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高阶多元 Nörlund Euler_Bernoulli 多项式*

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摘要: 给出了高阶多元 Nörlund Euler 多项式和高阶多元 Nörlund Bernoulli 多项式的定义,讨论了它们的一些重要性质,建立了一些包含递归序列和上述多项式的恒等式.

关 键 词: 高阶多元 Nörlund Euler 多项式; 高阶多元 Nörlund Bernoulli 多项式; 递归序列; 恒等式

中图分类号: O156; O174 文献标识码: A

引 言

k 阶 Nörlund Euler_Bernoulli 多项式 $E_v^{(k)}(x \mid \omega_1, \dots, \omega_k)$ 和 $B_v^{(k)}(x \mid \omega_1, \dots, \omega_k)$ 分别由下列展开式给出[1]

$$\frac{2^k e^{xt}}{(e^{\omega_1 t}+1)\cdots(e^{\omega_k t}+1)} = \sum_{v=0}^{\infty} E_v^{(k)}(x \mid \omega_1, \dots, \omega_k)\frac{t^v}{v!}, \tag{1}$$

$$\frac{\omega_1\cdots\omega_k t^k e^{xt}}{(e^{\omega_1 t}-1)\cdots(e^{\omega_k t}-1)} = \sum_{v=0}^{\infty} B_v^{(k)}(x \mid \omega_1, \dots, \omega_k)\frac{t^v}{v!}. \tag{2}$$

显然 $E_v(x) := E_v^{(1)}(x \mid 1)$, $B_v(x) := B_v^{(1)}(x \mid 1)$ 分别是通常的 Euler 多项式和 Bernoulli 多项式. Euler_Bernoulli 多项式作为两类特殊函数,在函数论和解析数论中占有重要的地位,有着广泛的应用. 对 Euler_Bernoulli 多项式的研究,一直是国内外许多学者感兴趣的研究课题,并有了大量的研究成果. 本文作者在文献[2]中已将 Euler_Bernoulli 多项式推广到高阶多元;并在文献[3]中讨论了高阶多元 Euler_Bernoulli 多项式与递归序列的关系. 本文在文献[2]的基础上,进一步提出高阶多元 Nörlund Euler_Bernoulli 多项式,并对它们进行深入的研究,所得的结果是文献[1]~[6]相应结果的推广和深化.

1 定义和引理

定义 1 k 阶 n 元 Nörlund Euler 数 $E_{v_1\cdots v_n}^{(k)}[\omega_1, \dots, \omega_k]$ 由下列展开式给出

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$$\frac{2^k e^{(\sum_{i=1}^k \omega_i)(\sum_{j=1}^n t_j)}}{\prod_{i=1}^k (e^{2\omega_i \sum_{j=1}^n t_j} + 1)} = \sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} E^{(k)}_{v_1 \cdots v_n}[\omega_1, \cdots, \omega_k] \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!}. \tag{3}$$

定义 2　k 阶 n 元 Nörlund Euler 多项式 $E^{(k)}_{v_1 \cdots v_n}(x_1, \cdots, x_n \mid \omega_1, \cdots, \omega_k)$ 由下列展开式给出

$$\frac{2^k e^{\sum_{j=1}^n x_j t_j}}{\prod_{i=1}^k (e^{\omega_i \sum_{j=1}^n t_j} + 1)} = \sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} E^{(k)}_{v_1 \cdots v_n}(x_1, \cdots, x_n \mid \omega_1, \cdots, \omega_k) \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!}. \tag{4}$$

定义 3　k 阶 n 元 Nörlund Bernoulli 数 $B^{(k)}_{v_1 \cdots v_n}[\omega_1, \cdots, \omega_k]$ 由下列展开式给出

$$(\sum_{j=1}^n t_j)^k \prod_{i=1}^k \left(\frac{\omega_i}{e^{\omega_i \sum_{j=1}^n t_j} - 1} \right) = \sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} B^{(k)}_{v_1 \cdots v_n}[\omega_1, \cdots, \omega_k] \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!}. \tag{5}$$

定义 4　k 阶 n 元 Nörlund Euler 多项式 $B^{(k)}_{v_1 \cdots v_n}(x_1, \cdots, x_n \mid \omega_1, \cdots, \omega_k)$ 由下列展开式给出

$$(\sum_{j=1}^n t_j)^k e^{\sum_{j=1}^n x_j t_j} \prod_{i=1}^k \left(\frac{\omega_i}{e^{\omega_i \sum_{j=1}^n t_j} - 1} \right) = \sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} B^{(k)}_{v_1 \cdots v_n}(x_1, \cdots, x_n \mid \omega_1, \cdots, \omega_k) \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!}. \tag{6}$$

