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Klein_Gordon_Schrödinger 方程组的精确孤立波解*

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(李继彬推荐)

摘要: 利用齐次平衡原则导出 Klein_Gordon_Schrödinger 方程组的精确孤立波解· 该解在形式上比文献中纯理论的存在性证明的结果更一般, 文献中的解的形式是该结果的特殊情形·

关 键 词: Klein_Gordon_Schrödinger 方程组; 齐次平衡原则; 精确孤立波解
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引 言

本文考虑耦合 Klein_Gordon_Schrödinger(KGS) 方程组:

$$\begin{cases} \mathrm{i}\psi_t + \dfrac{1}{2}\Delta\psi = -\varphi\psi, & (1)\\ \varphi_{tt} - \Delta\varphi + m^2\varphi = |\psi|^2 \cdot & (2)\end{cases}$$

(1) ~ (2) 是核子场与介子场相互作用的经典模型[1], 其中 ψ 是标量复核子场, φ 是实介子场, Δ 是 Laplace 算子· 文献[2] 定性地证明了方程组(1) ~ (2) 的如下形式的定态解

$$(\psi(t,x),\varphi(t,x)) = (\mathrm{e}^{\mathrm{i}\omega t}u(x), v(x)) \qquad (x \in \mathbf{R}^3,\ \omega \in \mathbf{R}) \tag{3}$$

的存在性, 其中 u, v 满足:

$$-\Delta u + 2\omega u + 2uv = 0,$$
$$-\Delta v + m^2 v - u^2 = 0\cdot$$

本文的目的是用齐次平衡原则[3]~[5], 求出(1) ~ (2) 的精确孤立波解, 构造性地表明方程组(1) ~ (2) 确实存在比(3) 中表示的形式更为一般的解· (3) 中解的形式是我们所得结果的特殊情形· 本文的安排如下: 首先, 在 1 节讨论方程组(1) ~ (2) 的(1+1) 维情形的精确孤立波解; 然后, 在 2 节, 将 1 节的结果推广到 $(1+n)$ 维情形·

1 (1+1) 维情形

(1+1) 维的 KGS 方程组为

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$$\begin{cases} \mathrm{i}\phi_t + \dfrac{1}{2}\phi_{xx} = -\phi\varphi, & (4) \\ \varphi_{tt} - \varphi_{xx} + m^2\varphi = |\phi|^2. & (5) \end{cases}$$

既然 ϕ 是复的, 我们可设

$$\phi = \mathrm{e}^{\mathrm{i}\eta}u(x,t), \quad \eta = \alpha x + \beta t, \quad \alpha, \beta \text{ 待定} \tag{6}$$

将(6) 代入(4) ~ (5), 消去 $\mathrm{e}^{\mathrm{i}\eta}$ 后得:

$$u_t + \alpha u_x = 0, \tag{7}$$

$$u_{xx} + 2u\varphi - (\alpha^2 + 2\beta)u = 0, \tag{8}$$

$$\varphi_{tt} - \varphi_{xx} + m^2\varphi - u^2 = 0. \tag{9}$$

为解方程组(7) ~ (9), 根据齐次平衡原则[3]~[5], 可设(7) ~ (9) 的解具形式:

$$u = f''w_x^2 + f'w_{xx}, \tag{10}$$

$$\varphi = g''w_x^2 + g''w_{xx}, \tag{11}$$

其中函数 $f(w)$, $g(w)$ 待定, 而

$$w = 1 + \exp\xi, \quad \xi = kx + \omega t, \quad k, \omega \text{ 待定} \tag{12}$$

将(10) 代入(7) 的左端得:

$$u_t + au_x = \left[f'''w_x^2 + f''\left(2w_x\frac{\partial}{\partial x} + w_{xx}\right) + f'\frac{\partial^2}{\partial x^2}\right](w_t + \alpha w_x), \tag{13}$$

由此, 为使(10) 满足(7), 只需 $w(x,t)$ 满足:

$$w_t + \alpha w_x = 0. \tag{14}$$

将(12) 代入(14), 可确定出 $\omega = -k\alpha$.

