用(26), 从系统(23)也可以导出系统(3).

定理 1.3 系统(3)的守恒积分(4)规范等价于系统(23)的守恒积分(27).

$$F_m = \frac{1}{2}\langle\Lambda^m\Phi_2, \Phi_2\rangle - \frac{1}{2}\langle\Lambda^m\Phi_1, \Phi_1\rangle - \frac{1}{2}\sum_{j=1}^{m}\begin{vmatrix}\langle\Lambda^{j-1}\Phi_1, \Phi_1\rangle & \langle\Lambda^{j-1}\Phi_1, \Phi_2\rangle \\ \langle\Lambda^{m-j}\Phi_1, \Phi_2\rangle & \langle\Lambda^{m-j}\Phi_2, \Phi_2\rangle\end{vmatrix} \quad (m = 1, 2, \cdots). \tag{27}$$

证明 通过直接计算可得结论. 略.

注 (27)的对合性质不能由(21)的Jacobi映射得到, 参见文献[1], 但利用文献[7]的某些结果, 我们能够证明 $F_1, F_2, \cdots, F_m$ 是线性独立的, 并且 $\{F_m, F_n\} = 0, (m, n = 1, 2, \cdots, N)$; $\{H, F_m\} = 0, (m = 1, 2, \cdots N)$. 这意味着 $F_1, F_2, \cdots, F_N$ 也是系统(23)的守恒积分. 所以系统(23)在Liouville意义下完全可积[8,9].

2 约束流(23)的Lax表示与 r_矩阵

定理 2.1 约束流(3)的Lax表示规范等价于约束流(23)的Lax表示, (23)的Lax表示为

$$\boldsymbol{M}_x = [\boldsymbol{U}(\lambda, \Phi_1, \Phi_2), \boldsymbol{M}(\lambda)], \tag{28}$$

$$\boldsymbol{U}(\lambda, \Phi_1, \Phi_2) = \begin{pmatrix}\langle\Phi_1, \Phi_2\rangle & \lambda \\ \lambda + 2\langle\Phi_1, \Phi_1\rangle & -\langle\Phi_1, \Phi_2\rangle\end{pmatrix}, \tag{29}$$

$$\boldsymbol{M}(\lambda) = \begin{pmatrix}A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda)\end{pmatrix} = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} + \sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}\varphi_{1j}\varphi_{2j} & -\varphi_{1j}^2 \\ \varphi_{2j}^2 & -\varphi_{1j}\varphi_{2j}\end{pmatrix}. \tag{30}$$

证明 在定理1.2的证明中用 λ 代 Λ 得到表达式(29). 为证明(30), 利用规范变换(21), 有

$$\boldsymbol{M}(\lambda) = \begin{pmatrix}A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda)\end{pmatrix} = \boldsymbol{T}^{-1}\boldsymbol{M}(\lambda)\boldsymbol{T} = \frac{1}{\sqrt{2}}\begin{pmatrix}\beta^{-1} & \beta \\ -\beta^{-1} & \beta\end{pmatrix}\begin{pmatrix}A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda)\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}\beta & -\beta \\ \beta^{-1} & \beta^{-1}\end{pmatrix} =$$
$$\frac{1}{2}\begin{pmatrix}\beta^2C(\lambda) + \beta^{-2}B(\lambda) & -2A(\lambda) + \beta^{-2}B(\lambda) - \beta^2C(\lambda) \\ -2A(\lambda) + \beta^2C(\lambda) - \beta^{-2}B(\lambda) & -\beta^2C(\lambda) - \beta^{-2}B(\lambda)\end{pmatrix},$$

把(7)和(21)代入上式, 有

$$A(\lambda) = \sum_{j=1}^{N}\frac{\varphi_{1j}\varphi_{2j}}{\lambda-\lambda_j}, \quad B(\lambda) = 1 - \sum_{j=1}^{N}\frac{\varphi_{1j}^2}{\lambda-\lambda_j}, \quad C(\lambda) = 1 + \sum_{j=1}^{N}\frac{\varphi_{2j}^2}{\lambda-\lambda_j}. \tag{31}$$

为推导(28), 对 $\boldsymbol{TM}(\lambda) = \boldsymbol{M}(\lambda)\boldsymbol{T}$ 关于 x 求导后, 我们得到 $(\boldsymbol{M}(\lambda))_x = \boldsymbol{T}^{-1}((\boldsymbol{M}(\lambda))_x\boldsymbol{T} + \boldsymbol{M}(\lambda)\boldsymbol{T}_x) + \boldsymbol{T}_x^{-1}\boldsymbol{M}(\lambda)\boldsymbol{T}$. 使用此关系式以及 $((\boldsymbol{M}(\lambda))_x = [\boldsymbol{U}, \boldsymbol{M}(\lambda)]$, 有

$$[\boldsymbol{U}(\lambda, \Phi_1, \Phi_2), \boldsymbol{M}(\lambda)] = [\boldsymbol{T}^{-1}\boldsymbol{U}(\lambda, \Psi_1, \Psi_2)\boldsymbol{T} - \boldsymbol{T}^{-1}\boldsymbol{T}_x, \boldsymbol{T}^{-1}\boldsymbol{M}(\lambda)\boldsymbol{T}] =$$
$$\boldsymbol{T}^{-1}[\boldsymbol{U}(\lambda, \Psi_1, \Psi_2), \boldsymbol{M}]\boldsymbol{T} + \boldsymbol{T}^{-1}\boldsymbol{M}(\lambda)\boldsymbol{T}_x - \boldsymbol{T}^{-1}\boldsymbol{T}_x\boldsymbol{T}^{-1}\boldsymbol{M}(\lambda)\boldsymbol{T} =$$

