应用数学和力学编委会编
重 庆 出 版 社 出 版

文章编号: 1000_0887(2001) 06_0651_10

快变海底和自由表面流共振生成的弱非线性 Stokes 波*

黄 虎[1,2], 周锡[1]

(1. 天津大学 建筑工程学院, 天津 300072; 2. 上海大学 上海市应用数学和力学研究所, 上海 200072)

(惠昌年推荐)

摘要: 运用 WKBJ 型摄动逼近法, 对环境流同沙纹海底共振产生的自由表面水波的非线性效应进行了研究• 沙纹海底由缓变平均水深部分和快变海底部分叠加构成• 根据对快变海底波长的不同选取, 可以相应地激发环境流同非平整海底的同步共振、超谐波共振和次谐波共振, 由此产生自由表面波运动• 对次谐波共振进行了详细考察• 对于定常流自治动力系统, 对可能出现的非线性各种稳态及其稳定性进行了探讨• 假如环境流具有一个小振动分量, 动力系统成为非自治的, 则将发生混沌现象•

关 键 词: 非线性共振; 弱非线性 Stokes 波; 自由表面流; 沙纹海底; 动力系统
中图分类号: O353.2 文献标识码: A

引 言

表面波、环境流与非平整海底的相互作用对于海岸工程师和沉积学家具有根本的意义• 例如, 环境流同由潮流或波浪产生的海底地形(如近岸海床上的沙丘(dunes)和沙纹(ripples))共振而激发的表面波将会改变邻近海岸区域的波候• 目前, 对于波浪(或者环境流)与预先设定的非平整海底地形的相互作用, 人们已进行了大量的研究[1~4]• 但是对波_流_海底相互作用的考察, 尤其是探讨它们之间非线性共振的机制, 却为数不多[5~6]• 基于此, 本文从动力系统的观点出发, 对波_流_海底的非线性共振机制进行探讨, 这样也许能够解释沙纹海底地形的演变过程及其基本特征•

1 控制方程和 WKBJ 摄动方法

考虑自由表面流在水深为 $h(x)$ 的沙纹非平整海底上流动, x 表示 Cartesian 水平坐标• 垂直坐标 z 竖直向上, $z = 0$ 表示无扰动的自由表面, $z = \zeta(x, t)$ 表示自由表面高度• 假设水体无粘, 水流无旋, 则可引入总势 $\Phi(x, z, t)$ 和扰动势 $\phi(x, z, t)$• 假设水深由慢变分量 h_0 和快变分量 h_1 组成:

$$h = h_0(x) + h_1(x), \tag{1}$$

* 收稿日期: 1999_09_03; 修订日期: 2001_01_15
基金项目: 国家教委博士点基金资助项目(940562); 国家高性能计算基金资助项目(96103)
作者简介: 黄虎(1964—), 男, 新疆石河子人, 副教授, 博士.

用 Λ 表示$h_0(x)$ 的水平长度尺度, $\lambda = 2\pi/k$ 表示载波的典型波长• 慢变分量 h_0 和快变分量 h_1 遵守下列量级关系式:

$$O(\delta) = O\left(\frac{1}{kh_0}\frac{\partial h_0}{\partial x}\right) = O\left(\frac{\lambda}{\Lambda}\right) \ll 1, \tag{2}$$

$$O(kh_1) \approx O(\delta^3), \quad O\left(\frac{h_1}{\Lambda}\right) = O(\delta^4)• \tag{3}$$

引入下列慢变量

$$x_1 = \delta x, \quad t_1 = \delta t, \tag{4}$$

则水深可重新写为

$$h = h_0(x_1) + \delta^3 h_1(x) = h_0(x_1) + \delta^2 h_1(x_1)• \tag{5}$$

当不存在沙纹时, 以 U 表示均匀来流速度, 则

$$\Phi(x, z, t) = Ux + \phi(x, z, t)• \tag{6}$$

将(6) 代入 Laplace 方程, 可得

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (-h \leqslant z \leqslant \zeta)• \tag{7}$$

假设自由表面压力为零, Bernoulli 方程可写为

$$g\zeta + \frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] = 0 \qquad (z = \zeta), \tag{8}$$

运动边界条件可表示为

$$\left[\frac{\partial}{\partial t} + \left(U + \frac{\partial\phi}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}\right](z - \zeta) = 0 \qquad (z = \zeta)• \tag{9}$$