于是, (12) 变为:

$$w = 1 + \exp\xi, \quad \xi = k(x - \alpha t). \tag{15}$$

由(10) 及(11) 并利用(15) 可算出

$$u = f''k^2\exp[2\xi] + f'k^2\exp\xi, \tag{16}$$

$$u_{xx} = k^4(f^{(4)}\exp[4\xi] + 6f'''\exp[3\xi] + 7f''\exp[2\xi] + f'\exp\xi), \tag{17}$$

$$u\varphi = f''g''k^4\exp[4\xi] + (f''g' + f'g'')k^4\exp[3\xi] + f'g'k^4\exp[2\xi], \tag{18}$$

$$\varphi = k^2(g''\exp[2\xi] + g'\exp\xi), \tag{19}$$

$$u^2 = k^4(f''^2\exp[4\xi] + 2f''f'\exp[3\xi] + f'^2\exp[2\xi]), \tag{20}$$

$$\varphi_{tt} = k^4\alpha^2(g^{(4)}\exp[4\xi] + 6g'''\exp[3\xi] + 7g''\exp[2\xi] + g'\exp\xi), \tag{21}$$

$$\varphi_{xx} = k^4(g^{(4)}\exp[4\xi] + 6g'''\exp[3\xi] + 7g''\exp[2\xi] + g'\exp\xi). \tag{22}$$

将(6) ~ (8) 代入(8), 合并 $\exp[4\xi]$, $\exp[3\xi]$, $\exp[2\xi]$ 及 $\exp\xi$ 的同类项, 并令它们的系数为零(因为它们线性无关) 而得 f, g 满足的 ODE 组及 k, α, β 满足的关系:

$$\begin{cases} f^{(4)} + 2f''g'' = 0, & (23) \\ 6f''' + 2(f''g' + f'g'') = 0, & (24) \\ (7f'' + 2f'g')k^2 - (\alpha^2 + 2\beta)f'' = 0, & (25) \\ k^2 = \alpha^2 + 2\beta. & (26) \end{cases}$$

同样, 将(19) ~ (22) 代入(9), 合并 $\exp[4\xi]$, $\exp[3\xi]$, $\exp[2\xi]$ 及 $\exp\xi$ 的同类项, 并令它们的系数为零, 又可得 f, g 满足的 ODE 组及 k, α 满足的关系:

$$\begin{cases}(1-\alpha^2)g^{(4)}+f''^2=0, & (27)\\ 6(1-\alpha^2)g'''+2f'f''=0, & (28)\\ 7(1-\alpha^2)k^2g''+k^2f'^2-m^2g''=0, & (29)\\ g'(1-\alpha^2)k^2-m^2g'=0 \text{ 或 } k^2=\dfrac{m^2}{1-\alpha^2}\quad(0\leqslant\alpha^2<1)\cdot & (30)\end{cases}$$

将(23)及(27)联立,可解得:

$$f(w)=\pm3\sqrt{2(1-\alpha^2)}\ln w, \tag{31}$$

$$g(w)=3\ln w, \tag{32}$$

其中 $0\leqslant\alpha^2<1$, w 由(15)表示•

经验证,(31)及(32)确实满足(24)及(28);将(31)~(32)代入(25)及(29),分别得结果(25)及(30),由此知 α, β 满足条件:

$$k^2=\frac{m^2}{1-\alpha^2}=\alpha^2+2\beta \text{ 或}$$

$$\beta=\frac{1}{2}\left(\frac{m^2}{1-\alpha^2}-\alpha^2\right)\quad(0\leqslant\alpha^2<1)\cdot \tag{33}$$

将(31)及(32)分别代入(10)及(11),得方程组(7)~(9)的精确行波解如下:

$$\begin{cases}u=\pm\dfrac{3}{4}\sqrt{2(1-\alpha^2)}k^2\operatorname{sech}^2\dfrac{1}{2}k(x-\alpha t), & (34)\\ \varphi=\dfrac{3}{4}k^2\operatorname{sech}^2\dfrac{1}{2}k(x-\alpha t), & (35)\end{cases}$$

其中 $$k^2=\frac{m^2}{1-\alpha^2},$$

$$\beta=\frac{1}{2}\left(\frac{m^2}{1-\alpha^2}-\alpha^2\right)\quad(0\leqslant\alpha^2<1), \tag{36}$$

