文章编号: 1000_0887(2002) 09_0881_08

正交各向异性板的非对称大变形问题*

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(我刊编委叶开沅来稿)

摘要: 从各向异性板的基本理论出发, 推导出正交各向异性圆板的非对称大变形基本方程, 利用 Fourier 级数把问题的偏微分方程转化为一组可积分求解的非线性常微分方程, 并给出利用迭代法求解该问题的基本方法•

关 键 词: 正交各向异性; 圆薄板; 非对称; 大变形; 迭代法
中图分类号: O405 文献标识码: A

引 言

正交各向异性复合材料由于其独特的优异性能, 作为航空、航天工程中的结构材料正得到重视和应用• 正交各向异性板的理论是进一步研究复合板材料、层合板、加筋板、多层板壳的基础• 有关这方面的研究不少学者已涉及到[1~5], 但对于正交各向异性圆薄板的非对称大变形问题尚未查到有人问津• 本文是我们对各向同性板壳非对称大变形问题[6~8]的延拓, 这对正交各向异性复合材料结构性能的研究和广泛应用具有重要意义•

1 正交各向异性板的基本方程

考虑一半径为 a, 厚度为 h 的正交各向异性圆薄板, 受外载荷 q 的作用• 则考虑大变形情况时, 我们有

1.1 中面变形几何方程

$$\varepsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, \tag{1}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)^2, \tag{2}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial u}{\partial r} - \frac{v}{r} + \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}, \tag{3}$$

$$\chi_r = -\frac{\partial^2 w}{\partial r^2}, \tag{4}$$

$$\chi_\theta = -\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r}\frac{\partial w}{\partial r}, \tag{5}$$

* 收稿日期: 2001_04_03; 修订日期: 2001_11_28
基金项目: 甘肃工业大学科研基金(92114)
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$$\chi_{r\theta} = -\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial w}{\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right). \tag{6}$$

1.2 物理方程

$$\sigma_r = \frac{E_r}{1-\mu_r\mu_\theta}(\varepsilon_r + \mu_\theta\varepsilon_\theta), \tag{7}$$

$$\sigma_\theta = \frac{E_\theta}{1-\mu_r\mu_\theta}(\varepsilon_\theta + \mu_r\varepsilon_r), \tag{8}$$

$$\tau_{r\theta} = \frac{E}{2(1+\mu)}\varepsilon_{r\theta}, \tag{9}$$

$$M_r = D_r(\chi_r + \mu_\theta\chi_\theta), \tag{10}$$

$$M_\theta = D_\theta(\chi_\theta + \mu_r\chi_r), \tag{11}$$

$$M_{r\theta} = 2D_k\chi_{r\theta}, \tag{12}$$

其中, u, v 分别表示圆板径向和环向上的位移, w 表示圆板的挠度, μ_r, μ_θ 分别表示径向和环向上的波松比, E_r, E_θ 分别表示径向和环向上的弹性模量, D_r, D_θ 分别表示径向和环向上的抗弯刚度, G 为剪切弹性模量. 且有 $D_r = \frac{E_r h^3}{12(1-\mu_r\mu_\theta)}$, $D_\theta = \frac{E_\theta h^3}{12(1-\mu_r\mu_\theta)}$, $D_k = \frac{1}{12}Gh^3$, $G = \frac{E}{2(1+\mu)}$, $D_{r\theta} = D_r\mu_\theta + 2D_k$. 对于正交各向异性板有 $E_r\mu_\theta = E_\theta\mu_r$.

1.3 平衡方程

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} = 0, \tag{13}$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} = 0, \tag{14}$$

$$-\frac{1}{h}\left(\frac{\partial Q_r}{\partial r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial\theta} + \frac{Q_r}{r}\right) = \sigma_r\frac{\partial^2 w}{\partial r^2} + \sigma_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) + 2\tau_{r\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right) + \frac{q}{h}, \tag{15}$$

其中

$$Q_r = \frac{\partial M_r}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta} + \frac{M_r - M_\theta}{r}, \quad Q_\theta = \frac{\partial M_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial M_\theta}{\partial\theta} + \frac{2M_{r\theta}}{r}.$$

把(1)~(12)代入(13)~(15), 可得本问题的基本方程.