从(8) 和(9) 中消去自由表面位移 ζ, 可得

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} + \frac{\partial}{\partial t}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + \frac{1}{2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\phi \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + U^2\frac{\partial^2\phi}{\partial x^2} + 2U\frac{\partial^2\phi}{\partial x\partial t} + U\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] = 0 \qquad (z = \zeta), \tag{10}$$

沿着海底的无通量边界条件为

$$\frac{\partial\phi}{\partial z} + \frac{\partial h}{\partial x}\frac{\partial\phi}{\partial x} = 0 \qquad (z = -h)• \tag{11}$$

假设非线性参数 ε 与调制参数 δ 为同一量级• 将非线性和调制速率按照 WKBJ 型摄动法展开:

$$\phi = \sum_{n=1}\varepsilon^n\sum_{m=-n}^{m=+n}\phi^{(n,m)}E_0^m, \quad \zeta = \sum_{n=1}\varepsilon^n\sum_{m=-n}^{m=+n}\zeta^{(n,m)}E_0^m, \tag{12}$$

其中

$$E_0 = \exp[\mathrm{i}\chi_0(x_1)/\varepsilon], \quad \chi_0 = \int k_0\,\mathrm{d}x_1, \tag{13}$$

$$\phi^{(n,m)} = \phi^{(n,m)}(x_1, z, t_1, x_2, t_2, \cdots), \quad \zeta^{(n,m)} = \zeta^{(n,m)}(x_1, t_1, x_2, t_2, \cdots)• \tag{14}$$

允许入射流速度 U 在共振点处存在微小的解调性

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots• \tag{15}$$

将(8) 和(10) 在 $z = 0$ 处按 Taylor 级数展开; (11) 式在 $z = -h_0$ 处按 Taylor 级数展开, 最后将 (12) 中的 ϕ 表达式代入 Laplace 方程(7)• 由以上则得关于每一个指标(n, m) 的一组常微分方程:

$$\left(\frac{\partial^2}{\partial z^2} - m^2 k_0^2\right) \phi^{(n,m)} = R^{(n,m)} \qquad (0 \geqslant z \geqslant -h_0), \tag{16}$$

$$g\frac{\partial \phi^{(n,m)}}{\partial z} - m^2 (k_0 U_0)^2 \phi^{(n,m)} = G^{(n,m)} \qquad (z = 0), \tag{17}$$

$$\frac{\partial \phi^{(n,m)}}{\partial z} = F^{(n,m)} \qquad (z = -h_0), \tag{18}$$

$$\zeta^{(n,m)} = -\frac{1}{g}[\mathrm{i} m U_0 k_0 \phi^{(n,m)} + H^{(n,m)}] \qquad (z = 0), \tag{19}$$

其中强迫函数 $R^{(n,m)}$、$G^{(n,m)}$、$F^{(n,m)}$ 和 $H^{(n,m)}$ 的显示表达形式在下一节需要时给出•

现在考虑在 $O(\varepsilon)$ 上能够产生与 $\varepsilon\exp(\pm \mathrm{i}\chi_0/\varepsilon)$ 成正比例关系的自由表面波• 我们可以做类似的证明[5]: 如果快变水深分量与下列任何一种谐波量成正比例, 则将发生由非线性引起的共振:

$$\exp(\pm \mathrm{i} p \chi_0/\varepsilon), \quad p = \left(\frac{1}{3}, \frac{1}{2}, 1, 2, 3\right), \tag{20}$$

与此对应的共振可分别称为同步共振 $(p = 1)$; 超谐波共振 $(p = 1/3, 1/2)$; 次谐波共振 $(p = 2, 3)$• 在下一节里我们将详细地考察次谐波共振•

2 次谐波共振

次谐波共振可由小于自由表面波运动一阶量大小的快变水深诱发产生:

$$h_1 = \frac{1}{2} a E_0^2, \tag{21}$$

其中 $a = a(x_1, x_2, \cdots)$•

假定水流可表示为

$$U = U_0 + \varepsilon^2 U_2(t_2), \tag{22}$$

对于一阶问题 $(n = 1)$, 所有强迫项均为零• (16) ~ (19) 的齐次解为

$$\frac{\partial \phi^{(1,0)}}{\partial z} = \zeta^{(1,0)} = 0, \tag{23}$$

$$\phi^{(1,1)} = \frac{\mathrm{i} g A}{2 k_0 U_0} \frac{\operatorname{ch} Q}{\operatorname{ch} q}, \tag{24}$$

$$\zeta^{(1,1)} = \frac{1}{2} A• \tag{25}$$

其中

$$(k_0 U_0)^2 = g k_0 \operatorname{th} q, \quad Q = k_0 (z + h_0), \quad q = k_0 h_0• \tag{26}$$