将(34)及(36)代入(6),可得方程组(1)~(2)的精确孤立波解:

$$\begin{cases}\psi=\pm\dfrac{3\sqrt{2}}{4}\dfrac{m^2}{\sqrt{1-\alpha^2}}\operatorname{sech}^2\dfrac{1}{2}\dfrac{m}{\sqrt{1-\alpha^2}}(x-\alpha t)\cdot\exp\left[i\left(\alpha x+\dfrac{1}{2}\left(\dfrac{m^2}{1-\alpha^2}-\alpha^2\right)t\right)\right], & (37)\\ \varphi=\dfrac{3}{4}\dfrac{m^2}{1-\alpha^2}\operatorname{sech}^2\dfrac{1}{2}\dfrac{m}{\sqrt{1-\alpha^2}}(x-\alpha t)\quad(0\leqslant\alpha^2<1)\cdot & (38)\end{cases}$$

特别地,当 $\alpha=0$ 时得

$$\psi=\pm\frac{3\sqrt{2}}{4}m^2\operatorname{sech}^2 mx\cdot\exp\left[i\frac{m^2}{2}t\right], \tag{37'}$$

$$\varphi=\frac{3}{4}m^2\operatorname{sech}^2\frac{1}{2}mx\cdot \tag{38'}$$

前边提到,文献[2]已定性地证明了有形如(37)及(38)的定态解存在,但那里并没有具体给出这种解•

2 $(1+n)$ 维情形

本段将(1+1)维情形的结果推广到 $(1+n)$ 维情形•

$(1+n)$ 维情形的 KGS 方程组为:

$$\begin{cases}i\psi_t+\dfrac{1}{2}\Delta\psi=-\varphi\psi, & (39)\\ \varphi_{tt}-\Delta\varphi+m^2\varphi=|\psi|^2, & (40)\end{cases}$$

其中　$\Delta = \frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_2^2} + \cdots + \frac{\partial}{\partial x_n^2}$•

由于 Laplace 算子关于空间变量的对称性, 仿照(1+ 1) 维情形, 我们可设

$$\phi = e^{i\eta} u(\boldsymbol{x}, t), \tag{41}$$

这里　$\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$,　$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$,

$$\eta = \alpha \cdot \boldsymbol{x} + \beta t = \sum_{i=1}^{n} \alpha_i x_i + \beta t \cdot$$

将(41) 代入(39) ~ (40), 消去 $e^{i\eta}$ 后, 得,

$$u_t + \sum_{i=1}^{n} \alpha_i u_{x_i} = 0, \tag{42}$$

$$\Delta u + 2u\varphi - \left(\sum_{i=1}^{n} \alpha_i^2 + 2\beta\right) u = 0, \tag{43}$$

$$\varphi_{tt} - \Delta\varphi + m^2\varphi - u^2 = 0 \cdot \tag{44}$$

为解(42) ~ (44), 仿照(1+ 1) 维情形, 可设

$$u(\boldsymbol{x}, t) = f''\left(\sum_{=1}^{n} w_{x_i}^2\right) + f'\Delta w \tag{45}$$

$$\varphi(\boldsymbol{x}, t) = g''\left(\sum_{i=1}^{n} w_{x_i}^2\right) + g'\Delta w \tag{46}$$

其中　$w = 1 + \exp\xi,\ \xi = \sum_{i=1}^{n} k_i x_i + \omega t,$　(47)

k_i 不全为零, $f(w)$, $g(w)$, k_i 及 ω 待定•

利用(47), (45) 及(46) 可算出

$$u = f''\left(\sum_{i=1}^{n} k_i^2\right) e^{2\xi} + f'\left(\sum_{i=1}^{n} k_i^2\right) e^{\xi}, \tag{48}$$

$$u_t = \omega\left(\sum_{i=1}^{n} k_i^2\right) f''' e^{3\xi} + 3\omega\left(\sum_{i=1}^{n} k_i^2\right) f'' e^{2\xi} + \omega\left(\sum_{i=1}^{n} k_i^2\right) f' e^{\xi}, \tag{49}$$

$$u_{x_j} = k_j\left(\sum_{i=1}^{n} k_i^2\right)(f''' e^{3\xi} + 3f'' e^{2\xi} + f' e^{\xi}) \quad (j = 1, 2, \cdots, n) \cdot \tag{50}$$