1.5 基本方程

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\mu_\theta}{r}\frac{\partial^2 v}{\partial r\partial\theta} + \frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \frac{\mu_\theta}{r^2}\frac{\partial w}{\partial\theta}\frac{\partial^2 w}{\partial r\partial\theta} + \frac{1-\mu_\theta}{2r}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{\mu_\theta}{2r^3}\left(\frac{\partial w}{\partial\theta}\right)^2 - \frac{E_\theta}{E_r}\left[\frac{u}{r^2} + \frac{1}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{r^3}\left(\frac{\partial w}{\partial\theta}\right)^2\right] + \frac{(1-\mu_r\mu_\theta)E}{2(1+\mu)E_r}\left(\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta}\frac{\partial w}{\partial\theta}\right) = 0, \tag{16}$$

$$E(1-\mu_r\mu_\theta)\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2}\frac{\partial w}{\partial\theta} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial w}{\partial r}\frac{\partial w}{\partial\theta}\right) + 2(1+\mu)E_\theta\left(\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial u}{\partial\theta} + \frac{\mu_r}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{1}{r^3}\frac{\partial w}{\partial\theta}\frac{\partial^2 w}{\partial\theta^2} + \frac{\mu_r}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial\theta}\right) = 0, \tag{17}$$

$$\frac{D_r}{h}\left(\frac{\partial^4 w}{\partial r^4}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}\right)+\frac{D_\theta}{h}\left(\frac{2}{r^4}\frac{\partial^2 w}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^3}\frac{\partial w}{\partial r}+\frac{1}{r^4}\frac{\partial^4 w}{\partial\theta^4}\right)+$$
$$\frac{2D_{r\theta}}{h}\left(\frac{1}{r^4}\frac{\partial^2 w}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial^4 w}{\partial r^2\partial\theta^2}-\frac{1}{r^3}\frac{\partial^3 w}{\partial r\partial\theta^2}\right)=\frac{q}{h}+\frac{E_r}{1-\mu_r\mu_\theta}\frac{\partial^2 w}{\partial r^2}\left[\frac{\partial u}{\partial r}+\frac{\mu_\theta}{r}\frac{\partial v}{\partial\theta}+\right.$$
$$\left.\frac{\mu_\theta}{r}u+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2+\frac{\mu_\theta}{2r^2}\left(\frac{\partial w}{\partial\theta}\right)^2\right]+\frac{E_\theta}{1-\mu_r\mu_\theta}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\left[\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}+\right.$$
$$\left.\mu_r\frac{\partial u}{\partial r}+\frac{1}{2r^2}\left(\frac{\partial w}{\partial\theta}\right)^2+\frac{\mu_r}{2}\left(\frac{\partial w}{\partial r}\right)^2\right]+\frac{E}{1+\mu}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial w}{\partial\theta}\right)\times$$
$$\left(\frac{1}{r}\frac{\partial u}{\partial\theta}-\frac{v}{r}+\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial\theta}\right). \tag{18}$$

1.5 边界条件

$$r=0\text{ 时},\ u,v,w,\frac{\partial u}{\partial r},\frac{\partial v}{\partial r},\frac{\partial w}{\partial r}\text{ 有限} \tag{19}$$

1) 周边固支情况

$$r=a\text{ 时},\ w=u=v=\frac{\partial w}{\partial r}=0. \tag{20}$$

2) 周边简支情况

$$r=a\text{ 时},\ w=u=v=0,\ \frac{\partial^2 w}{\partial r^2}+\mu_\theta\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)=0. \tag{21}$$

1.6 无量纲基本方程

为了求解方便, 引入下列无量纲量

$$x=\frac{r}{a},\ y=\theta,\ V=\frac{a}{h^2}v,\ U=\frac{a}{h^2}u,\ W=\frac{w}{h},\ Q=\frac{a^4}{D_rh}q,$$

并且令 $D_{r\theta}=(1-\varepsilon_1)D_r$, $D_\theta=(1-\varepsilon_2)D_r$, 则 $E_\theta=(1-\varepsilon_2)E_r$, $\mu_\theta=(1-\varepsilon_2)\mu_r$, $E=\frac{(1-\varepsilon_1-\mu_\theta)(1+\mu)}{1-\mu_r\mu_\theta}E_r$, 可得圆板大挠度的无量纲基本方程为