当 $n \geqslant 2$ 时, $m = 0$ 和 $m = 1$ 时的可解性条件可分别表示为

$$\int_{-h_0}^{0} R^{(n,0)} \mathrm{d} z + F^{(n,0)} = \frac{1}{g} G^{(n,0)}, \tag{27}$$

$$\frac{1}{k_0} \int_0^q R^{(n,1)} \operatorname{ch} Q \, \mathrm{d} Q + F^{(n,1)} = \frac{1}{g} G^{(n,1)} \operatorname{ch} q; \tag{28}$$

当 $n = 2$, $m = 0$ 时, 存在下列强迫函数

$$\left.\begin{aligned} & R^{(2,0)} = F^{(2,0)} = G^{(2,0)} = 0, \\ & H^{(2,0)} = \left[\frac{\partial \phi^{(1,0)}}{\partial t_1} + U_0 \frac{\partial \phi^{(1,0)}}{\partial x_1}\right]_{z=0} + \frac{g^2 |A|^2}{4 U_0^2} (1 + \operatorname{th}^2 q)• \end{aligned}\right\} \tag{29}$$

将(29) 代入(16) ~ (19), 可得

$$\frac{\partial \phi^{(2,0)}}{\partial z} = 0, \tag{30}$$

$$\zeta^{(2,0)} = -\frac{1}{g}\left[\frac{\partial \phi^{(1,0)}}{\partial t_1} + U_0 \frac{\partial \phi^{(1,0)}}{\partial x_1}\right]_{z=0} - \frac{g\,|A|^2}{4U_0^2}(1+\mathrm{th}^2 q)· \tag{31}$$

当 $n=2$, $m=1$ 时, 其强迫函数为

$$\left.\begin{aligned}
&R^{(2,1)} = -\mathrm{i}\frac{\partial}{\partial x_1}(k_0\phi^{(1,1)}) - \mathrm{i}k_0\frac{\partial \phi^{(1,1)}}{\partial x_1},\quad F^{(2,1)} = -\frac{\partial h_0}{\partial x_1}[\mathrm{i}k_0\phi^{(1,1)}]_{z=-h_0},\\
&G^{(2,1)} = -U_0^2\left[\mathrm{i}k_0\frac{\partial \phi^{(1,1)}}{\partial x_1} + i\frac{\partial(k_0\phi^{(1,1)})}{\partial x_1}\right]_{z=0} - 2\mathrm{i}U_0k_0\left[\frac{\partial \phi^{(1,1)}}{\partial t_1}\right]_{z=0},\\
&H^{(2,1)} = \left[U_0\frac{\partial \phi^{(1,1)}}{\partial x_1} + \frac{\partial \phi^{(1,1)}}{\partial t_1}\right]_{z=0}·
\end{aligned}\right\} \tag{32}$$

将(32)代入(16)~(19), 可得 $\phi^{(2,1)}$ 的一般解为

$$\begin{aligned}
\phi^{(2,1)} = {}& \frac{\mathrm{i}gD\,\mathrm{ch}Q}{2k_0U_0} + \frac{gA}{4k_0U_0\mathrm{ch}q}\Big\{\alpha_1[2Q\,\mathrm{ch}Q + 2\mathrm{sh}Q(\mathrm{ch}^2q - \mathrm{ch}^2Q)] + {}\\
&\alpha_2[-\mathrm{sh}Q\,\mathrm{sh}2Q + \mathrm{sh}Q(\mathrm{sh}2q + 2q)] + \alpha_3[Q^2\mathrm{ch}Q + 2Q\,\mathrm{sh}Q\,\mathrm{ch}^2q + {}\\
&\mathrm{sh}Q(\mathrm{sh}2q + 2q - 2q\,\mathrm{ch}^2q)] - 2\frac{\partial h_0}{\partial x_1}\mathrm{sh}Q\Big\},
\end{aligned} \tag{33}$$

其中

$$\alpha_1 = \frac{\partial Q}{k_0\partial x_1},\quad \alpha_2 = \frac{\mathrm{ch}q}{k_0A}\frac{\partial}{\partial x_1}\left(\frac{A}{\mathrm{ch}q}\right),\quad \alpha_3 = \frac{\partial}{\partial x_1}\left(\frac{1}{k_0}\right)· \tag{34}$$