定义 5　负 k 阶 n 元 Nörlund Bernoulli 数 $B^{(-k)}_{v_1 \cdots v_n}[\omega_1, \cdots, \omega_k]$ 由下列展开式给出

$$(\sum_{j=1}^n t_j)^{-k} \prod_{i=1}^k \left(\frac{e^{\omega_i \sum_{j=1}^n t_j} - 1}{\omega_i} \right) = \sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} B^{(-k)}_{v_1 \cdots v_n}[\omega_1, \cdots, \omega_k] \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!}. \tag{7}$$

在上述定义中, k 是非负整数, n 是自然数, $\omega_1, \cdots, \omega_k$ 是任意常数.

定义 6　二阶线性递归序列 $\{H_s\}$ 定义为 $H_s = pH_{s-1} - qH_{s-2}$, 这里 H_0, H_1, q, p 为任意实数, 且 $\Delta = p^2 - 4q > 0$. 若取 $H_0 = 0, H_1 = 1$ 或 $H_0 = 2, H_1 = p$, 则 $\{H_s\}$ 分别为广义 Fibonacci 序列 $\{F_s^*\}$ 与广义 Lucas 序列 $\{L_s^*\}$, 显然 $H_s = (H_1 - pH_0/2)F_s^* + H_0 L_s^*/2$.

引理 1　设方程 $x^2 - px + q = 0$ 的两根为 α, β, 且 $\alpha > \beta$, 则

$$(\text{i})\quad F_s^* = \frac{\alpha^s - \beta^s}{\alpha - \beta}, \quad L_s^* = \alpha^s + \beta^s; \tag{8}$$

(ii) 若假定 $H_0 = \sum_{i=1}^k \omega_i, H_1 = \frac{1}{2}p \sum_{i=1}^k \omega_i + (x - \frac{1}{2}\sum_{i=1}^k \omega_i)\Delta^{\frac{1}{2}}$, 则有

$$H_s = (x - \frac{1}{2}\sum_{i=1}^k \omega_i)\Delta^{\frac{1}{2}} F_s^* + \frac{1}{2} L_s^* \sum_{i=1}^k \omega_i = x\alpha^s + (\sum_{i=1}^k \omega_i - x)\beta^s. \tag{9}$$

(此时记 $H_s = H_s(x)$).

引理 2[2]

$$\left(\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} f(v_1, \cdots, v_n) \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!} \right) \left(\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} g(v_1, \cdots, v_n) \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!} \right) =$$

$$\sum_{v_1=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} \left(\sum_{j_1=0}^{v_1} \cdots \sum_{j_n=0}^{v_n} \binom{v_1}{j_1} \cdots \binom{v_n}{j_n} f(j_1, \cdots, j_n) \times \right.$$

$$\left. g(v_1 - j_1, \cdots, v_n - j_n) \right) \frac{t_1^{v_1}}{v_1!} \cdots \frac{t_n^{v_n}}{v_n!}. \tag{10}$$

引理 3[3] (i) $$\sum_{j_1=0}^{v_1}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}\left(1+(-1)^{\sum_{s=1}^{n}(v_s-j_s)}\right)E_{j_1\cdots j_n}=\begin{cases}2\ (v_1=\cdots v_n=0)\\0\ (v_1+\cdots+v_n>0)\end{cases};\tag{11}$$

$$\text{(ii) } B_{v_1\cdots v_n}=\begin{cases}1 & (v_1=\cdots v_n=0),\\ \sum_{j_1=0}^{v_1}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}B_{j_1\cdots j_n} & (v_1+\cdots+v_n>0)\end{cases}\cdot\tag{12}$$

2 主 要 结 果

定理 1 (i) $$E^{(1)}_{v_1\cdots v_n}[\omega]=\omega^{\sum_{j=1}^{n}v_j}E_{v_1\cdots v_n};\tag{13}$$

(ii) $$E^{(k)}_{v_1\cdots v_n}[\omega_1,\cdots,\omega_k]=\sum_{j_1=0}^{v_1}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}E^{(i)}_{j_1\cdots j_n}[\omega_1,\cdots,\omega_i]\times E^{(k-i)}_{(v_1-j_1)\cdots(v_n-j_n)}[\omega_{i+1},\cdots,\omega_k]\quad(1<i<k);\tag{14}$$

