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非饱和土本构关系的混合物理论(Ⅱ)——线性本构方程和场方程*

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(我刊编委黄义来稿)

摘要: 通过对非饱和土非线性本构方程和场方程的线性化, 推导出了非饱和土的线性本构方程和场方程• 把线性方程表示为与 Biot 饱和多孔介质方程相似的形式; 证明了 Darcy 定律对非饱和土的适用性; 说明了 Biot 饱和多孔介质方程是这些线性方程的特例• 所有这些都表明用混合物理论处理非饱和土本构问题的正确性•

关 键 词: 混合物理论; 非饱和土; 线性本构方程; 线性场方程; Darcy 定律; Biot 方程

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引 言

前文[1]应用混合物理论建立了非饱和土的非线性本构方程和场方程以及非饱和土混合物热力学系统完备方程组• 本文对非线性本构方程和场方程进行线性化, 推导非饱和土线性本构方程和场方程; 并且把线性方程表示为与 Biot[2]饱和多孔介质方程相似的形式; 证明了 Darcy 定律对非饱和土的适用性; 最后指出 Biot 方程是本文线性方程的特例•

1 本构方程耗散部分的线性化

若非饱和土的耗散特性为各向同性, 则耗散势函数 Θ_0 是系统热力学力 $\boldsymbol{Y}_0$ 中张量和矢量不变量的函数• 为了得到线性本构方程, 首先把 Θ_0 表示成 $\boldsymbol{Y}_0$ 中独立本构变量二阶不变量的一次多项式• $\boldsymbol{Y}_0$ 中独立本构变量的二阶不变量有

$$I_A = (\mathrm{tr}\boldsymbol{d}_s)^2,\ I_B = (\mathrm{tr}\boldsymbol{d}_l)^2,\ I_C = (\mathrm{tr}\boldsymbol{d}_g)^2,\ I_D = \mathrm{tr}\boldsymbol{d}_s^2,\ I_E = \mathrm{tr}\boldsymbol{d}_l^2,\ I_F = \mathrm{tr}\boldsymbol{d}_g^2,$$

$$I_G = \mathrm{tr}(\boldsymbol{d}_s\boldsymbol{d}_l),\ I_H = \mathrm{tr}(\boldsymbol{d}_l\boldsymbol{d}_g),\ I_I = \mathrm{tr}(\boldsymbol{d}_g\boldsymbol{d}_s),\ I_J = \mathrm{tr}(\boldsymbol{w}_l - \boldsymbol{w}_s)^2,\ I_K = \mathrm{tr}(\boldsymbol{w}_g - \boldsymbol{w}_s)^2,$$

$$I_L = \mathrm{tr}[(\boldsymbol{w}_l - \boldsymbol{w}_s)(\boldsymbol{w}_g - \boldsymbol{w}_s)],\ I_M = \frac{\boldsymbol{g}\cdot\boldsymbol{g}}{\theta^2},\ I_N = |\boldsymbol{v}_l - \boldsymbol{v}_s|^2,\ I_O = |\boldsymbol{v}_g - \boldsymbol{v}_s|^2,$$

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$$I_P = \frac{\boldsymbol{g}}{\theta}\cdot(\boldsymbol{v}_l - \boldsymbol{v}_s),\ I_Q = \frac{\boldsymbol{g}}{\theta}\cdot(\boldsymbol{v}_g - \boldsymbol{v}_s),\ I_R = (\boldsymbol{v}_l - \boldsymbol{v}_s)\cdot(\boldsymbol{v}_g - \boldsymbol{v}_s),$$
$$I_S = (\phi'_g)^2,\ I_T = \phi'_g \operatorname{tr}\boldsymbol{d}_s,\ I_U = \phi'_g \operatorname{tr}\boldsymbol{d}_l,\ I_V = \phi'_g \operatorname{tr}\boldsymbol{d}_g \cdot \tag{1a~v}$$

设

$$\Theta_0 = \sum_Z k_Z I_Z \qquad (Z = A, B, \cdots, V), \tag{2}$$

式中

$$k_Z = k_Z(\boldsymbol{Y}_{R0};\ \boldsymbol{Y}_{I0}) = \frac{\partial\Theta_0}{\partial I_Z}\cdot \tag{3}$$

把(2)给出的 Θ_0 代入本构方程耗散部分表示式([1] 中(107)~(111)),得

$$\boldsymbol{t}^s_{sD} = \lambda_{ss}(\operatorname{tr}\boldsymbol{d}_s)\boldsymbol{I} + 2\mu_{ss}\boldsymbol{d}_s + 2\mu_{sl}\boldsymbol{d}_l + 2\mu_{sg}\boldsymbol{d}_g + k_T\phi'_g\boldsymbol{I}, \tag{4}$$
$$\boldsymbol{t}^s_{lD} = \lambda_{ll}(\operatorname{tr}\boldsymbol{d}_l)\boldsymbol{I} + 2\mu_{ls}\boldsymbol{d}_s + 2\mu_{ll}\boldsymbol{d}_l + 2\mu_{lg}\boldsymbol{d}_g + k_U\phi'_g\boldsymbol{I}, \tag{5}$$
$$\boldsymbol{t}^s_{gD} = \lambda_{gg}(\operatorname{tr}\boldsymbol{d}_g)\boldsymbol{I} + 2\mu_{gs}\boldsymbol{d}_s + 2\mu_{gl}\boldsymbol{d}_l + 2\mu_{gg}\boldsymbol{d}_g + k_V\phi'_g\boldsymbol{I}, \tag{6}$$
$$\boldsymbol{M}_{lD} = -2\varphi_{ll}(\boldsymbol{w}_l - \boldsymbol{w}_s) - 2\varphi_{lg}(\boldsymbol{w}_g - \boldsymbol{w}_s), \tag{7}$$
$$\boldsymbol{M}_{gD} = -2\varphi_{gl}(\boldsymbol{w}_l - \boldsymbol{w}_s) - 2\varphi_{gg}(\boldsymbol{w}_g - \boldsymbol{w}_s), \tag{8}$$
$$\boldsymbol{M}_{sD} = -\sum_f \boldsymbol{M}_{fD} = 2(\varphi_{ll} + \varphi_{gl})(\boldsymbol{w}_l - \boldsymbol{w}_s) + 2(\varphi_{gg} + \varphi_{lg})(\boldsymbol{w}_g - \boldsymbol{w}_s), \tag{9}$$
$$\boldsymbol{q}_I = -2k_M\frac{\boldsymbol{g}}{\theta} - k_P(\boldsymbol{v}_l - \boldsymbol{v}_s) - K_Q(\boldsymbol{v}_g - \boldsymbol{v}_s), \tag{10}$$
$$\boldsymbol{f}_l = -2k_N(\boldsymbol{v}_l - \boldsymbol{v}_s) - k_R(\boldsymbol{v}_g - \boldsymbol{v}_s) - k_P\frac{\boldsymbol{g}}{\theta}, \tag{11}$$
$$\boldsymbol{f}_g = -2k_O(\boldsymbol{v}_g - \boldsymbol{v}_s) - k_R(\boldsymbol{v}_l - \boldsymbol{v}_s) - k_Q\frac{\boldsymbol{g}}{\theta}, \tag{12}$$
$$\boldsymbol{f}_s = -\sum_f \boldsymbol{f}_f = (2k_N + k_R)(\boldsymbol{v}_l - \boldsymbol{v}_s) + (2k_O + k_R)(\boldsymbol{v}_g - \boldsymbol{v}_s) + (k_P + k_Q)\frac{\boldsymbol{g}}{\theta}, \tag{13}$$
$$\sigma_g = 2k_S\phi'_g + k_T\operatorname{tr}\boldsymbol{d}_s + k_U\operatorname{tr}\boldsymbol{d}_l + k_V\operatorname{tr}\boldsymbol{d}_g, \tag{14}$$

其中

$$\begin{cases}\lambda_{ss} = 2k_A,\ \lambda_{ll} = 2k_B,\ \lambda_{gg} = 2k_C,\ \mu_{ss} = k_D,\ \mu_{ll} = k_E,\ \mu_{gg} = k_F,\\ \mu_{sl} = \mu_{ls} = \dfrac{1}{2}k_G,\ \mu_{lg} = \mu_{gl} = \dfrac{1}{2}k_H,\ \mu_{gs} = \mu_{sg} = \dfrac{1}{2}k_I,\end{cases} \tag{15}$$
$$\lambda_{ls} = \lambda_{sl} = \lambda_{lg} = \lambda_{gl} = \lambda_{gs} = \lambda_{sg} = 0, \tag{16}$$
$$\varphi_{ll} = 2k_J,\ \varphi_{gg} = 2k_K,\ \varphi_{lg} = \varphi_{gl} = k_L\cdot \tag{17}$$

方程(4)~(14)就是非饱和土本构方程耗散部分线性化的结果• 应该指出:耗散部分线性方程中的系数 $k_Z(Z = A, B, \cdots, V)$ 仍然可以是非耗散独立本构变量 $\boldsymbol{Y}_{R0}$ 和混合物内部状态参量 $\boldsymbol{Y}_{I0}$ 的非线性函数•

为了确定 $k_Z(Z = A, B, \cdots, V)$ 满足的条件,把(4)~(14)式代入熵不等式([1] 中(88)),有

$$\sum_a\lambda_{aa}(\operatorname{tr}\boldsymbol{d}_a)^2 + 2\sum_a\mu_{aa}\operatorname{tr}\boldsymbol{d}_a^2 + 4[\mu_{sl}\operatorname{tr}(\boldsymbol{d}_s\boldsymbol{d}_l) + \mu_{lg}\operatorname{tr}(\boldsymbol{d}_l\boldsymbol{d}_g) + \mu_{gs}\operatorname{tr}(\boldsymbol{d}_g\boldsymbol{d}_s)] +$$
$$2k_S\phi'^2_g + 2k_T\phi'_g\operatorname{tr}\boldsymbol{d}_s + 2k_U\phi'_g\operatorname{tr}\boldsymbol{d}_l + 2k_V\phi'_g\operatorname{tr}\boldsymbol{d}_g + \varphi_{ll}\operatorname{tr}[(\boldsymbol{w}_l - \boldsymbol{w}_s)^{\mathrm{T}}(\boldsymbol{w}_l - \boldsymbol{w}_s)] +$$
$$\varphi_{gg}\operatorname{tr}[(\boldsymbol{w}_g - \boldsymbol{w}_s)^{\mathrm{T}}(\boldsymbol{w}_g - \boldsymbol{w}_s)] + 2\varphi_{lg}\operatorname{tr}[(\boldsymbol{w}_l - \boldsymbol{w}_s)^{\mathrm{T}}(\boldsymbol{w}_g - \boldsymbol{w}_s)] +$$

$$2k_M\left|\frac{\boldsymbol{g}}{\theta}\right|^2+2k_N|\boldsymbol{v}_l-\boldsymbol{v}_s|^2+2k_O|\boldsymbol{v}_g-\boldsymbol{v}_s|^2+2k_P\frac{\boldsymbol{g}}{\theta}\cdot(\boldsymbol{v}_l-\boldsymbol{v}_s)+$$
$$2k_Q\frac{\boldsymbol{g}}{\theta}\cdot(\boldsymbol{v}_g-\boldsymbol{v}_s)+2k_R(\boldsymbol{v}_l-\boldsymbol{v}_s)\cdot(\boldsymbol{v}_g-\boldsymbol{v}_s)\geqslant 0. \tag{18}$$