(33)右端的第一项和第二项分别表示 $\phi^{(2,1)}$ 的齐次解和特解· 相应地, 自由表面位移 $\zeta^{(2,1)}$ 可表示为

$$\zeta^{(2,1)} = -\frac{1}{g}[\mathrm{i}U_0k_0\phi^{(2,1)} + H^{(2,1)}]_{z=0}· \tag{35}$$

当 $n=2$, $m=1$ 时, 由可解性条件(28) 得到

$$\frac{\partial A}{\partial t_1} + U_0(1-r)\frac{\partial A}{\partial x_1} = \frac{AU_0}{2k_0}\frac{\partial k_0}{\partial x_1}(1-r) + \frac{1}{2}AU_0\frac{\partial r}{\partial x_1}, \tag{36}$$

其中

$$r = \frac{1}{2}\left(1 + \frac{2q}{\mathrm{sh}2q}\right)·$$

当 $n=2$, $m=2$ 时, 其强迫项为

$$\left.\begin{aligned}
&R^{(2,2)} = F^{(2,2)} = 0,\quad G^{(2,2)} = \mathrm{i}A^2(k_0U_0)^3\left(\frac{1}{4} - \frac{1}{2\mathrm{sh}^2q}\right),\\
&H^{(2,2)} = \left\{\mathrm{i}\zeta^{(1,1)}k_0U_0\frac{\partial \phi^{(1,1)}}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial \phi^{(1,1)}}{\partial z}\right)^2 - k_0^2(\phi^{(1,1)})^2\right]\right\}_{z=0}·
\end{aligned}\right\} \tag{37}$$

由(16)~(19)即得 $\phi^{(2,2)}$ 的解

$$\phi^{(2,2)} = \frac{\mathrm{i}k_0U_0A^2}{16\mathrm{sh}^4q}(2 - \mathrm{sh}^2q)\,\mathrm{ch}2Q, \tag{38}$$

$$\zeta^{(2,2)} = \frac{(k_0U_0A\,\mathrm{ch}q)^2}{4g\mathrm{sh}^4q}· \tag{39}$$

当 $n=3,\ m=0$ 时，强迫项为

$$\left.\begin{aligned}&R^{(3,0)}=-\frac{\partial^2\phi^{(1,0)}}{\partial x_1^2},\quad F^{(3,0)}=-\frac{\partial h_0}{\partial x_1}\left[\frac{\partial\phi^{(1,0)}}{\partial x_1}\right],\\&G^{(3,0)}=-\frac{\partial^2\phi^{(1,0)}}{\partial t_1^2}-U_0^2\frac{\partial^2\phi^{(1,0)}}{\partial x_1^2}-2U_0\frac{\partial^2\phi^{(1,0)}}{\partial x_1\partial t_1}-2\left[k_0^2\frac{\partial|\phi^{(1,1)}|^2}{\partial t_1}+\right.\\&\quad\left.\frac{\partial}{\partial t_1}\left|\frac{\partial\phi^{(1,1)}}{\partial z}\right|^2\right]-U_0\left[4k_0\frac{\partial k_0}{\partial x_1}|\phi^{(1,1)}|^2+2k_0^2\frac{\partial|\phi^{(1,1)}|^2}{\partial x_1}+\right.\\&\quad\left.2\frac{\partial}{\partial x_1}\left|\frac{\partial\phi^{(1,1)}}{\partial z}\right|^2\right]+\left\{(k_0U_0)^2\zeta^{(2,1)}\frac{\partial\phi^{(1,1)}}{\partial z}-2\mathrm{i}k_0U_0\zeta^{(1,1)}\frac{\partial^2\phi^{(1,1)}}{\partial z\partial t_1}-\right.\\&\quad\left.\zeta^{(1,1)}\left[-(k_0U_0)^2\frac{\partial\phi^{(2,1)}}{\partial z}+U_0^2\left(\mathrm{i}k_0\frac{\partial^2\phi^{(1,1)}}{\partial z\partial x_1}+\mathrm{i}\frac{\partial^2(\phi^{(1,1)}k_0)}{\partial z\partial x_1}\right)\right]\right\}+\mathrm{c.c.},\end{aligned}\right\}\tag{40}$$