(iii) $$B^{(1)}_{v_1\cdots v_n}[\omega]=\omega^{\sum_{j=1}^{n}v_j}B_{v_1\cdots v_n};\tag{15}$$

(iv) $$B^{(k)}_{v_1\cdots v_n}[\omega_1\cdots\omega_k]=\sum_{j_1=0}^{v_1}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}B^{(i)}_{j_1\cdots j_n}[\omega_1,\cdots,\omega_i]\times B^{(k-i)}_{(v_1-j_1)\cdots(v_n-j_n)}[\omega_{i+1},\cdots,\omega_k]\quad(1<i<k)\cdot\tag{16}$$

证明 (i) 由定义 1, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E^{(1)}_{v_1\cdots v_n}[\omega]\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}=\frac{2e^{\omega(t_1+\cdots+t_n)}}{e^{2\omega(t_1+\cdots+t_n)}+1}=$$
$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E^{(1)}_{v_1\cdots v_n}[\omega]\frac{(\omega t_1)^{v_1}}{v_1!}\cdots\frac{(\omega t_n)^{v_n}}{v_n!}=$$
$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\omega^{\sum_{j=1}^{n}v_j}E^{(1)}_{v_1\cdots v_n}[\omega]\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!},\tag{17}$$

由(17), 我们有(13)·

(ii) 由引理 2 和定义 1, 我们有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\Big(\sum_{j_1=0}^{v_n}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}E^{(i)}_{j_1\cdots j_n}[\omega_1,\cdots,\omega_i]\times$$
$$E^{(k-i)}_{(v_1-j_1)\cdots(v_n-j_n)}[\omega_{i+1},\cdots,\omega_k]\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}=$$
$$\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E^{(i)}_{v_1\cdots v_n}[\omega_1,\cdots,\omega_i]\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)\times$$
$$\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E^{(k-i)}_{v_1\cdots v_n}[\omega_{i+1},\cdots,\omega_n]\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)=$$

$$\frac{2^k e^{(\sum_{s=1}^{k}\omega_s)(\sum_{j=1}^{n}t_j)}}{\prod_{s=1}^{k}(e^{2\omega_s\sum_{j=1}^{n}t_j}+1)} = \sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}E_{v_1\cdots v_n}^{(k)}[\omega_1,\cdots,\omega_k]\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}, \tag{18}$$

由(18),我们有(14)。(iii)和(iv)的证明类似于(i)、(ii)。

注1 定理1是高阶多元 Nörlund Euler 数和高阶多元 Nörlund Bernoulli 数的递推公式。由定理(i)、(ii)和引理3(i)可逐一求出高阶多元 Nörlund Euler 数。由定理1(iii)、(iv)引理3ii)可逐一求出高阶多元 Nörlund Bernoulli 数。

定理2 (i) $E_{v_1\cdots v_n}^{(k)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_k)=$

$$\sum_{j_1=0}^{v_1}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}\left(\frac{1}{2}\right)^{\sum_{s=1}^{n}j_s}\left(x_1-\frac{1}{2}\sum_{i=1}^{k}\omega_i\right)^{v_1-j_1}\cdots\times$$
$$\left(x_n-\frac{1}{2}\sum_{i=1}^{k}\omega_i\right)^{v_n-j_n}E_{j_1\cdots j_n}^{(k)}[\omega_1,\cdots,\omega_k]; \tag{19}$$

(ii) $B_{v_1\cdots v_n}^{(k)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_k)=$

$$\sum_{j_1=0}^{v_1}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}x_1^{v_1-j_1}\cdots x_n^{v_n-j_n}B_{j_1\cdots j_n}^{(k)}[\omega_1,\cdots,\omega_k]. \tag{20}$$

证明 由定义1,定义2和引理2即可得(19)。由定义3,定义4和引理2即可得(20)。

注2 由定理2可逐一求出高阶多元 Nörlund Euler多项式和高阶多元 Nörlund BErnoulli 多项式。

定理3 (i) $E_{v_1\cdots v_n}^{(k)}(\sum_{i=1}^{k}\omega_i-x_1,\cdots,\sum_{i=1}^{k}\omega_i-x_n\mid\omega_1,\cdots,\omega_k)=$

$$(-1)^{\sum_{j=1}^{n}v_j}E_{v_1\cdots v_n}^{(k)}(x_1,\cdots x_n\mid\omega_1,\cdots,\omega_k); \tag{21}$$