$$\frac{\partial^2 U}{\partial x^2}+\frac{1}{x}\frac{\partial U}{\partial x}+\frac{(1-\varepsilon_2)\mu_r}{x}\frac{\partial^2 V}{\partial x\partial y}+\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2}+\frac{(1-\varepsilon_2)\mu_r}{x^2}\frac{\partial W}{\partial y}\frac{\partial^2 W}{\partial x\partial y}+\frac{1-\mu_r+\varepsilon_2\mu_r}{2x}\times$$
$$\left(\frac{\partial W}{\partial x}\right)^2-\frac{\mu_r-\varepsilon_2\mu_r}{2x^3}\left(\frac{\partial W}{\partial y}\right)^2-(1-\varepsilon_2)\left[\frac{U}{x^2}+\frac{1}{x^2}\frac{\partial V}{\partial y}+\frac{1}{x^3}\left(\frac{\partial W}{\partial y}\right)^2\right]+$$
$$\frac{1-\varepsilon_1-\mu_r+\varepsilon_2\mu_r}{2}\left(\frac{1}{x^2}\frac{\partial^2 U}{\partial y^2}-\frac{1}{x^2}\frac{\partial V}{\partial y}+\frac{1}{x}\frac{\partial^2 V}{\partial x\partial y}+\right.$$
$$\left.\frac{1}{x}\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial y^2}+\frac{1}{x}\frac{\partial^2 W}{\partial x\partial y}\frac{\partial W}{\partial y}\right)=0, \tag{22}$$

$$(1-\varepsilon_1-\mu_r+\varepsilon_2\mu_r)\left(\frac{\partial^2 V}{\partial x^2}+\frac{1}{x}\frac{\partial V}{\partial x}-\frac{V}{x^2}+\frac{1}{x}\frac{\partial^2 U}{\partial x\partial y}+\frac{1}{x^2}\frac{\partial U}{\partial y}+\frac{1}{x}\frac{\partial^2 W}{\partial x^2}\frac{\partial W}{\partial y}+\right.$$
$$\left.\frac{1}{x}\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x\partial y}+\frac{1}{x^2}\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}\right)+2(1-\varepsilon_2)\left(\frac{1}{x^2}\frac{\partial^2 V}{\partial y^2}+\frac{1}{x^2}\frac{\partial U}{\partial y}+\frac{\mu_r}{x}\frac{\partial^2 U}{\partial x\partial y}+\right.$$
$$\left.\frac{1}{x^3}\frac{\partial W}{\partial y}\frac{\partial^2 W}{\partial y^2}+\frac{\mu_r}{x}\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x\partial y}\right)=0, \tag{23}$$

$$\frac{\partial^4 W}{\partial x^4}+\frac{2}{x}\frac{\partial^3 W}{\partial x^3}+(1-\varepsilon_2)\left(\frac{2}{x^4}\frac{\partial^2 W}{\partial y^2}-\frac{1}{x^2}\frac{\partial^2 W}{\partial x^2}+\frac{1}{x^3}\frac{\partial W}{\partial x}+\frac{1}{x^4}\frac{\partial^4 W}{\partial y^4}\right)+$$
$$2(1-\varepsilon_1)\left(\frac{1}{x^4}\frac{\partial^2 W}{\partial y^2}+\frac{1}{x^2}\frac{\partial^4 W}{\partial x^2\partial y^2}-\frac{1}{x^3}\frac{\partial^3 W}{\partial x\partial y^2}\right)=$$

$$
Q+\frac{\partial^2 W}{\partial x^2}\left[\frac{\partial U}{\partial x}+\frac{(1-\varepsilon_2)\mu_r}{x}\frac{\partial V}{\partial y}+\right.
$$
$$
\left.\frac{(1-\varepsilon_2)\mu_r}{x}U+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2+\frac{(1-\varepsilon_2)\mu_r}{2x^2}\left(\frac{\partial W}{\partial y}\right)^2\right]+(1-\varepsilon_2)\left(\frac{1}{x}\frac{\partial W}{\partial x}+\frac{1}{x^2}\frac{\partial^2 W}{\partial y^2}\right)
$$
$$
\left[\frac{1}{x}\frac{\partial V}{\partial y}+\frac{U}{x}+\mu_r\frac{\partial U}{\partial x}+\frac{1}{2x^2}\left(\frac{\partial W}{\partial y}\right)^2+\frac{\mu_r}{2}\left(\frac{\partial W}{\partial x}\right)^2\right]+(1-\varepsilon_1-\mu_r+\varepsilon_2\mu_r)\times
$$
$$
\left(\frac{1}{x}\frac{\partial^2 W}{\partial x\partial y}-\frac{1}{r^2}\frac{\partial W}{\partial y}\right)\left(\frac{1}{x}\frac{\partial U}{\partial y}-\frac{V}{x}+\frac{\partial V}{\partial x}+\frac{1}{x}\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}\right). \tag{24}
$$