上式等价于下面几个不等式

$$\sum_{a,c=l,g}\varphi_{ac}\mathrm{tr}[(\boldsymbol{w}_a-\boldsymbol{w}_s)^{\mathrm{T}}(\boldsymbol{w}_c-\boldsymbol{w}_s)]\geqslant 0, \tag{19}$$

$$\sum_{a,c=s,l,g}\mu_{ac}\mathrm{tr}\left\{\left[\boldsymbol{d}_a-\frac{1}{3}(\mathrm{tr}\boldsymbol{d}_a)\boldsymbol{I}\right]\left[\boldsymbol{d}_c-\frac{1}{3}(\mathrm{tr}\boldsymbol{d}_c)\boldsymbol{I}\right]\right\}\geqslant 0, \tag{20}$$

$$k_M\left|\frac{\boldsymbol{g}}{\theta}\right|^2+k_N|\boldsymbol{v}_l-\boldsymbol{v}_s|^2+k_O|\boldsymbol{v}_g-\boldsymbol{v}_s|^2+k_P\frac{\boldsymbol{g}}{\theta}\cdot(\boldsymbol{v}_l-\boldsymbol{v}_s)+$$
$$k_Q\frac{\boldsymbol{g}}{\theta}\cdot(\boldsymbol{v}_g-\boldsymbol{v}_s)+k_R(\boldsymbol{v}_l-\boldsymbol{v}_s)\cdot(\boldsymbol{v}_g-\boldsymbol{v}_s)\geqslant 0, \tag{21}$$

$$\sum_{a,c=s,l,g}\left(\lambda_{ac}+\frac{2}{3}\mu_{ac}\right)(\mathrm{tr}\boldsymbol{d}_a)(\mathrm{tr}\boldsymbol{d}_c)+2k_S(\phi_g')^2+$$
$$2k_T\phi_g'\mathrm{tr}\boldsymbol{d}_s+2k_U\phi_g'\mathrm{tr}\boldsymbol{d}_l+2k_V\phi_g'\mathrm{tr}\boldsymbol{d}_g\geqslant 0. \tag{22}$$

它们成立的必要条件是下面的对称矩阵是半正定的

$$\frac{1}{2}[\varphi_{ac}]=\begin{bmatrix}k_J & \frac{1}{2}k_L\\ \frac{1}{2}k_L & k_K\end{bmatrix}, \tag{23}$$

$$[\mu_{ac}]=\begin{bmatrix}k_D & \frac{1}{2}k_G & \frac{1}{2}k_I\\ \frac{1}{2}k_G & k_E & \frac{1}{2}k_H\\ \frac{1}{2}k_I & \frac{1}{2}k_H & k_F\end{bmatrix}, \tag{24}$$

$$\begin{bmatrix}k_M & \frac{1}{2}k_P & \frac{1}{2}k_Q\\ \frac{1}{2}k_P & k_N & \frac{1}{2}k_R\\ \frac{1}{2}k_Q & \frac{1}{2}k_R & k_O\end{bmatrix}, \tag{25}$$

$$\frac{1}{2}\begin{bmatrix}\lambda_{ss}+\frac{2}{3}\mu_{ss} & \lambda_{sl}+\frac{2}{3}\mu_{sl} & \lambda_{sg}+\frac{2}{3}\mu_{sg} & k_T\\ \lambda_{ls}+\frac{2}{3}\mu_{ls} & \lambda_{ll}+\frac{2}{3}\mu_{ll} & \lambda_{lg}+\frac{2}{3}\mu_{lg} & k_U\\ \lambda_{gs}+\frac{2}{3}\mu_{gs} & \lambda_{gl}+\frac{2}{3}\mu_{gl} & \lambda_{gg}+\frac{2}{3}\mu_{gg} & k_V\\ k_T & k_U & k_V & 2k_S\end{bmatrix}=$$

$$\begin{bmatrix} k_A + \frac{1}{3}k_D & \frac{1}{6}k_G & \frac{1}{6}k_I & \frac{1}{2}k_T \\ \frac{1}{6}k_G & k_B + \frac{1}{3}k_E & \frac{1}{6}k_H & \frac{1}{2}k_U \\ \frac{1}{6}k_I & \frac{1}{6}k_H & k_C + \frac{1}{3}k_F & \frac{1}{2}k_V \\ \frac{1}{2}k_T & \frac{1}{2}k_U & \frac{1}{2}k_V & k_S \end{bmatrix}, \tag{26}$$

这就是系数 $k_Z(Z = A, B, \cdots, V)$ 满足的必要条件•

2 本构方程非耗散部分的线性化

讨论均匀非饱和土本构方程非耗散部分在某静力平衡状态处的线性化问题• 静力平衡状态就是系统完备方程组的静态解, 把这个状态作为组分参考构形• 因此, 对于静力平衡状态有

$$\boldsymbol{F}_s = \boldsymbol{I}, \tag{27}$$

线性非耗散本构方程, 是对混合物偏离静力平衡状态很小时, 非线性本构方程非耗散部分的线性近似• 为此, 把非饱和土混合物组分自由能密度

$$\Psi_a = \Psi_a(\theta, \boldsymbol{C}_s, \rho_g, \phi_l, \phi_g) \tag{28}$$

在静力平衡状态处展开

$$\begin{aligned} \Psi_a = {} & \Psi_a^+ + \left(\frac{\partial \Psi_a}{\partial \theta}\right)^+ (\theta - \theta^+) + 2\mathrm{tr}\left[\left(\frac{\partial \Psi_a}{\partial \boldsymbol{C}_s}\right)^+ \boldsymbol{E}_s\right] + \left(\frac{\partial \Psi_a}{\partial \rho_g}\right)^+ (\rho_g - \rho_g^+) + \\ & \sum_f \left(\frac{\partial \Psi_a}{\partial \phi_f}\right)^+ (\phi_f - \phi_f^+) + \frac{1}{2}\left(\frac{\partial^2 \Psi_a}{\partial \theta^2}\right)^+ (\theta - \theta^+)^2 + \\ & 2\mathrm{tr}\left[\left(\frac{\partial^2 \Psi_a}{\partial \theta \partial \boldsymbol{C}_s}\right)^+ \boldsymbol{E}_s\right](\theta - \theta^+) + \left(\frac{\partial^2 \Psi_a}{\partial \theta \partial \rho_g}\right)^+ (\rho_g - \rho_g^+)(\theta - \theta^+) + \\ & 2\mathrm{tr}\left[\left(\left(\frac{\partial^2 \Psi_a}{\partial \boldsymbol{C}_s^2}\right)^+ [\boldsymbol{E}_s]\right) \boldsymbol{E}_s\right] + 2\mathrm{tr}\left[\left(\frac{\partial^2 \Psi_a}{\partial \boldsymbol{C}_s \partial \rho_g}\right)^+ \boldsymbol{E}_s\right](\rho_g - \rho_g^+) + \\ & \frac{1}{2}\left(\frac{\partial^2 \Psi_a}{\partial \rho_g^2}\right)^+ (\rho_g - \rho_g^+)^2 + \frac{1}{2}\sum_{b,\, c = l,\, g}\left(\frac{\partial^2 \Psi_a}{\partial \phi_b \partial \phi_c}\right)^+ (\phi_b - \phi_b^+)(\phi_c - \phi_c^+) + \\ & 2\mathrm{tr}\left[\sum_f \left(\frac{\partial^2 \Psi_a}{\partial \boldsymbol{C}_s \partial \phi_f}\right)^+ \boldsymbol{E}_s\right](\phi_f - \phi_f^+) + \sum_f \left(\frac{\partial^2 \Psi_a}{\partial \rho_g \partial \phi_f}\right)^+ (\rho_g - \rho_g^+)(\phi_f - \phi_f^+) + \\ & \sum_f \left(\frac{\partial^2 \Psi_a}{\partial \theta \partial \phi_f}\right)^+ (\phi_f - \phi_f^+)(\theta - \theta^+), \end{aligned} \tag{29}$$

右上标"+"表示函数在静力平衡状态处取值; $\boldsymbol{E}_a$ 是第 a 组分的无穷小应变张量

$$\boldsymbol{E}_a = \frac{1}{2}[\mathrm{GRAD}\,\boldsymbol{W}_a(\boldsymbol{X}, t) + (\mathrm{GRAD}\,\boldsymbol{W}_a(\boldsymbol{X}, t))^{\mathrm{T}}], \tag{30}$$

式中,"GRAD"表示对参考构形坐标 $\boldsymbol{X}$ 取梯度• $\boldsymbol{W}_a$ 是第 a 组分位移,

$$\boldsymbol{W}_a(\boldsymbol{X}, t) = \boldsymbol{x}_a(\boldsymbol{X}, t) - \boldsymbol{X}• \tag{31}$$