(ii) $B_{v_1\cdots v_n}^{(k)}(\sum_{i=1}^{k}\omega_i-x_1,\cdots,\sum_{i=1}^{k}\omega_i-x_n\mid\omega_1,\cdots,\omega_k)=$

$$(-1)^{\sum_{j=1}^{n}v_j}B_{v_1\cdots v_n}^{(k)}(x_1,\cdots x_n\mid\omega_1,\cdots,\omega_k). \tag{22}$$

证明 由定义2和定义4即可得到(21),(22)。

定理4 (i) $E_{v_1\cdots v_n}^{(k)}(\omega_i+x_1,\cdots,\omega_i+x_n\mid\omega_1,\cdots,\omega_k)+E_{v_1\cdots v_n}^{(k)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_k)=$

$$2E_{v_1\cdots v_n}^{(k-1)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_{i-1},\omega_{i+1},\cdots,\omega_k)\qquad(1\leqslant i\leqslant k); \tag{23}$$

(ii) $B_{v_1\cdots v_n}^{(k)}(\omega_i+x_1,\cdots,\omega_i+x_n\mid\omega_1,\cdots,\omega_k)-B_{v_1\cdots v_n}^{(k)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_k)=$

$$\omega_i\sum_{j=1}^{n}v_jB_{v_1\cdots(v_j-1)\cdots v_n}^{(k-1)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_{i-1},\omega_{i+1},\cdots,\omega_k)\qquad(1\leqslant i\leqslant k). \tag{24}$$

证明 由定义2,我们有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_2=0}^{\infty}(E_{v_1\cdots v_n}^{(k)}(\omega_i+x_1,\cdots,\omega_i+x_n\mid\omega_1,\cdots,\omega_k)+$$
$$E_{v_1\cdots v_n}^{(k)}(x_1\cdots x_n\mid\omega_1,\cdots,\omega_k))\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}=$$

$$\frac{2^k e^{\sum_{j=1}^{n}(\omega_i+x_j)t_j}}{\prod_{s=1}^{k}(e^{\omega_s\sum_{j=1}^{n}t_j}+1)}+\frac{2^k e^{\sum_{j=1}^{n}x_jt_j}}{\prod_{s=1}^{k}(e^{\omega_s\sum_{j=1}^{n}t_j}+1)}=\frac{2^k e^{\sum_{j=1}^{n}x_jt_j}}{\prod_{s=1,\ s\neq i}^{k}(e^{\omega_s\sum_{j=1}^{n}t_j}+1)}=$$

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}2E_{v_1\cdots v_n}^{(k-1)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_{i-1},\omega_{i+1},\cdots,\omega_k)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!},\tag{25}$$

由(25),我们立即得到(23)• (24)的证明类似于(23)•

定理 5 (i) $E_{v_1\cdots v_n}^{(p+q)}(x_1+y_1,\cdots,x_n+y_n\mid\omega_1,\cdots,\omega_{p+q})=$

$$\sum_{j_1=0}^{v_1}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}E_{j_1\cdots j_n}^{(p)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_p)\times E_{(v_1-j_1)\cdots(v_n-j_n)}^{(q)}(y_1,\cdots,y_n\mid\omega_{p+1},\cdots,\omega_{p+q});\tag{26}$$

(ii) $B_{v_1\cdots v_n}^{(p+q)}(x_1+y_1,\cdots,x_n+y_n\mid\omega_1,\cdots,\omega_{p+q})=$

$$\sum_{j_1=0}^{v_1}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}B_{j_1\cdots j_n}^{(p)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_p)\times B_{(v_1-j_1)\cdots(v_n-j_n)}^{(q)}(y_1,\cdots,y_n\mid\omega_{p+1},\cdots,\omega_{p+q})\text{•}\tag{27}$$