无量纲边界条件

$$
x=0\text{ 时}, U, V, W, \frac{\partial U}{\partial x}, \frac{\partial V}{\partial x}, \frac{\partial W}{\partial x}\text{ 有限} \tag{25}
$$

1) 周边固支情况

$$
x=1\text{ 时}, W=U=V=\frac{\partial W}{\partial x}=0, \tag{26}
$$

2) 周边简支情况

$$
x=1\text{ 时}, W=U=V=0, \frac{\partial^2 W}{\partial x^2}+(1-\varepsilon_2)\mu_r\left(\frac{1}{x^2}\frac{\partial^2 W}{\partial y^2}+\frac{1}{x}\frac{\partial W}{\partial x}\right)=0. \tag{27}
$$

2 问题的求解

我们利用 Fourier 级数求解此非线性方程组, 设

$$
U=\sum_{k=-\infty}^{+\infty}[U_{rk}(x)+\mathrm{i}U_{ik}(x)]\mathrm{e}^{\mathrm{i}ky}, \tag{28}
$$
$$
V=\sum_{k=-\infty}^{+\infty}[V_{rk}(x)+\mathrm{i}V_{ik}(x)]\mathrm{e}^{\mathrm{i}ky}, \tag{29}
$$
$$
W=\sum_{k=-\infty}^{+\infty}[W_{rk}(x)+\mathrm{i}W_{ik}(x)]\mathrm{e}^{\mathrm{i}ky}, \tag{30}
$$
$$
Q=\sum_{k=-\infty}^{+\infty}[q_{rk}(x)+\mathrm{i}q_{ik}(x)]\mathrm{e}^{\mathrm{i}ky}. \tag{31}
$$

将(28)~(31)代入无量纲基本方程及边界条件, 利用 $\mathrm{e}^{\mathrm{i}ky}$ 的正交性质和复函数的性质, 可将非线性偏微分方程组化为如下非线性常微分方程组

$$
U''_{rk}+\frac{1}{x}U'_{rk}-\frac{1}{x^2}U_{rk}=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(xU_{rk})\right]=F_{1rk}, \tag{32}
$$
$$
U''_{ik}+\frac{1}{x}U'_{ik}-\frac{1}{x^2}U_{ik}=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(xU_{ik})\right]=F_{1ik}, \tag{33}
$$
$$
V''_{rk}+\frac{1}{x}V'_{rk}-\frac{1}{x^2}V_{rk}=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(xV_{rk})\right]=F_{2rk}, \tag{34}
$$
$$
V''_{ik}+\frac{1}{x}V'_{ik}-\frac{1}{x^2}V_{ik}=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(xV_{ik})\right]=F_{2ik}, \tag{35}
$$
$$
W_{rk}^{(4)}+\frac{2}{x}W'''_{rk}-\frac{1+2k^2}{x^2}W''_{rk}+\frac{1+2k^2}{x^3}W'_{rk}-\frac{k^4-4k^2}{x^4}W_{rk}=
$$
$$
\left(x^{k-1}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^{2k-1}}\frac{\mathrm{d}}{\mathrm{d}x}x^k\right)^2W_{rk}=\left(\frac{1}{x^{k+1}}\frac{\mathrm{d}}{\mathrm{d}x}x^{2k+1}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^k}\right)^2W_{rk}=F_{3rk}, \tag{36}
$$
$$
W_{ik}^{(4)}+\frac{2}{x}W'''_{ik}-\frac{1+2k^2}{x^2}W''_{ik}+\frac{1+2k^2}{x^3}W'_{ik}-\frac{k^4-4k^2}{x^4}W_{ik}=
$$

$$\left(x^{k-1}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^{2k-1}}\frac{\mathrm{d}}{\mathrm{d}x}x^{k}\right)^{2}W_{ik}=\left(\frac{1}{x^{k+1}}\frac{\mathrm{d}}{\mathrm{d}x}x^{2k+1}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^{k}}\right)^{2}W_{ik}=F_{3ik}\cdot \tag{37}$$

边界条件

$x=0$ 时, U_{rk}, U_{ik}, V_{rk}, V_{ik}, W_{rk}, W_{ik}, U'_{rk}, U'_{ik}, V'_{rk}, V'_{ik}, W'_{rk}, W'_{ik} 有限 (38)

1) 周边固支情况

$x=1$ 时, $W_{rk}=W_{ik}=U_{rk}=U_{ik}=V_{rk}=V_{ik}=W'_{rk}=W'_{ik}=0\cdot$ (39)