张量 $\boldsymbol{E}_s$ 与 $\boldsymbol{C}_s$ 的关系是

$$\boldsymbol{C}_s = \boldsymbol{I} + 2\boldsymbol{E}_s• \tag{32}$$

(29) 式对混合物组分求和就是

$$\begin{aligned} \Psi_{\mathrm{I}} = {} & \Psi_{\mathrm{I}}^+ - \rho^+ \eta^+ (\theta - \theta^+) - (\Psi_s^+ + \phi_s^+ \delta_g^+)\mathrm{tr}\boldsymbol{E}_s + \mu_g^+(\rho_g - \rho_g^+) + \\ & \delta_g^+(\phi_g - \phi_g^+) + \delta_l^+(\phi_l - \phi_l^+) - \frac{1}{2}\nu^+ (\theta - \theta^+)^2 - \mathrm{tr}(\boldsymbol{B}_s^+ \boldsymbol{E}_s)(\theta - \theta^+) + \end{aligned}$$

$$\tau_g^+\left(\frac{\rho_g-\rho_g^+}{\rho_g^+}\right)(\theta-\theta^+)+\frac{1}{2}\mathrm{tr}[\boldsymbol{E}_s(\boldsymbol{A}^+[\boldsymbol{E}_s])]-\mathrm{tr}(\boldsymbol{J}_{sg}^+\boldsymbol{E}_s)\left(\frac{\rho_g-\rho_g^+}{\rho_g^+}\right)+$$
$$\frac{1}{2}\lambda_{gg}^+\left(\frac{\rho_g-\rho_g^+}{\rho_g^+}\right)^2+\frac{1}{2}\lambda_{ll}^+\left(\frac{\phi_l-\phi_l^+}{\phi_l^+}\right)^2-\Gamma_{lg}^+\left(\frac{\phi_l-\phi_l^+}{\phi_l^+}\right)(\phi_g-\phi_g^+)+$$
$$\frac{1}{2}\Phi_g^+(\phi_g-\phi_g^+)^2-\mathrm{tr}(\boldsymbol{J}_{sl}^+\boldsymbol{E}_s)\left(\frac{\phi_l-\phi_l^+}{\phi_l^+}\right)+\mathrm{tr}(\boldsymbol{P}_{sg}^+\boldsymbol{E}_s)(\phi_g-\phi_g^+)+$$
$$\lambda_{lg}^+\left(\frac{\rho_g-\rho_g^+}{\rho_g^+}\right)\left(\frac{\phi_l-\phi_l^+}{\phi_l^+}\right)-\Gamma_{gg}^+\left(\frac{\rho_g-\rho_g^+}{\rho_g^+}\right)(\phi_g-\phi_g^+)+$$
$$\tau_l^+\left(\frac{\phi_l-\phi_l^+}{\phi_l^+}\right)(\theta-\theta^+)+\Theta_g^+(\phi_g-\phi_g^+)(\theta-\theta^+),\tag{33}$$

其中

$$\begin{cases}\rho^+\eta^+=-\left(\dfrac{\partial\Psi_{\mathrm{I}}}{\partial\theta}\right)^+,\ (\Psi_s^++\phi_s^+\delta_g^+)\boldsymbol{I}=-2\left(\dfrac{\partial\Psi_{\mathrm{I}}}{\partial\boldsymbol{C}_s}\right)^+,\ \mu_g^+=\left(\dfrac{\partial\Psi_{\mathrm{I}}}{\partial\rho_g}\right)^+=\dfrac{\Psi_g^+}{\rho_g^+},\\ \delta_g^+=\left(\dfrac{\partial\Psi_{\mathrm{I}}}{\partial\phi_g}\right)^+,\ \delta_l^+=\left(\dfrac{\partial\Psi_{\mathrm{I}}}{\partial\phi_l}\right)^+=\dfrac{1}{\phi_l^+}(\Psi_{\mathrm{I}}^++\phi_l^+\delta_g^+),\ \nu^+=-\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\theta^2}\right)^+,\\ \boldsymbol{B}_s^+=(\boldsymbol{B}_s^+)^{\mathrm{T}}=-2\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\theta\partial\boldsymbol{C}_s}\right)^+,\ \tau_g^+=\rho_g^+\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\theta\partial\rho_g}\right)^+,\ \boldsymbol{A}^+=4\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\boldsymbol{C}_s^2}\right)^+,\\ \boldsymbol{J}_{sg}^+=-2\rho_g^+\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\boldsymbol{C}_s\partial\rho_g}\right)^+,\ \lambda_{gg}^+=(\rho_g^+)^2\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\rho_g^2}\right)^+,\ \lambda_{ll}^+=(\phi_l^+)^2\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\phi_l^2}\right)^+,\\ \Gamma_{lg}^+=-\phi_l^+\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\phi_g\partial\phi_l}\right)^+,\ \Phi_g^+=\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\phi_g^2}\right)^+,\ \boldsymbol{J}_{sl}^+=-2\phi_l^+\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\boldsymbol{C}_s\partial\phi_l}\right)^+,\\ \boldsymbol{P}_{sg}^+=2\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\boldsymbol{C}_s\partial\phi_g}\right)^+,\ \lambda_{lg}^+=\phi_l^+\rho_g^+\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\rho_g\partial\phi_l}\right)^+,\ \Gamma_{gg}^+=-\rho_g^+\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\rho_g\partial\phi_g}\right)^+,\\ \tau_l^+=\phi_l^+\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\theta\partial\phi_l}\right)^+,\ \Theta_g^+=\left(\dfrac{\partial^2\Psi_{\mathrm{I}}}{\partial\theta\partial\phi_g}\right)^+\cdot\end{cases}\tag{34}$$

把(33)式分别代入公式([1]中(73)、(69)、(78)和(70)),混合物熵密度和组分偏应力张量非耗散部分的线性形式分别是

$$\rho\eta=\rho^+\eta^++\nu^+(\theta-\theta^+)+\mathrm{tr}(\boldsymbol{B}_s^+\boldsymbol{E}_s)+\tau_g^+\mathrm{tr}(\boldsymbol{E}_g)+$$
$$\tau_l^+\mathrm{tr}(\boldsymbol{E}_l)-\Theta_g^+(\phi_g-\phi_g^+),\tag{35}$$

$$\boldsymbol{t}_{s\mathrm{R}}=-(\Psi_s^++\phi_s^+\delta_g^+)\boldsymbol{I}-2(\Psi_s^++\phi_s^+\delta_g^+)\boldsymbol{E}_s-\boldsymbol{B}_s^+(\theta-\theta^+)+\boldsymbol{A}^+[\boldsymbol{E}_s]+$$
$$\boldsymbol{J}_{sg}^+\mathrm{tr}\boldsymbol{E}_g+\boldsymbol{J}_{sl}^+\mathrm{tr}\boldsymbol{E}_l+\boldsymbol{P}_{sg}^+(\phi_g-\phi_g^+),\tag{36}$$

$$\boldsymbol{t}_{l\mathrm{R}}=-[\phi_l^+\delta_l^+(1-\mathrm{tr}\boldsymbol{E}_l)-\lambda_{ll}^+\mathrm{tr}\boldsymbol{E}_l-\Gamma_{lg}^+(\phi_g-\phi_g^+)-$$
$$\mathrm{tr}(\boldsymbol{J}_{sl}^+\boldsymbol{E}_s)-\lambda_{lg}^+\mathrm{tr}\boldsymbol{E}_g+\tau_l^+(\theta-\theta^+)]\boldsymbol{I},\tag{37}$$

$$\boldsymbol{t}_{g\mathrm{R}}=-[\Psi_g^+(1-\mathrm{tr}\boldsymbol{E}_g)-\lambda_{lg}^+\mathrm{tr}\boldsymbol{E}_l-\Gamma_{gg}^+(\phi_g-\phi_g^+)-\mathrm{tr}(\boldsymbol{J}_{sg}^+\boldsymbol{E}_s)-$$
$$\lambda_{gg}^+\mathrm{tr}\boldsymbol{E}_g+\tau_g^+(\theta-\theta^+)]\boldsymbol{I}\cdot\tag{38}$$

推导过程中用到了气体组分和液体组分质量守恒方程([1]中(24)和(77))的线性形式

$$\left(\frac{\rho_g-\rho_g^+}{\rho_g^+}\right)=-\mathrm{tr}\boldsymbol{E}_g,\tag{39}$$

$$\left(\frac{\phi_l-\phi_l^+}{\phi_l^+}\right)=-\mathrm{tr}\boldsymbol{E}_l\cdot\tag{40}$$

从公式([1]中(84))导出 Lagrange 乘子 P 为

$$P = -\frac{\partial \Psi_I}{\partial \phi_g} - \sigma_g, \tag{41}$$

其中 $\partial \Psi_I/\partial \phi_g$ 的线性形式是

$$\frac{\partial \Psi_I}{\partial \phi_g} = \delta_g^+ + \Gamma_{lg}^+ \mathrm{tr}\boldsymbol{E}_l + \Phi_g^+(\phi_g - \phi_g^+) + \mathrm{tr}(\boldsymbol{P}_{sg}^+\boldsymbol{E}_s) + \Gamma_{gg}^+\mathrm{tr}\boldsymbol{E}_g + \Theta_g^+(\theta - \theta^+)\bullet \tag{42}$$

定义

$$\boldsymbol{t}_{s\mathrm{r}} = \boldsymbol{t}_{s\mathrm{R}} + \phi_s\frac{\partial \Psi_I}{\partial \phi_g}\boldsymbol{I}, \tag{43}$$

$$\boldsymbol{t}_{l\mathrm{r}} = \boldsymbol{t}_{l\mathrm{R}} + \phi_l\frac{\partial \Psi_I}{\partial \phi_g}\boldsymbol{I}, \tag{44}$$

则([1]中(85)和(86))式表示成

$$\boldsymbol{t}_s = \Psi_s\boldsymbol{I} + \boldsymbol{t}_{s\mathrm{r}} + \boldsymbol{t}_{s\mathrm{D}} + \phi_s\sigma_g\boldsymbol{I},\ \boldsymbol{t}_l = \Psi_l\boldsymbol{I} + \boldsymbol{t}_{l\mathrm{r}} + \boldsymbol{t}_{l\mathrm{D}} + \phi_l\sigma_g\boldsymbol{I}\bullet \tag{45}$$