其中$(^{-})$表示所在项的复共轭，c.c. 则表示前项的复共轭• 由可解性条件(27)得到

$$\begin{aligned}&\frac{\partial^2\phi^{(1,0)}}{\partial t_1^2}+2U_0\frac{\partial^2\phi^{(1,0)}}{\partial x_1\partial t_1}+(U_0^2-\mathrm{g}h_0)\frac{\partial^2\phi^{(1,0)}}{\partial x_1^2}-\mathrm{g}\frac{\partial\phi^{(1,0)}}{\partial x_1}\frac{\partial h_0}{\partial x_1}+\frac{1}{4}\mathrm{g}k_0U_0\left[-\frac{q}{\mathrm{ch}^2q}+\right.\\&\quad\left.3\mathrm{th}q+2\mathrm{th}q\,\mathrm{sh}^2q+\frac{4}{\mathrm{sh}2q}\right]\frac{\partial|A|^2}{\partial x_1}+\mathrm{g}k_0\frac{\partial|A|^2}{\partial t_1}\frac{2+\mathrm{th}^2q}{4\mathrm{th}q}+(k_0U_0)^3|A|^2\times\\&\quad\left\{-\frac{1}{2k_0}\frac{\partial q}{\partial x_1}\left[\frac{1}{\mathrm{sh}2q}(-2\mathrm{sh}^4q-\mathrm{sh}^2q+2)-\frac{q}{2\mathrm{ch}^2q}\right]+\frac{1}{k_0^2}\frac{\partial k_0}{\partial x_1}\left[\frac{q}{\mathrm{sh}2q}+\right.\right.\\&\quad\left.\left.\frac{1}{2\mathrm{sh}^2q}(\mathrm{sh}^4q+5\mathrm{sh}^2q+2)\right]+\frac{1}{\mathrm{sh}2q}\frac{\partial h_0}{\partial x_1}\right\}-\frac{1}{2}(k_0U_0)^2\frac{\partial\phi^{(1,1)}}{\partial x_1}\mathrm{th}q(\mathrm{i}A+\mathrm{c.c.})=0•\end{aligned}\tag{41}$$

从物理意义上讲，(41)描述了短波群在非平整海底上传播而产生的二阶长波运动•

当 $n=3,\ m=1$ 时，强迫项为

$$R^{(3,1)}=-\frac{\partial^2\phi^{(1,1)}}{\partial x_1^2}-\mathrm{i}\left\{\frac{\partial}{\partial x_1}[k_0\phi^{(2,1)}]+k_0\frac{\partial\phi^{(2,1)}}{\partial x_1}\right\}-\mathrm{i}\left\{\frac{\partial}{\partial x_2}[k_0\phi^{(1,1)}]+k_0\frac{\partial\phi^{(1,1)}}{\partial x_2}\right\},\tag{42}$$

$$F^{(3,1)}=-\frac{\partial h_0}{\partial x_1}\left[\mathrm{i}k_0\phi^{(2,1)}+\frac{\partial\phi^{(1,1)}}{\partial x_1}\right]_{z=-h_0}-\frac{3}{2}ak_0^2[\phi^{(1,1)}]_{z=-h_0},\tag{43}$$