将(49) ~ (50) 代入(42) 得,

$$u_t + \sum_{i=1}^{n} \alpha_i u_{x_i} = \left(\sum_{i=1}^{n} k_i^2\right)(f''' e^{3\xi} + 3f'' e^{2\xi} + f' e^{\xi}) \cdot \left(\omega + \sum_{i=1}^{n} k_i \alpha_i\right) = 0 \cdot$$

从而　$\omega = -\sum_{i=1}^{n} k_i \alpha_i \cdot$　(51)

利用(51), (47) 可写为:

$$w = 1 + \exp\xi = 1 + \exp\sum_{i=1}^{n} k_i(x_i - \alpha_i t) \cdot \tag{52}$$

由(45), 及(46) 并利用(52) 可算得

$$u_{x_j x_j} = k_j^2\left(\sum_{i=1}^{n} k_i^2\right)(f^{(4)} e^{4\xi} + 6f''' e^{3\xi} + 7f'' e^{2\xi} + f' e^{\xi}), \tag{53}$$

$$\Delta u = \left(\sum_{i=1}^{n} k_i^2\right)(f^{(4)} e^{4\xi} + 6f''' e^{3\xi} + 7f'' e^{2\xi} + f' e^{\xi}), \tag{54}$$

$$\varphi=\left(\sum_{i=1}^{n}k_i^2\right)(g''e^{2\xi}+g'e^{\xi}),\tag{55}$$

$$u\varphi=\left(\sum_{i=1}^{n}k_i^2\right)^2[f''g''e^{4\xi}+(f''g'+f'g'')e^{3\xi}+f'g'e^{2\xi}],\tag{56}$$

$$\varphi_{tt}=\left(\sum_{i=1}^{n}k_i^2\right)\left(\sum_{i=1}^{n}k_i\alpha_i\right)^2(g^{(4)}e^{4\xi}+6g'''e^{3\xi}+7g''e^{2\xi}+g'e^{\xi}),\tag{57}$$

$$\Delta\varphi=\left(\sum_{i=1}^{n}k_i^2\right)^2(g^{(4)}e^{4\xi}+6g'''e^{3\xi}+7g''e^{2\xi}+g'e^{\xi}),\tag{58}$$

$$u^2=\left(\sum_{i=1}^{n}k_i^2\right)^2(f''^2e^{4\xi}+2f''f'e^{3\xi}+f'^2e^{2\xi})\cdot\tag{59}$$

类似于(1+1)维情形,将(54)~(59)分别代入(43)及(44),合并 $\exp[4\xi]$, $\exp[3\xi]$, $\exp[2\xi]$ 及 $\exp\xi$ 的同类项,并令它们的系数为零,又可得 f, g 满足的 ODE 组及 k, α, β 所满足的关系:

$$\begin{cases}f^{(4)}+2f''g''=0, & (60)\\ 6f'''+2(f''g'+f'g'')=0, & (61)\\ \left[7\sum_{i=1}^{n}k_i^2-\left(\sum_{i=1}^{n}\alpha_i^2+2\beta\right)\right]f''+2\sum_{i=1}^{n}k_i^2f'g'=0, & (62)\\ \sum_{i=1}^{n}k_i^2-\left(\sum_{i=1}^{n}\alpha_i^2+2\beta\right)=0\cdot & (63)\end{cases}$$

$$\begin{cases}\left[\left(\sum_{i=1}^{n}k_i\alpha_i\right)^2-\sum_{i=1}^{n}k_i^2\right]g^{(4)}-\sum_{i=1}^{n}k_i^2f''^2=0, & (64)\\ 6\left[\left(\sum_{i=1}^{n}k_i\alpha_i\right)^2-\sum_{i=1}^{n}k_i^2\right]g'''-2\left(\sum_{i=1}^{n}k_i^2\right)f''f'=0, & (65)\\ \left[7\left(\sum_{i=1}^{n}k_i\alpha_i\right)^2-7\sum_{i=1}^{n}k_i^2+m^2\right]g''-\sum_{i=1}^{n}k_i^2f'^2=0, & (66)\\ \left(\sum_{i=1}^{n}k_i\alpha_i\right)^2-\sum_{i=1}^{n}k_i^2+m^2=0\cdot & (67)\end{cases}$$