把(36)式、(37)式、(42)式和(40)式以及

$$\phi_s = \phi_s^+(1 - \mathrm{tr}\boldsymbol{E}_s) \tag{46}$$

代入(43)式和(44)式, 它们的线性形式分别是

$$\begin{aligned}\boldsymbol{t}_{s\mathrm{r}} = &-\Psi_s^+\boldsymbol{I} + \boldsymbol{A}[\boldsymbol{E}_s] + \mathrm{tr}[(\Psi_s^+\boldsymbol{I} + \boldsymbol{P}_{sg}^+)\boldsymbol{E}_s]\boldsymbol{I} + (\phi_s^+\Gamma_{lg}^+\boldsymbol{I} + \boldsymbol{J}_{sl}^+)\mathrm{tr}\boldsymbol{E}_l + \\ &(\phi_s^+\Gamma_{gg}^+\boldsymbol{I} + \boldsymbol{J}_{sg}^+)\mathrm{tr}\boldsymbol{E}_g + (\phi_s^+\Phi_g^+\boldsymbol{I} + \boldsymbol{P}_{sg}^+)(\phi_g - \phi_g^+) + (\phi_s^+\Theta_g^+\boldsymbol{I} - \boldsymbol{B}_s^+)(\theta - \theta^+),\end{aligned} \tag{47}$$

$$\begin{aligned}\boldsymbol{t}_{l\mathrm{r}} = \Big\{&-\Psi_l^+ + (\Psi_l^+ + \phi_l^+\Gamma_{lg}^+ + \lambda_{ll}^+)\mathrm{tr}\boldsymbol{E}_l + \mathrm{tr}[(\phi_l^+\boldsymbol{P}_{sg}^+ + \boldsymbol{J}_{sl}^+)\boldsymbol{E}_s] + \\ &(\phi_l^+\Gamma_{gg}^+ + \lambda_{lg}^+)\mathrm{tr}\boldsymbol{E}_g + (\phi_l^+\Phi_g^+ + \Gamma_{lg}^+)(\phi_g - \phi_g^+) + (\phi_l^+\Theta_g^+ - \tau_l^+)(\theta - \theta^+)\Big\}\boldsymbol{I},\end{aligned} \tag{48}$$

式中的 $\boldsymbol{A}$ 由下式给出

$$\boldsymbol{A}[\boldsymbol{E}_s] = \boldsymbol{A}^+[\boldsymbol{E}_s] - 2(\Psi_s^+ + \phi_s^+\delta_g^+)\boldsymbol{E}_s - (\Psi_s^+ + \phi_s^+\delta_g^+)(\mathrm{tr}\boldsymbol{E}_s)\boldsymbol{I}\bullet \tag{49}$$

再利用饱和条件([1]中(60))的线性形式

$$\phi_g - \phi_g^+ = -(\phi_l - \phi_l^+) - (\phi_s - \phi_s^+) = \phi_l^+\mathrm{tr}\boldsymbol{E}_l + \phi_s^+\mathrm{tr}\boldsymbol{E}_s, \tag{50}$$

(47)式给出的 $\boldsymbol{t}_{s\mathrm{r}}$、(48)式给出的 $\boldsymbol{t}_{l\mathrm{r}}$ 和(38)式给出的 $\boldsymbol{t}_{g\mathrm{R}}$ 分别转化为

$$\begin{aligned}\boldsymbol{t}_{s\mathrm{r}} = &-\Psi_s^+\boldsymbol{I} + \mathrm{tr}\Big\{[(\Psi_s^+ + \phi_s^{+2}\Phi_g^+)\boldsymbol{I} + \boldsymbol{P}_{sg}^+]\boldsymbol{E}_s\Big\}\boldsymbol{I} + \phi_s^+\boldsymbol{P}_{sg}^+\mathrm{tr}\boldsymbol{E}_s + \boldsymbol{A}[\boldsymbol{E}_s] + \\ &(\phi_s^+\Gamma_{gg}^+\boldsymbol{I} + \boldsymbol{J}_{sg}^+)\mathrm{tr}\boldsymbol{E}_g + [\phi_s^+(\Gamma_{lg}^+ + \phi_l^+\Phi_g^+)\boldsymbol{I} + \\ &(\boldsymbol{J}_{sl}^+ + \phi_l^+\boldsymbol{P}_{sg}^+)]\mathrm{tr}\boldsymbol{E}_l + (\phi_s^+\Theta_g^+\boldsymbol{I} - \boldsymbol{B}_s^+)(\theta - \theta^+),\end{aligned} \tag{51}$$

$$\begin{aligned}\boldsymbol{t}_{l\mathrm{r}} = \Big\{&-\Psi_l^+ + [\Psi_l^+ + \phi_l^+(\Phi_g^+\phi_l^+ + 2\Gamma_{lg}^+) + \lambda_{ll}^+]\mathrm{tr}\boldsymbol{E}_l + \mathrm{tr}[((\phi_l^+\boldsymbol{P}_{sg}^+ + \boldsymbol{J}_{sl}^+) + \\ &\phi_s^+(\Gamma_{lg}^+ + \phi_l^+\Phi_g^+)\boldsymbol{I})\boldsymbol{E}_s] + (\phi_l^+\Gamma_{gg}^+ + \lambda_{lg}^+)\mathrm{tr}\boldsymbol{E}_g + (\phi_l^+\Theta_g^+ - \tau_l^+)(\theta - \theta^+)\Big\}\boldsymbol{I},\end{aligned} \tag{52}$$

$$\begin{aligned}\boldsymbol{t}_{g\mathrm{R}} = \Big\{&-\Psi_g^+ + (\Psi_g^+ + \lambda_{gg}^+)\mathrm{tr}\boldsymbol{E}_g + \mathrm{tr}[(\boldsymbol{J}_{sg}^+ + \phi_s^+\Gamma_{gg}^+\boldsymbol{I})\boldsymbol{E}_s] + \\ &(\phi_l^+\Gamma_{gg}^+ + \lambda_{lg}^+)\mathrm{tr}\boldsymbol{E}_l - \tau_g^+(\theta - \theta^+)\Big\}\boldsymbol{I}\bullet\end{aligned} \tag{53}$$

对于各向同性非饱和土, 有

$$\boldsymbol{B}_s^+ = B_s\boldsymbol{I},\ \boldsymbol{J}_{sg}^+ = J_{sg}\boldsymbol{I},\ \boldsymbol{J}_{sl}^+ = J_{sl}\boldsymbol{I},\ \boldsymbol{P}_{sg}^+ = P_{sg}\boldsymbol{I},\ \boldsymbol{A}[\boldsymbol{E}_s] = \lambda_s(\mathrm{tr}\boldsymbol{E}_s)\boldsymbol{I} + 2\mu_s\boldsymbol{E}_s\bullet \tag{54}$$

公式(51)~(53)简化为

$$\boldsymbol{t}_{s\mathrm{r}} = [-\Psi_s^+ + '\Lambda_{ss}\mathrm{tr}\boldsymbol{E}_s + \Lambda_{sl}\mathrm{tr}\boldsymbol{E}_l + \Lambda_{sg}\mathrm{tr}\boldsymbol{E}_g + k_{\theta s}(\theta - \theta^+)]\boldsymbol{I} + 2\mu_s\boldsymbol{E}_s, \tag{55}$$

$$\boldsymbol{t}_{l\mathrm{r}} = [-\Psi_l^+ + \Lambda_{ls}\mathrm{tr}\boldsymbol{E}_s + '\Lambda_{ll}\mathrm{tr}\boldsymbol{E}_l + \Lambda_{lg}\mathrm{tr}\boldsymbol{E}_g + k_{\theta l}(\theta - \theta^+)]\boldsymbol{I}, \tag{56}$$

$$\boldsymbol{t}_{g\mathrm{R}} = [-\Psi_g^+ + \Lambda_{gs}\mathrm{tr}\boldsymbol{E}_s + \Lambda_{gl}\mathrm{tr}\boldsymbol{E}_l + '\Lambda_{gg}\mathrm{tr}\boldsymbol{E}_g + k_{\theta g}(\theta - \theta^+)]\boldsymbol{I}, \tag{57}$$

其中

$$\begin{cases} '\Lambda_{ss} = \lambda_s + \Psi_s^+ + \phi_s^{+2}\Phi_g^+ + (1+\phi_s^+)P_{sg}, \\ '\Lambda_{ll} = \Psi_l^+ + \phi_l^+(\phi_l^+\Phi_g^+ + 2\Gamma_{lg}^+) + \lambda_{ll}^+, \\ '\Lambda_{gg} = \Psi_g^+ + \lambda_{gg}^+, \quad \Lambda_{sl} = \Lambda_{ls} = \phi_s^+\Gamma_{lg}^+ + \phi_l^+\phi_s^+\Phi_g^+ + J_{sl} + \phi_l^+P_{sg}, \\ \Lambda_{lg} = \Lambda_{gl} = \phi_l^+\Gamma_{gg}^+ + \lambda_{lg}^+, \quad \Lambda_{gs} = \Lambda_{sg} = \phi_s^+\Gamma_{gg}^+ + J_{sg}, \end{cases} \tag{58}$$

$$k_{\theta s} = \phi_s^+\Theta_g^+ - B_s, \quad k_{\theta l} = \phi_l^+\Theta_g^+ - \tau_l^+, \quad k_{\theta g} = -\tau_g^+ . \tag{59}$$

3 各向同性非饱和土的线性本构方程和场方程

从(35)式推导出非饱和土混合物熵密度的线性本构方程是

$$\eta = \eta^+ + \mathrm{tr}\left\{\left[\left(\rho_s^+\eta^+ + \frac{\phi_s^+\Theta_g^+}{\rho^+}\right)\boldsymbol{I} + \frac{\boldsymbol{B}_s^+}{\rho^+}\right]\boldsymbol{E}_s\right\} + \left(\rho_l^+\eta^+ + \frac{\tau_l^+ + \phi_l^+\Theta_g^+}{\rho^+}\right)\mathrm{tr}\boldsymbol{E}_l + \left(\rho_g^+\eta^+ + \frac{\tau_g^+}{\rho^+}\right)\mathrm{tr}\boldsymbol{E}_g + \frac{\nu^+}{\rho^+}(\theta - \theta^+), \tag{60}$$

推导过程中应用了公式(50)和混合物密度 ρ 的线性表示

$$\rho = \sum_a \rho_a = \rho^+ - \sum_a \rho_a^+ \mathrm{tr}\boldsymbol{E}_a . \tag{61}$$

混合物热流通量(10)式的线性形式是

$$\boldsymbol{q}_{\mathrm{I}} = -2k_M\frac{\boldsymbol{g}}{\theta} - k_P(\partial_t\boldsymbol{W}_l - \partial_t\boldsymbol{W}_s) - k_Q(\partial_t\boldsymbol{W}_g - \partial_t\boldsymbol{W}_s), \tag{62}$$

线性化时用到了近似式

$$(\ )' = \partial_t, \quad \boldsymbol{v}_a = \partial_t\boldsymbol{W}_a, \quad \mathrm{grad} = \mathrm{GRAD} . \tag{63}$$