证明 由定义 2, 定义 4 和引理 2 即得•

定理 6 $\omega_i\sum_{j=1}^{n}\frac{\partial}{\partial x_j}E_{v_1\cdots v_n}^{(k)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_k)=$

$$\sum_{j_1=0}^{v_1}\cdots\sum_{j_n=0}^{v_n}\binom{v_1}{j_1}\cdots\binom{v_n}{j_n}2^{\sum_{s=1}^{n}j_s}\left(B_{j_1\cdots j_n}^{(k)}\left(\frac{x_n}{2}+\frac{\omega_i}{2},\cdots,\frac{x_1}{2}+\frac{\omega_i}{2}\mid\omega_1,\cdots,\omega_k\right)-B_{j_1\cdots j_n}^{(k)}\left(\frac{x_1}{2},\cdots,\frac{x_n}{2}\mid\omega_1,\cdots,\omega_k\right)\right)\times B_{(v_1-j_1)\cdots(v_n-j_n)}^{(1-k)}[\omega_1,\cdots\omega_{i-1},\omega_{i+1},\cdots,\omega_k]\quad(1\leqslant i\leqslant k)\text{•}\tag{28}$$

证明 由定义 2, 定义 4, 定义 5 和引理 2 即得•

注 3 在本文定理 1~ 6 中, 令 $\omega_1=\cdots=\omega_k=1$, 则可得到文献[2]中的定理 2、定理 3、定理 4、定理 5、定理 7、定理 9•

定理 7 $2^kH_s^{m_1}(x_1)\cdots H_s^{m_n}(x_n)=\sum_{j_1=0}^{m_1}\cdots\sum_{j_n=0}^{m_n}\binom{m_1}{j_1}\cdots\binom{m_n}{j_n}(\Delta^{\frac{1}{2}}F_s^*)^{\sum_{i=1}^{n}j_i}(\sum_{i=1}^{n}(m_i-j_i))!\times$

$$\sum_{r_1+\cdots+r_k=\sum_{i=1}^{n}(m_i-j_i)}\frac{\omega_1^{r_1}L_{sr_1}^*\cdots\omega_k^{r_k}L_{sr_k}^*}{r_1!\cdots r_k!}E_{j_1\cdots j_n}^{(k)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_k)\text{•}\tag{29}$$

证明 在定义 2 中, 将 t_i 代换为 $\Delta^{\frac{1}{2}}F_s^*t_i(i=1,\cdots,n)$, 并结合引理 1, 有

$$\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}(\Delta^{\frac{1}{2}}F_S^*)^{\sum_{i=1}^{n}v_i}E_{v_1\cdots v_n}^{(k)}(x_1,\cdots,x_n\mid\omega_1,\cdots,\omega_k)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}=$$

$$\frac{2^k e^{\Delta^{\frac{1}{2}} F_s^* \sum_{j=1}^{n} x_j t_j}}{\prod_{i=1}^{k}\left(e^{\Delta^{\frac{1}{2}} F_s^* \omega_i \sum_{j=1}^{n} t_j}+1\right)}=\frac{2^k e^{\sum_{j=1}^{n} t_j H_s(x_j)}}{\prod_{i=1}^{k}\left(e^{\alpha^s \omega_i \sum_{j=1}^{n} t_j}+\left(e^{\beta^s \omega_i \sum_{j=1}^{n} t_j}\right)\right)},$$

因此,

$$2^k e^{\sum_{j=1}^{\infty} t_j H_s(x_j)}=\prod_{i=1}^{k}\left(e^{\alpha_s \omega_i \sum_{j=1}^{n} t_j}+e^{\beta_s \omega_i \sum_{j=1}^{n} t_j}\right)\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\Delta^{\frac{1}{2}} F_s^*\right)^{\sum_{i=1}^{n} v_i}\times\right.$$

$$\left.E_{v_1\cdots v_n}^{(k)}(x_1,\cdots,x_n \mid \omega_1,\cdots,\omega_k)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)=$$

$$\prod_{i=1}^{k}\left(\sum_{r_i=0}^{\infty}\frac{\omega_i^{r_i}}{r_i!}L_{sr_i}^*(t_1+\cdots+t_n)^{r_i}\right)\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\Delta^{\frac{1}{2}} F_s^*\right)^{\sum_{i=1}^{n} v_i}\times\right.$$

$$\left.E_{v_1\cdots v_n}^{(k)}(x_1,\cdots,x_n \mid \omega_1,\cdots,\omega_k)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)=$$

$$\left(\sum_{r=0}^{\infty}\left(\sum_{r_1+\cdots+r_k=r}\frac{\omega_1^{r_1}L_{sr_1}^*\cdots\omega_k^{r_k}L_{sr_k}^*}{r_1!\cdots r_k!}(t_1+\cdots+t_n)^r\right)\right)\times$$

$$\left(\sum_{v_1=0}^{\infty}\cdots\sum_{v_n=0}^{\infty}\left(\Delta^{\frac{1}{2}} F_s^*\right)^{\sum_{i=1}^{n} v_i}E_{v_1\cdots v_n}^{(k)}(x_1,\cdots,x_n \mid \omega_1,\cdots,\omega_k)\frac{t_1^{v_1}}{v_1!}\cdots\frac{t_n^{v_n}}{v_n!}\right)$$