$$T^{-1}(M(\lambda))_x T + T^{-1}M(\lambda)T_x - T^{-1}T_xT^{-1}M(\lambda)T$$

通过直接计算, 有

$$T^{-1}T_xT^{-1} = -(T^{-1})_x \cdot$$

所以

$$[U(\lambda, \Phi_1, \Phi_2), M(\lambda)] = (M(\lambda))_x \cdot$$

证毕•

定理 2.2 由式(30) 确定的 $A(\lambda)$, $B(\lambda)$ 和 $C(\lambda)$ 满足关系式

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0,$$

$$\{A(\lambda), B(\mu)\} = \frac{2}{\mu-\lambda}(B(\mu) - B(\lambda)),$$

$$\{A(\lambda), C(\mu)\} = \frac{2}{\mu-\lambda}(C(\lambda) - C(\mu)),$$

$$\{B(\lambda), C(\mu)\} = \frac{4}{\mu-\lambda}(A(\mu) - A(\lambda)) \cdot$$

证明 通过直接计算可得结论• 略•

定理 2.3 约束流(23) 有 r_矩阵

$$r^{(ij)}(\lambda_i, \lambda_j) = \frac{2}{\lambda_j-\lambda_i}P^{(ij)}$$

且满足经典的 Poisson 结构(9)•

证明 利用定理 2.2, 直接计算可得

$$\{M^{(1)}(\lambda_1) \overset{\leftarrow}{,} M^{(2)}(\lambda_2)\} =$$

$$\frac{2}{\lambda_2-\lambda_1}\begin{pmatrix} 0 & B(\lambda_2)-B(\lambda_1) & B(\lambda_1)-B(\lambda_2) & 0 \\ C(\lambda_1)-C(\lambda_2) & 0 & 2(A(\lambda_2)-A(\lambda_1)) & B(\lambda_2)-B(\lambda_1) \\ C(\lambda_2)-C(\lambda_1) & 2(A(\lambda_1)-A(\lambda_2)) & 0 & B(\lambda_1)-B(\lambda_2) \\ 0 & C(\lambda_1)-C(\lambda_2) & C(\lambda_2)-C(\lambda_1) & 0 \end{pmatrix} =$$

$$[r^{(12)}(\lambda_1, \lambda_2), M^{(1)}(\lambda_1)] - [r^{(21)}(\lambda_2, \lambda_1), M^{(2)}(\lambda_2)] \cdot$$

通过选择不同的指标 (i, j), 很容易完成证明•

定理 2.4 系统(23) 在 Liouville 意义下是完全可积的•

证明 系统(23) 的第二组守恒积分 $F^{(j)}(j = 1, 2, \cdots, N)$ 可由下式给出

$$\frac{1}{2}\mathrm{Tr}M^2(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda) = \frac{1}{2} + \sum_{j=1}^{N}\frac{F^{(j)}}{\lambda-\lambda_j}, \tag{32}$$

$$F^{(j)} = -\frac{1}{2}\varphi_{1j}^2 + \frac{1}{2}\varphi_{2j}^2 - \frac{1}{2}\sum_{\substack{k=1\\k\neq j}}^{N}\frac{(\varphi_{1k}\varphi_{2j} - \varphi_{2k}\varphi_{1j})^2}{\lambda_k-\lambda_j} \quad (j = 1, 2, \cdots, N) \cdot \tag{33}$$

正如文[4] 中指出, 若 r_矩阵满足经典 Poisson 结构(9), 则母函数 $\mathrm{Tr}M^2(\lambda)$ 满足关系式

$$\{\mathrm{Tr}M^2(\lambda), \mathrm{Tr}M^2(\mu)\} = 0, \tag{34}$$

把(32) 代入(34), 有

$$\{F^{(j)}, F^{(k)}\} = 0 \qquad (j, k = 1, 2, \cdots, N) \cdot$$

对于确定的 N, 容易验证

$$\frac{\partial(F^{(1)}, F^{(2)}, \cdots, F^{(N)})}{\partial(\varphi_{11}, \varphi_{12}, \cdots, \varphi_{1N}, \psi_{11}, \psi_{12}, \cdots, \psi_{1N})} \neq 0 \cdot$$

所以, $\mathrm{grad}F^{(j)}(j = 1, 2 \cdots, N)$ 是线性独立的, 这就意味着系统在 Liouville 意义下是完全可积

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[参 考 文 献]

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On Gauge Equivalent Integrable Systems and r_matrices for AKNS Hierarchy and A Coupled MKdV Hierarchy

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Abstract: The new coupled MKdV hierarchy is obtained. By using gauge transformation, the constrained flow, the integrable system and Lax representation for the coupled MKdV hierarchy were first constructed from the AKNS hierarchy and then using the Lax representation the r_matrix for the constrained flow of the coupled MKdV hierarchy was constructed. The second set of conserved integrals of this constrained flow and their involutivity were also given.

Key words: gauge equivalence; r_matrix; integrable system