结合公式(41)、(14)、(42)、(50)和线性近似

$$\partial_t\boldsymbol{E}_a = \boldsymbol{d}_a = \frac{1}{2}\left\{\mathrm{GRAD}(\partial_t\boldsymbol{W}_a) + [\mathrm{GRAD}(\partial_t\boldsymbol{W}_a)]^{\mathrm{T}}\right\}, \tag{64}$$

Lagrange 乘子 P 的线性形式为

$$\mathrm{P} = -\left\{\delta_g^+ + (\Gamma_{lg}^+ + \phi_l^+\Phi_g^+)\mathrm{tr}\boldsymbol{E}_l + \mathrm{tr}[(\phi_s^+\Phi_g^+\boldsymbol{I} + \boldsymbol{P}_{sg}^+)\boldsymbol{E}_s] + \Gamma_{gg}^+\mathrm{tr}\boldsymbol{E}_g + \Theta_g^+(\theta - \theta^+) + (2\phi_s^+k_S + k_T)\mathrm{tr}\boldsymbol{d}_s + (2\phi_l^+k_S + k_U)\mathrm{tr}\boldsymbol{d}_l + k_V\mathrm{tr}\boldsymbol{d}_g\right\} . \tag{65}$$

用公式(45)、(4)、(5)、(9)、(7)、(14)、(51)和(52),得出固体组分和液体组分偏应力本构方程的线性形式分别是

$$\begin{aligned} \boldsymbol{t}_s = {} & (\Psi_s - \Psi_s^+)\boldsymbol{I} + \boldsymbol{A}[\boldsymbol{E}_s] + [(\Psi_s^+ + \phi_s^{+2}\Phi_g^+)\boldsymbol{I} + (1+\phi_s^+)\boldsymbol{P}_{sg}^+]\mathrm{tr}\boldsymbol{E}_s + \\ & (\phi_s^+\Gamma_{gg}^+\boldsymbol{I} + \boldsymbol{J}_{sg}^+)\mathrm{tr}\boldsymbol{E}_g + [(\phi_s^+\Gamma_{lg}^+ + \phi_l^+\phi_s^+\Phi_g^+)\boldsymbol{I} + (\boldsymbol{J}_{sl}^+ + \phi_l^+\boldsymbol{P}_{sg}^+)]\mathrm{tr}\boldsymbol{E}_l + \\ & (\Theta_g^+\phi_s^+\boldsymbol{I} - \boldsymbol{B}_s^+)(\theta - \theta^+) + 2\mu_{ss}\boldsymbol{d}_s + 2\mu_{sl}\boldsymbol{d}_l + 2\mu_{sg}\boldsymbol{d}_g + \\ & (\Lambda_{ss\mathrm{D}}\mathrm{tr}\boldsymbol{d}_s + \Lambda_{sl\mathrm{D}}\mathrm{tr}\boldsymbol{d}_l + \Lambda_{sg\mathrm{D}}\mathrm{tr}\boldsymbol{d}_g)\boldsymbol{I} + \\ & (\varphi_{ll} + \varphi_{gl})(\boldsymbol{w}_l - \boldsymbol{w}_s) + (\varphi_{lg} + \varphi_{gg})(\boldsymbol{w}_g - \boldsymbol{w}_s), \end{aligned} \tag{66}$$

$$\begin{aligned} \boldsymbol{t}_l = {} & \{(\Psi_l - \Psi_l^+) + [\Psi_l^+ + \phi_l^+(\phi_l^+\Phi_g^+ + 2\Gamma_{lg}^+) + \lambda_{ll}^+]\mathrm{tr}\boldsymbol{E}_l + \\ & (\phi_l^+\Gamma_{gg}^+ + \lambda_{lg}^+)\mathrm{tr}\boldsymbol{E}_g + \mathrm{tr}([(\phi_s^+\Gamma_{lg}^+ + \phi_l^+\phi_s^+\Phi_g^+)\boldsymbol{I} + (\boldsymbol{J}_{sl}^+ + \phi_l^+\boldsymbol{P}_{sg}^+)]\boldsymbol{E}_s) + \\ & (\Theta_g^+\phi_l^+ - \tau_l^+)(\theta - \theta^+)\}\boldsymbol{I} + 2\mu_{ls}\boldsymbol{d}_s + 2\mu_{ll}\boldsymbol{d}_l + 2\mu_{lg}\boldsymbol{d}_g + \\ & (\Lambda_{ls\mathrm{D}}\mathrm{tr}\boldsymbol{d}_s + \Lambda_{ll\mathrm{D}}\mathrm{tr}\boldsymbol{d}_l + \Lambda_{lg\mathrm{D}}\mathrm{tr}\boldsymbol{d}_g)\boldsymbol{I} - \varphi_{ll}(\boldsymbol{w}_l - \boldsymbol{w}_s) - \varphi_{lg}(\boldsymbol{w}_g - \boldsymbol{w}_s), \end{aligned} \tag{67}$$

式中

$$\begin{cases} \Lambda_{ss\mathrm{D}} = \lambda_{ss} + 2\phi_s^+k_T + 2\phi_s^{+2}k_S, \quad \Lambda_{ll\mathrm{D}} = \lambda_{ll} + 2\phi_l^+k_U + 2\phi_l^{+2}k_S, \\ \Lambda_{sl\mathrm{D}} = \Lambda_{ls\mathrm{D}} = \phi_l^+k_T + 2\phi_l^+\phi_s^+k_S + \phi_s^+k_U, \quad \Lambda_{sg\mathrm{D}} = \phi_s^+k_V, \quad \Lambda_{lg\mathrm{D}} = \phi_l^+k_V . \end{cases} \tag{68}$$

从公式([1]中(118))、(6)、(8)、(57)和(50)等推导出气体组分偏应力张量线性形式

$$\begin{aligned}t_g = &\Big\{(\Psi_g - \Psi_g^+) + (\Psi_g^+ + \lambda_{gg}^+)\mathrm{tr}E_g + (\phi_l^+\Gamma_{gg}^+ + \lambda_g^+)\mathrm{tr}E_l + \\ &\mathrm{tr}[(J_{sg}^+ + \phi_s^+\Gamma_{gg}^+ I)E_s]\Big\}I + (\Lambda_{gsD}\mathrm{tr}d_s + \Lambda_{glD}\mathrm{tr}d_l + \Lambda_{ggD}\mathrm{tr}d_g)I + \\ &2\mu_{gs}d_s + 2\mu_{gl}d_l + 2\mu_{gg}d_g - \varphi_{gl}(w_l - w_s) - \\ &\varphi_{gg}(w_g - w_s) - \tau_g^+(\theta - \theta^+)I,\end{aligned} \tag{69}$$

其中

$$\Lambda_{gsD} = \Lambda_{sgD},\quad \Lambda_{glD} = \Lambda_{lgD},\quad \Lambda_{ggD} = \lambda_{gg}\cdot \tag{70}$$

对于各向同性非饱和土混合物,公式(60)、(66)、(67)和(69)分别简化为

$$\begin{aligned}\eta = &\eta^+ + \left(\rho_s^+\eta^+ + \frac{\phi_s^+\Theta_g^+ + B_s}{\rho^+}\right)\mathrm{tr}E_s + \left(\rho_l^+\eta^+ + \frac{\phi_l^+\Theta_g^+ + \tau_l^+}{\rho^+}\right)\mathrm{tr}E_l + \\ &\left(\rho_g^+\eta^+ + \frac{\tau_g^+}{\rho^+}\right)\mathrm{tr}E_g + \frac{\nu^+}{\rho^+}(\theta - \theta^+),\end{aligned} \tag{71}$$

$$\begin{aligned}t_s = &(\Psi_s - \Psi_s^+)I + [{}'\Lambda_{ss}\mathrm{tr}E_s + \Lambda_{sl}\mathrm{tr}E_l + \Lambda_{sg}\mathrm{tr}E_g]I + (\Lambda_{ssD}\mathrm{tr}d_s + \\ &\Lambda_{slD}\mathrm{tr}d_l + \Lambda_{sgD}\mathrm{tr}d_g)I + 2\mu_s E_s + 2\mu_{ss}d_s + 2\mu_{sl}d_l + 2\mu_{sg}d_g + \\ &(\varphi_{ll} + \varphi_{gl})(w_l - w_s) + (\varphi_{lg} + \varphi_{gg})(w_g - w_s) + k_{\theta s}(\theta - \theta^+)I,\end{aligned} \tag{72}$$

$$\begin{aligned}t_l = &(\Psi_l - \Psi_l^+)I + [\Lambda_{ls}\mathrm{tr}E_s + {}'\Lambda_{ll}\mathrm{tr}E_l + \Lambda_{lg}\mathrm{tr}E_g]I + (\Lambda_{lsD}\mathrm{tr}d_s + \\ &\Lambda_{llD}\mathrm{tr}d_l + \Lambda_{lgD}\mathrm{tr}d_g)I + 2\mu_{ls}d_s + 2\mu_{ll}d_l + 2\mu_{lg}d_g - \\ &\varphi_{ll}(w_l - w_s) - \varphi_{lg}(w_g - w_s) + k_{\theta l}(\theta - \theta^+)I,\end{aligned} \tag{73}$$

$$\begin{aligned}t_g = &(\Psi_g - \Psi_g^+)I + [\Lambda_{gs}\mathrm{tr}E_s + \Lambda_{gl}\mathrm{tr}E_l + {}'\Lambda_{gg}\mathrm{tr}E_g]I + \\ &(\Lambda_{gsD}\mathrm{tr}d_s + \Lambda_{glD}\mathrm{tr}d_l + \Lambda_{ggD}\mathrm{tr}d_g)I + 2\mu_{gs}d_s + 2\mu_{gl}d_l + 2\mu_{gg}d_g - \\ &\varphi_{gl}(w_l - w_s) - \varphi_{gg}(w_g - w_s) + k_{\theta g}(\theta - \theta^+)I\cdot\end{aligned} \tag{74}$$