2) 周边简支情况

$x=1$ 时, $W_{rk}=W_{ik}=U_{rk}=U_{ik}=V_{rk}=V_{ik}=0$, $W''_{rk}+(1-\varepsilon_2)\mu_r W'_{rk}/x=0$,

$W''_{ik}+(1-\varepsilon_2)\mu_r W'_{ik}/x=0$, (40)

其中

$$F_{1rk}=\frac{C_1}{2x^2}U_{rk}-\frac{C_2k}{2x^2}V_{ik}-\frac{C_3k}{2x}V'_{ik}+\frac{C_4}{2x^3}\sum_{n=-\infty}^{+\infty}[(k-n)n(W_{rn}W_{r,k-n}-W_{in}W_{i,k-n})]-$$
$$\frac{1}{2x^2}\sum_{n=-\infty}^{+\infty}\left\{[C_3(k-n)n-C_5n^2](W_{rn}W'_{r,k-n}-W_{in}W'_{i,k-n})\right\}-$$
$$\frac{C_6}{2x}\sum_{n=-\infty}^{+\infty}(W'_{rn}W'_{r,k-n}-W'_{in}W'_{i,k-n})-\sum_{n=-\infty}^{+\infty}(W'_{rn}W''_{r,k-n}-W'_{in}W''_{i,k-n}),$$

$$F_{1ik}=\frac{C_1}{2x^2}U_{ik}+\frac{C_2k}{2x^2}V_{rk}+\frac{C_3k}{2x}V'_{rk}+\frac{C_4}{x^3}\sum_{n=-\infty}^{+\infty}[(k-n)nW_{rn}W_{i,k-n}]-$$
$$\frac{1}{2x^2}\sum_{n=-\infty}^{+\infty}\left\{[C_3(k-n)n-C_5n^2](W_{rn}W'_{i,k-n}+W_{in}W'_{r,k-n})\right\}-$$
$$\frac{C_6}{x}\sum_{n=-\infty}^{+\infty}(W'_{rn}W'_{i,k-n})-\sum_{n=-\infty}^{+\infty}(W'_{rn}W''_{i,k-n}+W'_{in}W''_{r,k-n})$$

$$F_{2rk}=\frac{1}{\mu_r-1}\left\{-\frac{C_2k}{x^2}U_{ik}+\frac{C_3k}{x}U'_{ik}-\frac{C_7}{x^2}V_{rk}+\frac{C_8}{x}V'_{rk}+C_8V''_{rk}-\right.$$
$$\frac{2C_9}{x^3}\sum_{n=-\infty}^{+\infty}\left[(k-n)^2n(W_{rn}W_{i,k-n}+W_{in}W_{r,k-n})\right]-\frac{C_{10}}{x^2}\sum_{n=-\infty}^{+\infty}n(W_{rn}W'_{i,k-n}+W_{in}W'_{r,k-n})+$$
$$\left.\frac{C_3}{x}\sum_{n=-\infty}^{+\infty}n(W'_{rn}W'_{r,k-n}+W_{in}W'_{r,k-n})-\frac{C_{10}}{x}\sum_{n=-\infty}^{+\infty}n(W_{rn}W''_{i,k-n}+W_{in}W''_{r,k-n})\right\},$$

$$F_{2ik}=\frac{1}{\mu_r-1}\left\{\frac{C_2k}{x^2}U_{rk}-\frac{C_3k}{x}U'_{rk}-\frac{C_7}{x^2}V_{ik}+\frac{C_8}{x}V'_{ik}+C_8V''_{ik}+\right.$$
$$\frac{2C_9}{x^3}\sum_{n=-\infty}^{+\infty}[(k-n)^2n(W_{rn}W_{r,k-n}-W_{in}W_{i,k-n})]+\frac{C_{10}}{x^2}\sum_{n=-\infty}^{+\infty}n(W_{rn}W'_{r,k-n}-W_{in}W'_{i,k-n})-$$
$$\left.\frac{C_3}{x}\sum_{n=-\infty}^{+\infty}n(W'_{rn}W'_{r,k-n}-W'_{in}W'_{i,k-n})+\frac{C_{10}}{x}\sum_{n=-\infty}^{+\infty}n(W_{rn}W''_{r,k-n}-W_{in}W''_{i,k-n})\right\},$$