展开上式,比较 $t_1^{m_1}\cdots t_n^{m_n}$ 的系数即知结论成立•

定理 8 $k!\,\omega_1\cdots\omega_k(F_s^*)^k\sum_{j_1+\cdots+j_n=k}\binom{m_1}{j_1}\cdots\binom{m_n}{j_n}H_s^{m_1-j_1}(x_1)\cdots H_s^{m_n-j_n}(x_n)=$

$$\sum_{j_1=0}^{m_1}\cdots\sum_{j_n=0}^{m_n}\binom{m_1}{j_1}\cdots\binom{m_n}{j_n}\left(\Delta^{\frac{1}{2}} F_s^*\right)^{\sum_{i=1}^{n} j_i}\left(\sum_{i=1}^{n}(m_i-j_i)\right)!\times$$

$$\sum_{r_1+\cdots+r_k=\sum_{i=1}^{n}(m_i-j_i)}\frac{\omega_1^{r_1}F_{sr_1}^*\cdots\omega_k^{r_k}F_{sr_k}^*}{r_1!\cdots r_k!}B_{j_1\cdots j_n}^{(k)}(x_1,\cdots,x_n \mid \omega_1,\cdots,\omega_k)•$$

证明 证法与定理 7 类似•

注 4 在本文定理 7、定理 8 中,令 $\omega_1=\cdots=\omega_k=1$,则可得到文献[3]中的定理 3.1、定理 3.3•

3 应 用

例 1 计算 $E_{11}^{(2)}(x_1,x_2 \mid \omega_1,\omega_2)$•

1° 在定理 7 中,令 $n=k=2$, $m_1=m_2=0$,可得到 $E_{00}^{(2)}(x_1,x_2 \mid \omega_1,\omega_2)=1$;

2° 在定理 7 中,令 $n=k=2$, $m_1=1$, $m_2=0$,并结合 1° 可得到

$$E_{10}^{(2)}(x_1,x_2 \mid \omega_1,\omega_2)=x_1-(\omega_1+\omega_2)/2;$$

若令 $n=k=2$, $m_1=0$, $m_2=1$,并结合 1° 则可得

$$E_{01}^{(2)}(x_1,x_2 \mid \omega_1,\omega_2)=x_2-(\omega_1+\omega_2)/2;$$

3° 在定理 7 中,令 $n=k=2$, $m_1=m_2=1$,注意到 $L_0^*=2$, $L_{2n}^*=(L_n^*)^2/2+(\Delta^{1/2}F_n^*)^2/2$,并结合 1° 和 2° 可得到

$$E_{11}^{(2)}(x_1,x_2 \mid \omega_1,\omega_2)=(x_1-(\omega_1+\omega_2)/2)(x_2-(\omega_1+\omega_2)/2)-(\omega_1^2+\omega_2^2)/4•$$

例 2 计算 $B_{11}^{(2)}(x_1, x_2 \mid \omega_1, \omega_2)$•

4° 在定理 8 中, 令 $n = k = 2, m_1 = m_2 = 1$, 可得到

$$B_{00}^{(2)}(x_1, x_2 \mid \omega_1, \omega_2) = 1;$$

5° 在定理 8 中, 令 $n = k = 2, m_1 = 3, m_2 = 0$, 并结合 4° 可得到

$$B_{10}^{(2)}(x_1, x_2 \mid \omega_1, \omega_2) = x_1 - (\omega_1 + \omega_2)/2,$$

若令 $n = k = 2, m_1 = 0, m_2 = 3$, 并结合 4° 则可得到

$$B_{01}^{(2)}(x_1, x_2 \mid \omega_1, \omega_2) = x_2 - (\omega_1 + \omega_2)/2;$$

6° 在定理 8 中, 令 $n = k = 2, m_1 = 4, m_2 = 0$, 并结合 4° 和 5° 可得到

$$B_{20}^{(2)}(x_1, x_2 \mid \omega_1, \omega_2) = (x_1 - (\omega_1 + \omega_2)/2)^2 - (\omega_1^2 + \omega_2^2)/12•$$