混合物各组分动量供给量本构方程([1]中(113)~(115))的线性形式是

$$\begin{aligned}p_l = &-\mathrm{GRAD}\,\Psi_l - \Psi_l^+\mathrm{GRAD}(\mathrm{tr}E_l) - \left(\rho_l^+\eta_l^+ + \frac{k_P}{\theta^+}\right)\mathrm{GRAD}\theta - \\ &2k_N(\partial_t W_l - \partial_t W_s) - k_R(\partial_t W_g - \partial_t W_s),\end{aligned} \tag{75}$$

$$\begin{aligned}p_g = &-\mathrm{GRAD}\,\Psi_g - \Psi_g^+\mathrm{GRAD}(\mathrm{tr}E_g) - \left(\rho_g^+\eta_g^+ + \frac{k_Q}{\theta^+}\right)\mathrm{GRAD}\theta - \\ &2k_O(\partial_t W_g - \partial_t W_s) - k_R(\partial_t W_l - \partial_t W_s),\end{aligned} \tag{76}$$

$$\begin{aligned}p_s = &-\mathrm{GRAD}\,\Psi_s - \Psi_s^+\mathrm{GRAD}(\mathrm{tr}E_s) - \left(\rho_s^+\eta_s^+ - \frac{k_P + k_Q}{\theta^+}\right)\mathrm{GRAD}\theta + \\ &(2k_N + k_R)(\partial_t W_l - \partial_t W_s) + (2k_O + k_R)(\partial_t W_g - \partial_t W_s)\cdot\end{aligned} \tag{77}$$

在对 $p_a\,(a = s, l, g)$ 的线性化过程中应用了公式(33)和(65)•

把各向同性非饱和土各组分偏应力张量和动量供给量的线性本构方程(72)~(74)和(75)~(77)代入组分动量守恒方程([1]中(25))就得出混合物各组分线性场方程

$$\rho_a^+\partial_t^2 W_a = \mathrm{DIV}\,t_{ae} + p_{ae} + \rho_a^+ b_a, \tag{78}$$

式中

$$\begin{aligned}t_{se} = &\left(\sum_a \Lambda_{sa}\mathrm{tr}E_a + \sum_a \Lambda_{saD}\mathrm{tr}d_a\right)I + 2\mu_s E_s + 2\sum_a \mu_{sa}d_a + \\ &(\varphi_{ll} + \varphi_{gl})(w_l - w_s) + (\varphi_{lg} + \varphi_{gg})(w_g - w_s) + K_{\theta s}(\theta - \theta^+)I,\end{aligned} \tag{79}$$

$$t_{le} = \left(\sum_a \Lambda_{la}\mathrm{tr}E_a + \sum_a \Lambda_{laD}\mathrm{tr}d_a\right)I + 2\sum_a \mu_{la}d_a -$$

$$\varphi_{ll}(\boldsymbol{w}_l-\boldsymbol{w}_s)-\varphi_{lg}(\boldsymbol{w}_g-\boldsymbol{w}_s)+K_{\theta l}(\theta-\theta^+)\boldsymbol{I},\tag{80}$$

$$\boldsymbol{t}_{ge}=\left(\sum_a\Lambda_{ga}\mathrm{tr}\boldsymbol{E}_a+\sum_a\Lambda_{ga\mathrm{D}}\mathrm{tr}\boldsymbol{d}_a\right)\boldsymbol{I}+2\sum_a\mu_{ga}\boldsymbol{d}_a-$$
$$\varphi_{gl}(\boldsymbol{w}_l-\boldsymbol{w}_s)-\varphi_{gg}(\boldsymbol{w}_g-\boldsymbol{w}_s)+K_{\theta g}(\theta-\theta^+)\boldsymbol{I},\tag{81}$$

$$\boldsymbol{p}_{se}=(2k_N+k_R)(\partial_t\boldsymbol{W}_l-\partial_t\boldsymbol{W}_s)+(2k_O+k_R)(\partial_t\boldsymbol{W}_g-\partial_t\boldsymbol{W}_s),\tag{82}$$

$$\boldsymbol{p}_{le}=-2k_N(\partial_t\boldsymbol{W}_l-\partial_t\boldsymbol{W}_s)-k_R(\partial_t\boldsymbol{W}_g-\partial_t\boldsymbol{W}_s),\tag{83}$$

$$\boldsymbol{p}_{ge}=-k_R(\partial_t\boldsymbol{W}_l-\partial_t\boldsymbol{W}_s)-2k_O(\partial_t\boldsymbol{W}_g-\partial_t\boldsymbol{W}_s),\tag{84}$$

并且

$$\begin{cases}\Lambda_{ss}={}'\Lambda_{ss}-\Psi_s^+=\lambda_s+\phi_s^{+2}\Phi_g^++(1+\phi_s^+)P_{sg},\\ \Lambda_{ll}={}'\Lambda_{ll}-\Psi_l^+=\phi_l^+(\phi_l^+\Phi_g^++2\Gamma_{lg}^+)+\lambda_{ll}^+,\\ \Lambda_{gg}={}'\Lambda_{gg}-\Psi_g^+=\lambda_{gg}^+,\ K_{\theta s}=k_{\theta s}-\left(\rho_s^+\eta_s^+-\dfrac{k_Q+k_P}{\theta^+}\right),\\ K_{\theta l}=k_{\theta l}-\left(\rho_l^+\eta_l^++\dfrac{k_P}{\theta^+}\right),\ K_{\theta g}=k_{\theta g}-\left(\rho_g^+\eta_g^++\dfrac{k_Q}{\theta^+}\right).\end{cases}\tag{85}$$

把场方程(78)对组分求和就是

$$\sum_a\rho_a^+\partial_t^2\boldsymbol{W}_a=\mathrm{DIV}\,\boldsymbol{t}_{\mathrm{I}e}+\rho^+\boldsymbol{b},\tag{86}$$

其中

$$\boldsymbol{t}_{\mathrm{I}e}=\left(\sum_{a,b=s,l,g}\Lambda_{ab}\mathrm{tr}\boldsymbol{E}_b+\sum_{a,b=s,l,g}\Lambda_{ab\mathrm{D}}\mathrm{tr}\boldsymbol{d}_b\right)\boldsymbol{I}+2\mu_s\boldsymbol{E}_s+$$
$$2\sum_{a,b=s,l,g}\mu_{ab}\boldsymbol{d}_b+K(\theta-\theta^+)\boldsymbol{I},\tag{87}$$

$$K=\sum_a k_{\theta a}.\tag{88}$$

方程(78)和(86)反映出:对各组分和混合物运动起决定作用的是应力 $\boldsymbol{t}_{ae}$ 和 $\boldsymbol{t}_{\mathrm{I}e}$ 以及动量供给量 $\boldsymbol{p}_{ae}$. 把 $\boldsymbol{t}_{ae}$ 称为第 a 组分的有效应力; $\boldsymbol{t}_{\mathrm{I}e}$ 是混合物的总应力; $\boldsymbol{p}_{ae}$ 是第 a 组分动量供给量的有效部分.

4 各向同性非饱和土 Biot 型方程

把(79)~(81)式和(87)式给出的应力张量 $\boldsymbol{t}_{ae}$ 和 $\boldsymbol{t}_{\mathrm{I}e}$ 表示成

$$\boldsymbol{t}_{le}=-\phi_l^+P_l\boldsymbol{I}+2\sum_a\mu_{la}\left[\boldsymbol{d}_a-\frac{1}{3}(\mathrm{tr}\boldsymbol{d}_a)\boldsymbol{I}\right]-\varphi_{ll}(\boldsymbol{w}_l-\boldsymbol{w}_s)-\varphi_{lg}(\boldsymbol{w}_g-\boldsymbol{w}_s),\tag{89}$$

$$\boldsymbol{t}_{ge}=-\phi_g^+P_g\boldsymbol{I}+2\sum_a\mu_{ga}\left[\boldsymbol{d}_a-\frac{1}{3}(\mathrm{tr}\boldsymbol{d}_a)\boldsymbol{I}\right]-\varphi_{lg}(\boldsymbol{w}_l-\boldsymbol{w}_s)-\varphi_{gg}(\boldsymbol{w}_g-\boldsymbol{w}_s),\tag{90}$$

$$\boldsymbol{t}_{se}=\left(\sum_a\Lambda_{sa}\mathrm{tr}\boldsymbol{E}_a+\sum_a\left(\Lambda_{sa\mathrm{D}}+\frac{2}{3}\mu_{sa}\right)\mathrm{tr}\boldsymbol{d}_a\right)\boldsymbol{I}+2\mu_s\boldsymbol{E}_s+2\sum_a\mu_{sa}\left[\boldsymbol{d}_a-\frac{1}{3}(\mathrm{tr}\boldsymbol{d}_a)\boldsymbol{I}\right]+$$
$$K_{\theta s}(\theta-\theta^+)\boldsymbol{I}+(\varphi_{ll}+\varphi_{gl})(\boldsymbol{w}_l-\boldsymbol{w}_s)+(\varphi_{lg}+\varphi_{gg})(\boldsymbol{w}_g-\boldsymbol{w}_s)=$$
$$\boldsymbol{t}_{\mathrm{I}e}+\left(\sum_f\phi_f^+P_f\right)\boldsymbol{I}-2\sum_{f,a}\mu_{fa}\left[\boldsymbol{d}_a-\frac{1}{3}(\mathrm{tr}\boldsymbol{d}_a)\boldsymbol{I}\right]+$$
$$\sum_f\varphi_{lf}(\boldsymbol{w}_l-\boldsymbol{w}_s)+\sum_f\varphi_{gf}(\boldsymbol{w}_g-\boldsymbol{w}_s),\tag{91}$$

$$\boldsymbol{t}_{\mathrm{I}e}=-\left(\sum_f\phi_f^+P_f\right)\boldsymbol{I}+\left[\sum_a\Lambda_{sa}\mathrm{tr}\boldsymbol{E}_a+\sum_a\left(\Lambda_{sa\mathrm{D}}+\frac{2}{3}\mu_{sa}\right)\mathrm{tr}\boldsymbol{d}_a\right]\boldsymbol{I}+$$
$$2\mu_s\boldsymbol{E}_s+2\sum_{a,b}\mu_{ba}\left[\boldsymbol{d}_a-\frac{1}{3}(\mathrm{tr}\boldsymbol{d}_a)\boldsymbol{I}\right]+K_{\theta s}(\theta-\theta^+)\boldsymbol{I},\tag{92}$$