$$F_{3rk}=\frac{C_{11}k^2}{x^4}W_{rk}+\frac{C_{12}}{x^3}W'_{rk}-\frac{C_{12}}{x^2}W''_{rk}+\frac{12}{x^3}\sum_{n=-\infty}^{+\infty}\left\{[C_9(k-n)^2-C_{10}(k-n)n](U_{rn}W_{r,k-n}-\right.$$
$$\left.U_{in}W_{i,k-n})\right\}+\frac{12}{x^2}\sum_{n=-\infty}^{+\infty}\left\{[-C_9+C_{10}(k-n)n](U_{rn}W'_{r,k-n}-U_{in}W'_{i,k-n})\right\}+\frac{12\mu_rC_9}{x^2}\times$$
$$\sum_{n=-\infty}^{+\infty}[(k-n)^2(U'_{rn}W_{r,k-n}-U'_{in}W_{i,k-n})]-\frac{12\mu_rC_9}{x}\sum_{n=-\infty}^{+\infty}(U_{rn}W''_{r,k-n}-U_{in}W''_{i,k-n})-$$
$$\frac{12\mu_rC_9}{x}\sum_{n=-\infty}^{+\infty}(U'_{rn}W'_{r,k-n}-U'_{in}W'_{i,k-n})+12\sum_{n=-\infty}^{+\infty}(U'_{rn}W''_{r,k-n}-U'_{in}W''_{i,k-n})+$$

$$\frac{12}{x^3}\sum_{n=-\infty}^{+\infty}\Big\{[-C_9(k-n)^2n+C_{10}(k-n)](V_{rn}W_{i,k-n}+V_{in}W_{r,k-n})\Big\}+$$

$$\frac{12}{x^2}\sum_{n=-\infty}^{+\infty}\Big\{[C_9n-C_{10}(k-n)](V_{rn}W'_{i,k-n}+V_{in}W'_{r,k-n})\Big\}-$$

$$\frac{12C_{10}}{x^2}\sum_{n=-\infty}^{+\infty}[(k-n)(V'_{rn}W_{i,k-n}+V'_{in}W_{r,k-n})]+$$

$$\frac{12\mu_rC_9}{x}\sum_{n=-\infty}^{+\infty}[n(V_{rn}W''_{i,k-n}+V_{in}W''_{r,k-n})]+\frac{12C_{10}}{x}\sum_{n=-\infty}^{+\infty}[(k-n)(V'_{rn}W'_{i,k-n}+$$

$$V'_{in}W'_{r,k-n})]-\frac{6C_9}{x^4}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}[(k-m-n)mn^2(W_{rn}W_{rm}W_{r,k-m-n}-W_{rn}W_{im}W_{i,k-m-n}-$$

$$W_{in}W_{rm}W_{i,k-m-n}-W_{in}W_{im}W_{r,k-m-n})]-\frac{6C_{13}}{x^3}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}[mn(W_{rn}W_{rm}W'_{r,k-m-n}-$$

$$W_{rn}W_{im}W'_{i,k-m-n}-W_{in}W_{rm}W'_{i,k-m-n}-W_{in}W_{im}W'_{r,k-m-n})]+\frac{6}{x^2}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}[(2C_{10}mn+$$

$$\mu_rC_9n^2)(W_{rn}W'_{rm}W'_{r,k-m-n}-W_{rn}W'_{im}W'_{i,k-m-n}-W_{in}W'_{rm}W'_{i,k-m-n}-W_{in}W'_{im}W'_{r,k-m-n})]+$$

$$\frac{6\mu_rC_9}{x^2}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}[mn(W_{rn}W_{rm}W''_{r,k-m-n}-W_{rn}W_{im}W''_{i.k-m-n}-W_{in}W_{rm}W''_{i,k-m-n}-$$

$$W_{in}W_{im}W''_{r,k-m-n})]-\frac{6\mu_rC_9}{x}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}(W'_{rn}W'_{rm}W'_{r,k-m-n}-W'_{rn}W'_{im}W'_{i,k-m-n}-$$

$$W'_{in}W'_{rm}W'_{i,k-m-n}-W'_{in}W'_{im}W'_{r,k-m-n})+6\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}(W'_{rn}W'_{rm}W''_{r,k-m-n}-$$

$$W'_{rn}W'_{im}W''_{i,k-m-n}-W'_{in}W'_{rm}W''_{i,k-m-n}-W'_{in}W'_{im}W''_{r,k-m-n})+q_{rk},$$

$$F_{3ik}=\frac{C_{11}k^2}{x^4}W_{ik}+\frac{C_{12}}{x^3}W'_{ik}-\frac{C_{12}}{x^2}W''_{ik}+\frac{12}{x^3}\sum_{n=-\infty}^{+\infty}\Big\{[C_9(k-n)^2-C_{10}(k-n)n](U_{rn}W_{i,k-n}+$$