$$\begin{aligned}&G^{(3,1)}=-\frac{\partial^2\phi^{(1,1)}}{\partial t_1^2}+k_0^4|\phi^{(1,1)}|^2\phi^{(1,1)}-k_0^2\phi^{(1,1)}\left[\frac{\partial\phi^{(1,1)}}{\partial z}\right]^2-\frac{\partial\phi^{(1,1)}}{\partial z}\left\{k_0^2\frac{\partial|\phi^{(1,1)}|^2}{\partial z}+\right.\\&\quad\left.\frac{\partial}{\partial z}\left|\frac{\partial\phi^{(1,1)}}{\partial z}\right|^2\right\}-\frac{\partial\phi^{(1,1)}}{\partial z}\left[\frac{\partial\phi^{(1,1)}}{\partial z}\frac{\partial^2\phi^{(1,1)}}{\partial z^2}-k_0^2\phi^{(1,1)}\frac{\partial\phi^{(1,1)}}{\partial z}\right]-U_0^2\left\{\frac{\partial^2\phi^{(1,1)}}{\partial x_1^2}+\right.\\&\quad\left.\mathrm{i}k_0\frac{\partial\phi^{(2,1)}}{\partial x_1}+\mathrm{i}\frac{\partial}{\partial x_1}[k_0\phi^{(2,1)}]+2\mathrm{i}k_0\frac{\partial\phi^{(1,1)}}{\partial x_2}\right\}-2U_0\left\{\frac{\partial^2\phi^{(1,1)}}{\partial t_1\partial x_1}+\mathrm{i}k_0\frac{\partial\phi^{(2,1)}}{\partial t_1}\right\}-\\&\quad2\mathrm{i}k_0U_0\frac{\partial\phi^{(1,1)}}{\partial t_2}-U_0\left\{4\mathrm{i}k_0^3\phi^{(1,1)}\phi^{(2,2)}-2k_0^2\phi^{(1,1)}\frac{\partial\phi^{(1,0)}}{\partial x_1}+2\mathrm{i}k_0\frac{\partial\phi^{(1,1)}}{\partial z}\frac{\partial\phi^{(2,2)}}{\partial z}\right\}+\\&\quad(k_0U_0)^2\zeta^{(2,0)}\frac{\partial\phi^{(1,1)}}{\partial z}+4(k_0U_0)^2\zeta^{(1,1)}\frac{\partial\phi^{(2,2)}}{\partial z}+(k_0U_0)^2\zeta^{(2,2)}\frac{\partial\phi^{(1,1)}}{\partial z}+\\&\quad(k_0U_0)^2|\zeta^{(1,1)}|^2\frac{\partial^2\phi^{(1,1)}}{\partial z^2}+\frac{1}{2}(k_0U_0)^2[\zeta^{(1,1)}]^2\frac{\partial^2\phi^{(1,1)}}{\partial z^2}+2U_0U_2k_0^2\phi^{(1,1)},\end{aligned}\tag{44}$$

由(42)～(44)代入可解性条件(28)，并且忽略 $\phi^{(2,1)}$ 的齐次解效应，可得

$$\mathrm{i}\frac{\partial A}{\partial t_2}+\mathrm{i}U_0(1-r)\frac{\partial A}{\partial x_2}=-\frac{1+q\mathrm{th}q}{k_0}\frac{\partial^2A}{\partial t_1\partial x_1}-\frac{1}{2k_0U_0}\frac{\partial^2A}{\partial t_1^2}+k_0A\left[U_2+\frac{\partial\phi^{(1,0)}}{\partial t_1}+\right.$$

$$\frac{\text{th}^2 q}{2U_0}\left(\frac{\partial\phi^{(1,0)}}{\partial t_1}+U_0\frac{\partial\phi^{(1,0)}}{\partial x_1}\right)\Big]-k_0^2U_0Aa\frac{3}{2\text{sh}2q}+\mu_1\frac{\partial A}{\partial t_1}+\mu_2\frac{\partial^2 A}{\partial x_1^2}+\mu_3\frac{\partial A}{\partial x_1}+$$
$$A\mu_4-k_0^3U_0\frac{1}{16}\left(\frac{7}{\text{ch}^2q}+11\text{th}^2q+\frac{8}{\text{sh}^2 2q}+\frac{4}{\text{ch}^2q\,\text{sh}^4q}\right)|A|^2A, \tag{45}$$

其中 $\mu_j(j=1\sim 4)$ 的详细表达式在附录中给出• 我们强调指出(41) 和(45) 共同构成了一阶振幅 A 和平均流动势 $\phi^{(1,0)}$ 演化的控制方程• 从(36) 可得

$$\frac{\partial^2 A}{\partial t_1^2}=U_0^2(1-r)^2\frac{\partial^2 A}{\partial x_1^2}+\frac{1}{2}U_0^2\frac{\partial A}{\partial x_1}(r-1)\left[\frac{1}{k_0}\frac{\partial k_0}{\partial x_1}(1-r)+\frac{\partial r}{\partial x_1}\right]+$$
$$A\left\{\frac{1}{2}U_0\left[\frac{1}{k_0}\frac{\partial k_0}{\partial x_1}(1-r)+\frac{\partial r}{\partial x_1}\right]\right\}^2, \tag{46}$$

$$\frac{\partial^2 A}{\partial x_1\partial t_1}=-U_0(1-r)\frac{\partial^2 A}{\partial x_1^2}+\frac{1}{2}U_0\frac{\partial A}{\partial x_1}\left[\frac{1}{k_0}\frac{\partial k_0}{\partial x_1}(1-r)+\frac{\partial r}{\partial x_1}\right]+$$
$$\frac{1}{2}AU_0\left\{(1-r)\left[\frac{1}{k_0}\frac{\partial^2 k_0}{\partial x_1^2}-\frac{1}{k_0^2}\left(\frac{\partial k_0}{\partial x_1}\right)^2\right]-\frac{1}{k_0}\frac{\partial k_0}{\partial x_1}\frac{\partial r}{\partial x_1}+\frac{\partial^2 r}{\partial x_1^2}\right\}, \tag{47}$$