其中

$$P_l = -\frac{1}{\phi_l^{+}}\left[\sum_a \Lambda_{la}\mathrm{tr}\boldsymbol{E}_a + \sum_a\left(\Lambda_{la\mathrm{D}} + \frac{2}{3}\mu_{la}\right)\mathrm{tr}\boldsymbol{d}_a + K_{\theta l}(\theta - \theta^{+})\right], \tag{93}$$

$$P_g = -\frac{1}{\phi_g^{+}}\left[\sum_a \Lambda_{ga}\mathrm{tr}\boldsymbol{E}_a + \sum_a\left(\Lambda_{ga\mathrm{D}} + \frac{2}{3}\mu_{ga}\right)\mathrm{tr}\boldsymbol{d}_a + K_{\theta g}(\theta - \theta^{+})\right], \tag{94}$$

P_l 和 P_g 是液体组分和气体组分的真压力, 以压应力为正• $\boldsymbol{t}_{se}$是非饱和土混合物的有效应力•

定义 $\boldsymbol{U}_l$ 和 $\boldsymbol{U}_g$ 分别为

$$\boldsymbol{U}_l = \phi_l^{+}(\boldsymbol{W}_l - \boldsymbol{W}_s), \tag{95}$$

$$\boldsymbol{U}_g = \phi_g^{+}(\boldsymbol{W}_g - \boldsymbol{W}_s), \tag{96}$$

它们表示从单位混合物面元上的固体孔隙中流出流体组分的体积• $\boldsymbol{U}_l$ 和 $\boldsymbol{U}_g$ 的变化率

$$\partial_t\boldsymbol{U}_l = \phi_l^{+}(\partial_t\boldsymbol{W}_l - \partial_t\boldsymbol{W}_s), \tag{97}$$

$$\partial_t\boldsymbol{U}_g = \phi_g^{+}(\partial_t\boldsymbol{W}_g - \partial_t\boldsymbol{W}_s) \tag{98}$$

是混合物中流体组分从固体组分中流出的通量矢量, 通常称为流体组分的渗透速率(filtration velocity) 或比流量(specific discharge)• 引入

$$\zeta_l = -\nabla\cdot\boldsymbol{U}_l, \tag{99}$$

$$\zeta_g = -\nabla\cdot\boldsymbol{U}_g \tag{100}$$

表示流入单位体积混合物固体组分孔隙内的流体体积, 即代表固体孔隙中流体体积的增量• 用 ζ_l 和 ζ_g 把(92) ~ (94) 式表示成

$$\begin{aligned}\boldsymbol{t}_{\mathrm{I}e} = {}& [\lambda_e e - \lambda_{el}\zeta_l - \lambda_{eg}\zeta_g + \lambda_{e\mathrm{D}}\partial_t e - \lambda_{el\mathrm{D}}\partial_t\zeta_l - \lambda_{eg\mathrm{D}}\partial_t\zeta_g + \\ & K(\theta - \theta^{+})]\boldsymbol{I} + 2\mu_s\boldsymbol{E}_s + 2\sum_{a,b}\mu_{ba}\left[\boldsymbol{d}_a - \frac{1}{3}(\mathrm{tr}\boldsymbol{d}_a)\boldsymbol{I}\right],\end{aligned} \tag{101}$$

$$\begin{aligned}P_l = {}& -\lambda_{el}e + M_{ll}\zeta_l + M_{lg}\zeta_g - \frac{K_{\theta l}}{\alpha_l}(\theta - \theta^{+}) - \lambda_{el\mathrm{D}}\partial_t e + \\ & M_{ll\mathrm{D}}\partial_t\zeta_l + M_{lg\mathrm{D}}\partial_t\zeta_g,\end{aligned} \tag{102}$$

$$\begin{aligned}P_g = {}& -\lambda_{eg}e + M_{gl}\zeta_l + M_{gg}\zeta_g - \frac{K_{\theta g}}{\alpha_g}(\theta - \theta^{+}) - \lambda_{eg\mathrm{D}}\partial_t e + \\ & M_{gl\mathrm{D}}\partial_t\zeta_l + M_{gg\mathrm{D}}\partial_t\zeta_g,\end{aligned} \tag{103}$$

其中

$$\left\{\begin{aligned} & e = \nabla\cdot\boldsymbol{W}_s,\ \alpha_a = \phi_a^{+},\ M_{ab} = M_{ba} = \frac{\Lambda_{ab}}{\alpha_a\alpha_b}, \\ & M_{ab\mathrm{D}} = M_{ba\mathrm{D}} = \frac{\Lambda_{ab\mathrm{D}} + (2/3)\mu_{ab}}{\alpha_a\alpha_b}, \\ & \lambda_e = \sum_{a,b}\alpha_a\alpha_b M_{ab} = \sum_{a,b}\Lambda_{ab}, \\ & \lambda_{e\mathrm{D}} = \sum_{a,b}\alpha_a\alpha_b M_{ab\mathrm{D}} = \sum_{a,b}\left(\Lambda_{ab\mathrm{D}} + \frac{2}{3}\mu_{ab}\right), \\ & \lambda_{ef} = \sum_a\alpha_a M_{af} = \frac{1}{\alpha_f}\sum_a\Lambda_{af}, \\ & \lambda_{ef\mathrm{D}} = \sum_a\alpha_a M_{af\mathrm{D}} = \frac{1}{\alpha_f}\sum_a\left(\Lambda_{af\mathrm{D}} + \frac{2}{3}\mu_{af}\right), \\ & \qquad\qquad (a,\ b = s,\ l,\ g;\ f = l,\ g)\bullet \end{aligned}\right. \tag{104}$$

把公式(89)、(90)和(101)~(103)代入方程(78)和(86),场方程又表示为

$$m_l\partial_t^2\boldsymbol{U}_l+\gamma_l^+\partial_t^2\boldsymbol{W}_s=-\nabla P_l-\frac{1}{\alpha_l}\nabla\cdot\left\{2\sum_a\mu_{la}\left[\boldsymbol{d}_a-\frac{1}{3}(\mathrm{tr}\,\boldsymbol{d}_a)\boldsymbol{I}\right]-\varphi_{ll}(\boldsymbol{w}_l-\boldsymbol{w}_s)-\varphi_{lg}(\boldsymbol{w}_g-\boldsymbol{w}_s)\right\}-\frac{\eta_l}{K_{ll}}\partial_t\boldsymbol{U}_l-\frac{\eta_g}{K_{lg}}\partial_t\boldsymbol{U}_g+\gamma_l^+\boldsymbol{b}_l,\tag{105}$$

$$m_g\partial_t^2\boldsymbol{U}_g+\gamma_g^+\partial_t^2\boldsymbol{W}_s=-\nabla P_g-\frac{1}{\alpha_g}\nabla\cdot\left\{2\sum_a\mu_{ga}\left[\boldsymbol{d}_a-\frac{1}{3}(\mathrm{tr}\,\boldsymbol{d}_a)\boldsymbol{I}\right]-\varphi_{gl}(\boldsymbol{w}_l-\boldsymbol{w}_s)-\varphi_{gg}(\boldsymbol{w}_g-\boldsymbol{w}_s)\right\}-\frac{\eta_l}{K_{gl}}\partial_t\boldsymbol{U}_l-\frac{\eta_g}{K_{gg}}\partial_t\boldsymbol{U}_g+\gamma_g^+\boldsymbol{b}_g,\tag{106}$$

$$\rho^+\partial_t^2\boldsymbol{W}_s+\sum_f\gamma_f^+\partial_t^2\boldsymbol{U}_f=\mu_s\nabla^2\boldsymbol{W}_s+(\lambda_e+\mu_s)\nabla e-\lambda_{el}\nabla\zeta_l-\lambda_{eg}\nabla\zeta_g+\lambda_{eD}\nabla(\partial_t e)-\lambda_{elD}\nabla(\partial_t\zeta_l)-\lambda_{egD}\nabla(\partial_t\zeta_g)+K\nabla\theta+2\nabla\cdot\sum_{a,b}\mu_{ab}\left[\boldsymbol{d}_a-\frac{1}{3}(\mathrm{tr}\,\boldsymbol{d}_a)\boldsymbol{I}\right]+\rho^+\boldsymbol{b},\tag{107}$$

式中

$$m_f=\frac{\gamma_f^+}{\alpha_f},\quad\frac{\eta_l}{K_{ll}}=\frac{2k_N}{\alpha_l^2},\quad\frac{\eta_g}{K_{gg}}=\frac{2k_O}{\alpha_g^2},\quad\frac{\eta_l}{K_{gl}}=\frac{\eta_g}{K_{lg}}=\frac{k_R}{\alpha_l\alpha_g},\tag{108}$$

$K_{ab}(a,b=l,g)$ 是流体的渗透系数(coefficients of permeability),η_l 和 η_g 分别是液体和气体的粘度(viscosity).

本构方程(101)~(103)和场方程(105)~(107)与 Biot[2] 研究饱和多孔介质时得到的方程相似,把它们称为非饱和土的 Biot 型本构方程和场方程.

对于非饱和土的等温弹性变形过程,有

$$\begin{cases}\Lambda_{abD}=0,\ \mu_{ab}=0,\ M_{abD}=0,\ \lambda_{eD}=0,\ \lambda_{efD}=0,\ \theta=\theta^+,\\ \varphi_{ll}=\varphi_{lg}=\varphi_{gl}=\varphi_{gg}=0.\end{cases}\tag{109}$$

非饱和土的 Biot 型本构方程(101)~(103)和场方程(105)~(107)简化为

$$\boldsymbol{t}_{\mathrm{I}e}=\lambda_e e\boldsymbol{I}+2\mu_s\boldsymbol{E}_s-\lambda_{el}\zeta_l\boldsymbol{I}-\lambda_{eg}\zeta_g\boldsymbol{I},\tag{110}$$

$$P_l=-\lambda_{el}e+M_{ll}\zeta_l+M_{lg}\zeta_g,\tag{111}$$

$$P_g=-\lambda_{eg}e+M_{gl}\zeta_l+M_{gg}\zeta_g,\tag{112}$$

和

$$m_l\partial_t^2\boldsymbol{U}_l+\gamma_l^+\partial_t^2\boldsymbol{W}_s=-\nabla P_l-\frac{\eta_l}{K_{ll}}\partial_t\boldsymbol{U}_l-\frac{\eta_g}{K_{lg}}\partial_t\boldsymbol{U}_g+\gamma_l^+\boldsymbol{b}_l,\tag{113}$$

$$m_g\partial_t^2\boldsymbol{U}_g+\gamma_g^+\partial_t^2\boldsymbol{W}_s=-\nabla P_g-\frac{\eta_l}{K_{gl}}\partial_t\boldsymbol{U}_l-\frac{\eta_g}{K_{gg}}\partial_t\boldsymbol{U}_g+\gamma_g^+\boldsymbol{b}_g,\tag{114}$$

$$\rho^+\partial_t^2\boldsymbol{W}_s+\sum_f\gamma_f^+\partial_t^2\boldsymbol{U}_f=\nabla\cdot\boldsymbol{t}_{\mathrm{I}e}+\rho^+\boldsymbol{b},\tag{115}$$

在推导以上3式时,假设本构方程(101)~(103)中各参数为常数.