7° 在定理 8 中, 令 $n = k = 2, m_1 = 3, m_2 = 1$, 注意到 $F_0^* = 0, F_{2n}^* = F_n^* L_n^*, 4F_n^* F_{3n}^* = 3(F_{2n}^*)^2 + \Delta(F_n^*)^4$, 并结合 4°, 5° 和 6° 可得到

$$B_{11}^{(2)}(x_1, x_2 \mid \omega_1, \omega_2) = \left(x_1 - \frac{\omega_1 + \omega_2}{2}\right)\left(x_2 - \frac{\omega_1 + \omega_2}{2}\right) - \frac{\omega_1^2 + \omega_2^2}{12}•$$

注 5 在例 1 和例 2 中, 令 $\omega_1 = \omega_2 = 1$, 即可得到文献[3]中的例 1 例 2•

例 3 设 $f(x_1, x_2, \cdots x_n) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$, $g(x_1, x_2, \cdots x_n) = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$,
其中 v_j 是非负整数, $j = 1, 2, \cdots, n$• 则对任意正整数 k 和任意数 $\omega \neq 0$, 我们有

(a) $$\sum_{i=1}^{k-1} f(i\omega + x_1, \cdots, i\omega + x_n) = $$
$$1/\omega\left(B_{v_1 \cdots v_n}^{(1)}(k\omega + x_1, \cdots, k\omega + x_n \mid \omega) - B_{v_1, \cdots, v_n}^{(1)}(x_1, \cdots, x_n \mid \omega)\right);$$

(b) $$\sum_{i=0}^{k-1} (-1)^i g(i\omega + x_1, \cdots, i\omega + x_n) = $$
$$\frac{1}{2}\left((-1)^{k-1} E_{v_1 \cdots v_n}^{(1)}(k\omega + x_1, \cdots, k\omega + x_n \mid \omega) + E_{v_1 \cdots v_n}^{(1)}(x_1, \cdots, x_n \mid \omega)\right)•$$

8° 在定理 4 中令 $k = 1$, 我们有

$$E_{v_1 \cdots v_n}^{(1)}(\omega + x_1, \cdots, \omega + x_n \mid \omega) + E_{v_1 \cdots v_n}^{(1)}(x_1, \cdots, x_n \mid \omega) = 2E_{v_1 \cdots v_n}^{(0)}(x_1, \cdots, x_n);$$

$$B_{v_1 \cdots v_n}^{(1)}(\omega + x_1, \cdots, \omega + x_n \mid \omega) - B_{v_1 \cdots v_n}^{(1)}(x_1, \cdots, x_n \mid \omega) = $$
$$\omega \sum_{j=1}^{n} v_j B_{v_1 \cdots (v_j - 1) \cdots v_n}^{(0)}(x_1, \cdots, x_n)•$$

9° 在定义 2 和定义 4 中令 $k = 0$, 我们有

$$E_{v_1 \cdots v_n}^{(0)}(x_1, \cdots, x_n) = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n},\ B_{v_1 \cdots v_n}^{(0)}(x_1, \cdots, x_n) = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}•$$

由 8° 和 9°, 得

$$E_{v_1 \cdots v_n}^{(1)}(\omega + x_1, \cdots, \omega + x_n \mid \omega) + E_{v_1 \cdots v_n}^{(1)}(x_1, \cdots, x_n \mid \omega) = 2x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n},$$

$$B_{v_1 \cdots v_n}^{(1)}(\omega + x_1, \cdots, \omega + x_n \mid \omega) - B_{v_1 \cdots v_n}^{(1)}(x_1, \cdots, x_n \mid \omega) = \omega \sum_{j=1}^{n} \frac{\partial}{\partial x_j} x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}•$$

所以有(a)和(b)•

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Higher Order Multivariable N⌊rlund Euler_Bernoulli Polynomials

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Abstract: The definitions of higher order multivariable N⌊rlund Euler polynomials and N⌊rlund Bernoulli polynomials are presented and some of their important properties are expounded. Some identities involving recurrence sequences and higher order multivariable N⌊rlund Euler_Bernoulli polynomials are established.

Key words: higher order multivariable N⌊lund Euler polynomial; higher order multivariable N⌊rlund Bernoulli polynomial; recurrence sequence; identity