$$U_{in}W_{r,k-n})\Big\}+\frac{12}{x^2}\sum_{n=-\infty}^{+\infty}\Big\{[-C_9+C_{10}(k-n)n](U_{rn}W'_{i,k-n}+U_{in}W'_{r,k-n})\Big\}+\frac{12\mu_rC_9}{x^2}\times$$

$$\sum_{n=-\infty}^{+\infty}[(k-n)^2(U'_{rn}W_{i,k-n}+U'_{in}W_{r,k-n})]-\frac{12\mu_rC_9}{x}\sum_{n=-\infty}^{+\infty}(U_{rn}W''_{i,k-n}+U_{in}W''_{r,k-n})-$$

$$\frac{12\mu_rC_9}{x}\sum_{n=-\infty}^{+\infty}(U'_{rn}W'_{i,k-n}+U'_{in}W'_{r,k-n})+12\sum_{n=-\infty}^{+\infty}(U'_{rn}W''_{i,k-n}+U'_{in}W''_{r,k-n})+\frac{12}{x^3}\times$$

$$\sum_{n=-\infty}^{+\infty}\Big\{[C_9(k-n)^2n-C_{10}(k-n)](V_{rn}W_{ri,k-n}-V_{in}W_{i,k-n})\Big\}+\frac{12}{x^2}\sum_{n=-\infty}^{+\infty}\Big\{[-C_9n+$$

$$C_{10}(k-n)](V_{rn}W'_{r,k-n}-V_{in}W'_{i,k-n})\Big\}+\frac{12C_{10}}{x^2}\sum_{n=-\infty}^{+\infty}[(k-n)(V'_{rn}W_{r,k-n}-V'_{in}W_{i,k-n})]-$$

$$\frac{12\mu_rC_9}{x}\sum_{n=-\infty}^{+\infty}[n(V_{rn}W''_{r,k-n}-V_{in}W''_{i,k-n})]-\frac{12C_{10}}{x}\sum_{n=-\infty}^{+\infty}[(k-n)(V'_{rn}W'_{r,k-n}-$$

$$V'_{in}W'_{i,k-n})]-\frac{6C_9}{x^4}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}[(k-m-n)mn^2(W_{rn}W_{rm}W_{i,k-m-n}+W_{rn}W_{im}W_{r,k-m-n}+$$

$$W_{in}W_{rm}W_{r,k-m-n}-W_{in}W_{im}W_{i,k-m-n})]-\frac{6C_{13}}{x^3}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}[mn(W_{rn}W_{rm}W'_{i,k-m-n}+$$

$$W_{rn}W_{im}W'_{r,k-m-n}+W_{in}W_{rm}W'_{r,k-m-n}-W_{in}W_{im}W'_{i,k-m-n})]+\frac{6}{x^2}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}[(2C_{10}mm+$$

$$\mu_r C_9 n^2)(W_{rn}W'_{rm}W'_{i,k-m-n} + W_{rn}W'_{im}W'_{r,k-m-n} + W_{in}W'_{rm}W'_{r,k-m-n} - W_{in}W'_{im}W'_{i,k-m-n})] +$$

$$\frac{6\mu_r C_9}{x^2}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}[mn(W_{rn}W_{rm}W''_{i,k-m-n} + W_{rn}W_{im}W''_{r,k-m-n} + W_{in}W_{rm}W''_{r,k-m-n} -$$

$$W_{in}W_{im}W''_{i,k-m-n})] - \frac{6\mu_r C_9}{x}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}(W'_{rn}W'_{rm}W'_{i,k-m-n} + W'_{rn}W'_{im}W'_{r,k-m-n} +$$

$$W'_{in}W'_{rm}W'_{r,k-m-n} - W'_{in}W'_{im}W'_{i,k-m-n}) + 6\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}(W'_{rn}W'_{rm}W''_{i,k-m-n} +$$

$$W'_{rn}W'_{im}W''_{r,k-m-n} + W'_{in}W'_{rm}W''_{r,k-m-n} - W'_{in}W'_{im}W''_{i,k-m-n}) + q_{ik},$$