5 非饱和土的 Darcy 定律和饱和多孔介质的 Biot 方程

这里应用前面的结果证明 Darcy 定律描述非饱和土中流体流动的正确性,并且说明 Biot 的饱和多孔介质理论是本文理论的特例.

5.1 非饱和土的 **Darcy** 定律

大量试验证明非饱和土孔隙中液体和气体的运动可以用 Darcy 定律描述[3]. 现在从理论

上说明Darcy 定律对非饱和土孔隙内流体流动的适用性•

若气相和液相在非饱和土孔隙中稳定流动,同时忽略非饱和土各组分自身粘性,则场方程(105)和(106)简化成

$$\frac{\eta_l}{K_{ll}}\partial_t \boldsymbol{U}_l+\frac{\eta_g}{K_{lg}}\partial_t \boldsymbol{U}_g=-\nabla P_l+\gamma_l^{+}\boldsymbol{b}_l, \tag{116}$$

$$\frac{\eta_l}{K_{gl}}\partial_t \boldsymbol{U}_l+\frac{\eta_g}{K_{gg}}\partial_t \boldsymbol{U}_g=-\nabla P_g+\gamma_g^{+}\boldsymbol{b}_g• \tag{117}$$

注意到气相压力梯度对液相流动的影响与液相压力梯度对液相流动的决定性作用相比可以忽略,以及液相压力梯度对气相流动的影响与气相压力梯度对气相流动的决定性作用相比可以忽略,即(108)式中的 $k_R=0,\ K_{gl}=K_{lg}\to\infty$• 以上 2 式进一步写成

$$\partial_t \boldsymbol{U}_l=\frac{K_{ll}}{\eta_l}(-\nabla P_l+\gamma_l^{+}\boldsymbol{b}_l)=\frac{\alpha_l^2}{2k_N}(-\nabla P_l+\gamma_l^{+}\boldsymbol{b}_l), \tag{118}$$

$$\partial_t \boldsymbol{U}_g=\frac{K_{gg}}{\eta_g}(-\nabla P_g+\gamma_g^{+}\boldsymbol{b}_g)=\frac{\alpha_g^2}{2k_O}(-\nabla P_g+\gamma_g^{+}\boldsymbol{b}_g)• \tag{119}$$

推导过程中应用了公式(108)和(104b)• 以上两式与描述非饱和土中液相和气相流动的 Darcy 定律有相同的形式,比较得出液相和气相渗透系数分别是

$$k_l=\frac{\phi_l^{+2}}{2k_N}, \tag{120}$$

$$k_g=\frac{\phi_g^{+2}}{2k_O}, \tag{121}$$

ϕ_l^{+} 和 ϕ_g^{+} 与非饱和土孔隙率 n 和饱和度 S 的关系是

$$\phi_l^{+}=nS, \tag{122}$$

$$\phi_g^{+}=n(1-S), \tag{123}$$

代入(120)式和(121)式,有

$$k_l=\frac{(nS)^2}{2k_N}, \tag{124}$$

$$k_g=\frac{[n(1-S)]^2}{2k_O}• \tag{125}$$

可见,液相和气相渗透系数是孔隙率和饱和度的函数,与 Lloret[4] 的研究结论相同并且 $\lg k_l$ 和 $\lg k_g$ 与 $\lg n$ 是线性关系,这又与陈正汉[5] 的试验结果相同•

以上过程说明,混合物理论完全支持用 Darcy 定律描述非饱和土中流体的流动•

5.2 饱和多孔介质的 Biot 方程

Biot[2,6,7] 的饱和多孔介质理论在研究饱和多孔介质(包括饱和土)的工程性质和动力响应特性方面取得了许多成果[8],Terzaghi 的饱和土一维固结理是 Biot[9] 的饱和多孔介质一般固结理论的特例• 混合物理论[10] 进一步证明,在一定条件下,Biot 的饱和多孔介质理论与混合物理论对饱和多孔介质的研究结果相同• 现在把本文得到的非饱和土线性本构方程和场方程应用于饱和土,推导饱和土的本构方程和场方程• 所得方程与 Biot 的饱和多孔介质理论相同,因此 Biot 理论是本文理论的特例•

土骨架孔隙全部被液体充满的非饱和土是饱和土,所以饱和土混合物 3 种组分的体积分数分别是

$$\phi_g=0,\quad \phi_l=n,\quad \phi_s=1-n, \tag{126}$$

式中, n 是土的孔隙率• 由于 ϕ_g 是常数, 它不再是独立本构变量, 从(1) 式可知

$$\begin{cases} k_C = 0,\ k_F = 0,\ k_H = 0,\ k_I = 0,\ k_K = 0,\ k_L = 0,\ k_O = 0, \\ k_Q = 0,\ k_R = 0,\ k_S = 0,\ k_T = 0,\ k_U = 0,\ k_V = 0 \end{cases} \tag{127}$$

从(34) 式得出

$$\begin{cases} \mu_g^+ = 0,\ \delta_g^+ = 0,\ \tau_g^+ = 0,\ \boldsymbol{J}_{sg}^+ = \boldsymbol{0},\ \lambda_{gg}^+ = 0,\ \Gamma_{lg}^+ = 0, \\ \Phi_g^+ = 0,\ \boldsymbol{P}_{sg}^+ = \boldsymbol{0},\ \lambda_g^+ = 0,\ \Gamma_{gg}^+ = 0,\ \Theta_g^+ = 0 \end{cases} \tag{128}$$

应用(58) 式、(59) 式、(68) 式、(70) 式、(85) 式和(104) 式, 有

$$\begin{cases} \Lambda_{ga} = 0,\ \Lambda_{ga\mathrm{D}} = 0,\ k\theta_g = 0,\ K\theta_g = 0,\ \lambda_{eg} = 0, \\ \lambda_{sg\mathrm{D}} = 0,\ M_{ag} = 0,\ M_{ag\mathrm{D}} = 0 \qquad (a = s, l, g) \end{cases} \tag{129}$$

这样, 本构方程(110) ~ (112) 简化为

$$\boldsymbol{t}_{\mathrm{I}e} = \lambda_e e\boldsymbol{I} + 2\mu_s \boldsymbol{E}_s - \lambda_{el}\zeta_l\boldsymbol{I}, \tag{130}$$

$$P_l = -\lambda_{el}e + M_{ll}\zeta_l; \tag{131}$$

场方程(113) ~ (115) 简化为

$$m_l\partial_t^2\boldsymbol{U}_l + \gamma_l^+\partial_t^2\boldsymbol{W}_s = -\nabla P_l - \frac{\eta_l}{K_{ll}}\partial_t\boldsymbol{U}_l + \gamma_l^+\boldsymbol{b}_l, \tag{132}$$

$$\rho^+\partial_t^2\boldsymbol{W}_s + \gamma_l^+\partial_t^2\boldsymbol{U}_l = \nabla\cdot\boldsymbol{t}_{\mathrm{I}e} + \rho^+\boldsymbol{b}; \tag{133}$$

方程(130) ~ (133) 与 Biot[2]研究饱和多孔介质时得到的方程完全相同• 说明 Biot 的饱和多孔介质理论是本文理论的特例•

6 结　论

对非饱和土非线性本构方程和场方程进行线性化, 推导出线性本构方程和场方程• 把线性方程表示为与 Biot 饱和多孔介质方程相似的形式; 证明了用 Darcy 定律描述非饱和土中流体流动的可靠性; 最后指出 Biot 方程是本文线性方程的特例• 这些都说明了用混合物理论处理非饱和土本构问题的正确性•

[参 考 文 献]

[1] 黄义, 张引科. 非饱和土本构关系的混合物理论(Ⅰ)——非线性本构方程和场方程[J]. 应用数学和力学, 2003, **24**(2): 111—123.

[2] Biot M A. Mechanics of deformation and acoustic propagation in porous media[J]. J Appl Phys, 1962, **33**(4): 1482—1498.

[3] Fredlund D G, Rahardjo H. 非饱和土土力学[M]. 陈仲颐, 张在明, 陈愈炯, 等 译. 北京: 中国建筑工业出版社, 1997, 126—146.

[4] Lloret A, Alonso E E. Consolidation of unsaturated soils including swelling and collapse behavior [J]. Geotechnique, 1980, **30**(4): 449—457.

[5] 陈正汉. 非饱和土固结的混合物理论[D]. 博士学位论文, 西安: 陕西机械学院, 1991, 40—47.

[6] Biot M A. Theory of propagation of elastic wave in a fluid saturated porous solid Ⅰ: Low_frequency range[J]. J Acoust Soc Am, 1956, **28**(2): 168—178.

[7] Biot M A. Theory of propagation of elastic wave in a fluid saturated porous solid Ⅱ: Higher_frequency range[J]. J Acoust Soc Am, 1956, **28**(2): 179—191.

[8] 吴世明. 土介质中的波[M]. 北京: 科学出版社, 1997, 62—98.

[9] Biot M A. General theory of three_dimensional consolidation[J]. J Appl Phys, 1941, **12**(1): 155—

164.
[10] Bowen R M. 混合物理论[M]. 许慧已,董务民 译. 南京:江苏科学技术出版社, 1983, 98—170.

Constitutive Relation of Unsaturated Soil by Use of the Mixture Theory (Ⅱ)—Linear Constitutive Equations and Field Equations

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Abstract: The linear constitutive equations and field equations of unsaturated soils were obtained through linearizing the nonlinear equations given in the first part of this work. The linear equations were expressed in the forms similar to Biot's equations for saturated porous media. The Darcy's laws of unsaturated soil were proved. It is shown that Biot's equations of saturated porous media are the simplification of the theory. All these illustrate that constructing constitutive relation of unsaturated soil on the base of mixture theory is rational.

Key words: mixture theory; unsaturated soil; linear constitutive equation; linear field equation; Darcy's law; Biot's equations