因此(45) 可重新写为

$$\text{i}\frac{\partial A}{\partial t_2}+\text{i}U_0(1-r)\frac{\partial A}{\partial x_2}=-\frac{U_0}{2k_0}[r^2+r(2q\,\text{th}q-1)-1]\frac{\partial^2 A}{\partial x_1^2}+\left\{\mu_4+\frac{1}{2}U_0\mu_1\left[\frac{\partial r}{\partial x_1}+\right.\right.$$
$$\left.\frac{1}{k_0}\frac{\partial k_0}{\partial x_1}(1-r)\right]+k_0\left[\frac{\partial\phi^{(1,0)}}{\partial t_1}+U_2+\frac{\text{th}^2q}{2U_0}\left(\frac{\partial\phi^{(1,0)}}{\partial t_1}+U_0\frac{\partial\phi^{(1,0)}}{\partial x_1}\right)\right]-$$
$$\frac{U_0}{8k_0}\left[\frac{1}{k_0^2}\left(\frac{\partial k_0}{\partial x_1}\right)^2(1-r)(5+4q\,\text{th}q-r)+\frac{4}{k_0}\frac{\partial^2 k_0}{\partial x_1^2}(1+q\,\text{th}q)(1-r)+\left(\frac{\partial r}{\partial x_1}\right)^2+\right.$$
$$\left.\left.4(1+q\,\text{th}q)\frac{\partial^2 r}{\partial x^2}+\frac{1}{k_0}\frac{\partial k_0}{\partial x_1}\left(2\frac{\partial r}{\partial x_1}(1-r)-4\frac{\partial r}{\partial x_1}(1+q\,\text{th}q)\right)\right]\right\}A-\frac{3ak_0^2U_0}{2\text{sh}2q}A-$$
$$\frac{k_0^3U_0}{16}\left(11\text{th}^2q+\frac{7}{\text{ch}^2q}+\frac{8}{\text{sh}^2 2q}+\frac{4}{\text{ch}^2q\,\text{sh}^4q}\right)|A|^2A+\left\{\mu_3+\mu_1U_0(r-1)-\right.$$
$$\left.\frac{U_0}{4k_0}\left[\frac{2}{k_0}(1+q\,\text{th}q)+r-1\right]\left[\frac{1}{k_0}\frac{\partial k_0}{\partial x_1}(1-r)+\frac{\partial r}{\partial x_1}\right]\right\}\frac{\partial A}{\partial x_1}• \tag{48}$$

当振幅 a 独立于x_1和 x_2, A 相对于x_1和 x_2不存在空间调制时, 演化方程(48) 可简化为著名的 Stuart_Landau 方程:

$$\text{i}\frac{\partial A}{\partial t_2}=k_0U_2A-\frac{k_0^3U_0}{16}\left(11\text{th}^2q+\frac{7}{\text{ch}^2q}+\frac{8}{\text{sh}^2 2q}+\frac{4}{\text{ch}^2q\,\text{sh}^4q}\right)|A|^2A-$$
$$\frac{3ak_0^2U_0}{2\text{sh}2q}A• \tag{49}$$

3 动力系统

对于定常流、常量 U_2, 设

$$a'=k_0a, \tag{50}$$

$$A'=\left\{\frac{2\text{sh}2q}{3a'}\frac{1}{16}\left(11\text{th}^2q+\frac{7}{\text{ch}^2q}+\frac{8}{\text{sh}^2 2q}+\frac{4}{\text{ch}^2q\,\text{sh}^4q}\right)\right\}^{1/2}k_0A, \tag{51}$$

$$T=a'\frac{3}{2\text{sh}2q}k_0U_0t_2, \tag{52}$$

则(49) 可以无量纲化为

$$\mathrm{i}\,\frac{\partial A'}{\partial T} = \tau A' - |A'|^2 A' - A'. \tag{53}$$