并且

$$C_1 = (1-\mu_r)k^2 + \mu_r\varepsilon_2 k^2 - 2\varepsilon_2 - \varepsilon_1 k^2, \quad C_2 = \mu_r\varepsilon_2 - \mu_r - \varepsilon_1 - 2\varepsilon_2 + 3,$$

$$C_3 = \mu_r\varepsilon_2 - \mu_r + \varepsilon_1 - 1, \quad C_4 = \mu_r\varepsilon_2 - \mu_r + \varepsilon_2 - 1,$$

$$C_5 = \mu_r\varepsilon_2 - \mu_r + \varepsilon_1 + 1, \quad C_6 = \mu_r\varepsilon_2 - \mu_r + 1,$$

$$C_7 = \mu_r\varepsilon_2 + 2k^2 - 2\varepsilon_2 k^2 - \varepsilon_1, \quad C_8 = \mu_r\varepsilon_2 - \varepsilon_1,$$

$$C_9 = \varepsilon_2 - 1, \quad C_{10} = \mu_r\varepsilon_2 - \mu_r - \varepsilon_1 + 1,$$

$$C_{11} = \varepsilon_2 k^2 - 2\varepsilon_1 - 2\varepsilon_2, \quad C_{12} = \varepsilon_2 + 2\varepsilon_1 k^2,$$

$$C_{13} = 2\mu_r\varepsilon_2 - 2\mu_r - \varepsilon_2 - 2\varepsilon_1 + 3.$$

对此非线性常微分方程组可采用迭代法求解, 求解步骤如下:

1) 先将无量纲载荷 Q 写成Fourier展式, 求得 $q_{rk}, q_{ik}(k = 0, \pm 1, \pm 2, \ldots)$, 并令

$$U^{(0)} = V^{(0)} = W^{(0)} = 0,$$

则

$$U_{rk}^{(0)} = U_{ik}^{(0)} = V_{rk}^{(0)} = V_{ik}^{(0)} = W_{rk}^{(0)} = W_{ik}^{(0)} = 0;$$

2) 求得 $F_{1rk}^{(0)} = F_{1ik}^{(0)} = F_{2rk}^{(0)} = F_{2ik}^{(0)} = 0, \quad F_{3rk}^{(0)} = q_{rk}, F_{3ik}^{(0)} = q_{ik}$, 代入非线性常微分方程组并考虑边界条件, 可求解得 $U_{rk}^{(1)}, U_{ik}^{(1)}, V_{rk}^{(1)}, V_{ik}^{(1)}, W_{rk}^{(1)}, W_{ik}^{(1)}$;

3) 求得本文问题的一次近似解析解, 即小挠度解

$$U^{(1)} = \sum_{k=-\infty}^{+\infty}[U_{rk}^{(1)} + iU_{ik}^{(1)}]e^{iky},$$

$$V^{(1)} = \sum_{k=-\infty}^{+\infty}[V_{rk}^{(1)} + iV_{ik}^{(1)}]e^{iky},$$

$$W^{(1)} = \sum_{k=-\infty}^{+\infty}[W_{rk}^{(1)} + iW_{ik}^{(1)}]e^{iky}.$$

4) 由 $U_{rk}^{(1)}, U_{ik}^{(1)}, V_{rk}^{(1)}, V_{ik}^{(1)}, W_{rk}^{(1)}, W_{ik}^{(1)}, q_{rk}, q_{ik}(k = 0, \pm 1, \pm 2, \ldots)$ 求得 $F_{1rk}^{(1)}, F_{1ik}^{(1)}, F_{2rk}^{(1)}, F_{2ik}^{(1)}, F_{3rk}^{(1)}, F_{3ik}^{(1)}$, 代入非线性常微分方程组, 得本文问题的二次近似解析解;

5) 重复步骤4) 可求得本问题更高次近似的解析解.

3 讨 论

本文采用的修正迭代法是基于叶开沅、刘人怀两位教授早年提出的修正迭代法[9] ——它相当于以最大挠度为小参数的摄动法, 后又在多参数摄动法问题中提出的多参数修正迭代法[10]. 它的程序简明、计算量小. 关于本文问题的挠度小参数只要选择板内挠度不为零的点就可.

我们采用的求解方法推广到正交各向异性多层板壳非线性大变形问题中, 求其解析解都是可行的•

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Unsymmetrical Large Deformation Problem of Orthotropic Plates

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Abstract: Based upon the theory of anisotropic plates, the unsymmetrical large deformation equations of orthotropic circular plates were derived. By using Fourier series, the partial differential equations of this problem can be transformed into sets of non_linear differential equations. And the procedure to solve the problem using the iterative method is given.

Key words: orthotropic; circular thin plates; unsymmetrical; large deformation; iterative method