现在解(60)~(67)·

首先,联立(60)及(64)可解得:

$$\begin{cases}f=\pm3\sqrt{2}\cdot\sqrt{1-\dfrac{\left(\sum_{i=1}^{n}k_i\alpha_i\right)^2}{\sum_{i=1}^{n}k_i^2}}\ln w, & (68)\\ g=3\ln w, & (69)\end{cases}$$

其中 w 由(52)表示·

其次,经验证,(68)~(69)确实满足(61)及(65)· 再将(68)~(69)代入(62)及(66),得到与(63)及(67)一致的结果;由此得到 k_i, α_i, β 所满足的关系为:

$$\sum_{i=1}^{n}k_i^2=\sum_{i=1}^{n}\alpha_i^2+2\beta=m^2+\left(\sum_{i=1}^{n}k_i\alpha_i\right)^2,$$

或 $$\beta=\frac{1}{2}\sum_{i=1}^{n}(k_i^2-\alpha_i^2)\cdot\tag{70}$$

将(68),(69)及(70)代入(45)及(46),就可得到方程组(42)及(44)的精确行波解:

$$\begin{cases} u = \pm\frac{3\sqrt{2}}{4}m\cdot\sqrt{\sum_{i=1}^{n}k_i^2\operatorname{sech}^2\frac{1}{2}\sum_{i=1}^{n}k_i(x_i-\alpha_i t)}, & (71)\\ \varphi = \frac{3}{4}\sum_{i=1}^{n}k_i^2\operatorname{sech}^2\frac{1}{2}\sum_{i=1}^{n}k_i(x_i-\alpha_i t), & (72)\end{cases}$$

其中 k_i, α_i 满足:

$$\sum_{i=1}^{n}k_i^2 = m^2 + \left(\sum_{i=1}^{n}k_i\alpha_i\right)^2.$$

将(71)及(72)代入(41),得到 $(1+n)$ 维KGS方程组(39)~(40)的精确孤立波解:

$$\begin{cases} \psi(x_1, x_2, \cdots, x_n, t) = \pm\frac{3\sqrt{2}}{4}m\sqrt{\sum_{i=1}^{n}k_i^2}\cdot\operatorname{sech}^2\frac{1}{2}\sum_{i=1}^{n}k_i(x_i-\alpha_i t)\times \\ \qquad \exp\left[\mathrm{i}\left(\sum_{i=1}^{n}\alpha_i x_i + \frac{1}{2}\sum_{i=1}^{n}(k_i^2-\alpha_i^2)t\right)\right], & (73)\\ \varphi(x_1, x_2, \cdots, x_n, t) = \frac{3}{4}\left(\sum_{i=1}^{n}k_i^2\right)\cdot\operatorname{sech}^2\frac{1}{2}\sum_{i=1}^{n}k_i(x_i-\alpha_i t). & (74)\end{cases}$$

其中 k_i, α_i, β 满足(70).

特别的, 当 $n=1$ 记 $\alpha_1=\alpha$ 时, 有解:

$$\begin{cases} \psi(x, t) = \pm\frac{3\sqrt{2}}{4}m^2\frac{1}{\sqrt{1-\alpha^2}}\cdot\operatorname{sech}^2\frac{1}{2}\frac{m}{\sqrt{1-\alpha^2}}(x-\alpha t)\times \\ \qquad \exp\left[\mathrm{i}\left(\alpha x + \frac{1}{2}\left(\frac{m^2}{1-\alpha^2}-\alpha^2\right)t\right)\right], & (73)\\ \varphi = \frac{3}{4}\frac{m^2}{1-\alpha^2}\operatorname{sech}^2\frac{1}{2}\frac{m}{\sqrt{1-\alpha^2}}(x-\alpha t). & (74)\end{cases}$$

这正是前面我们求出的(1+1)维情形下的精确孤立波解.

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The Exact Solitary Wave Solution for the Klein_Gordon_Schr└dinger Equations

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Abstract: The solitary wave solutions for the Klein_Gordon_Schr└dinger Equations were obtained by using the homogeneous balance principle. The form of the solutions is more generalized than the result that has been proved by pure theoretical and qualitative method in literature; namely, the form of solutions in literature is a particular case of result of the present paper.

Key words: Klein_Gordon_Schr└dinger equations; homogeneous balance principle; exact solitary wave solution