假设 $A' = X + \mathrm{i}Y$, 通过分离实部和虚部, 可得一个二阶自治动力系统

$$\frac{\partial X}{\partial T} = (1+\tau)Y - (X^2+Y^2)Y, \tag{54}$$

$$\frac{\partial Y}{\partial \tau} = (1-\tau)X + (X^2+Y^2)X, \tag{55}$$

其中 τ 代表速度解调的分叉参数, 可表示为

$$\tau = \frac{2\mathrm{sh}2q}{3a'}\,\frac{U_2}{U_0}. \tag{56}$$

(54) 与 (55) 构成了一个可积的 Hamilton 系统:

$$H(X,Y) = \frac{1}{2}\tau(X^2+Y^2) - \frac{1}{4}(X^2+Y^2)^2 + \frac{1}{2}(Y^2-X^2). \tag{57}$$

(54) 与 (55) 存在下列不动点: 对任意 τ, 存在不动点 $(0, 0)$; 当 $\tau > -1$ 时, 存在不动点 $[0, \pm(\tau+1)^{1/2}]$; 当 $\tau > 1$ 时, 存在不动点 $[\pm(\tau-1)^{1/2}, 0]$. 由以上不动点的 Jacobians 可得到下列结果: 当 $\tau \in (-1, 1)$ 和 $\tau \notin (-1, 1)$ 时, $(0, 0)$ 分别表示一个鞍点和一个中心, 其振动的固有角频率为 $\Omega = (\tau^2-1)^{1/2}$; $[0, \pm(\tau+1)^{1/2}]$ 可写为 $(0, C^+)$ 和 $(0, C^-)$, C^+ 和 C^- 表示中心, 其对应的振动频率为 $\Omega = 2(1+\tau)^{1/2}$; $[\pm(\tau-1)^{1/2}, 0]$ 则可写为 $(S^+, 0)$ 和 $(S^-, 0)$, S^+ 和 S^- 表示鞍点. 该动力系统的分叉图如图 1 所示.

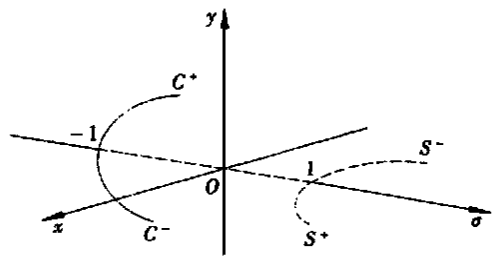

图 1 次谐波共振的分叉图

实际上[5], 原点处的不动点即表示水体自由平坦表面这样一种平凡状态, C^+ 和 C^- 代表某种水波运动, 它的可振荡的波峰与波谷位于海底沙波波谷的上方, S^+ 和 S^- 则表示自由表面波运动, 其波峰与波谷直接位于海底沙波波峰的上方.

对于弱振荡流, 依据 (15) 我们采用 Sammarco 等人的选择[5]:

$$U_2 = U_2 + U_2'\cos\omega T, \tag{58}$$

其中 U_2 为稳定部分, T 是由 (52) 正规化的无量纲时间, ω 为无量纲频率. 可以证明, (53) 可转化为

$$\mathrm{i}\,\frac{\partial A'}{\partial T} = (\tau + \gamma\cos\omega T)A' - |A'|^2 A' - A', \tag{59}$$

相应地, (54) 与 (55) 则转化为

$$\frac{\partial X}{\partial T} = (1+\tau)Y - (X^2+Y^2)Y + \gamma Y\cos\omega T, \tag{60}$$

$$\frac{\partial Y}{\partial \tau} = (1-\tau)X + (X^2+Y^2)X - \gamma X\cos\omega T, \tag{61}$$

其中 τ 仍由 (56) 给出, γ 则定义为

$$\gamma = \frac{2\mathrm{sh}2q}{3a'}\,\frac{U_2'}{U_0}, \tag{62}$$

即

$$\frac{\tau}{U_2} = \frac{\gamma}{U_2'}. \tag{63}$$

现在我们研究在上述不动点处最有可能发生的瞬态共振, 假设 $\gamma = \nu\gamma_1 = O(\nu)$, $\nu \ll 1$• 首先在中心(0, 0) 处展开:

$$X = \nu X_1 + \nu^2 X_2 + \nu^3 X_3 + \cdots, \tag{64}$$

$$Y = \nu Y_1 + \nu^2 Y_2 + \nu^3 Y_3 + \cdots \tag{65}$$

在 $O(\nu)$ 上, 由(60) 和(61) 可得

$$\frac{\partial X_1}{\partial T} - (1+\tau) Y_1 = 0, \tag{66}$$

$$\frac{\partial Y_1}{\partial T} - (1-\tau) X_1 = 0, \tag{67}$$

由此得到一阶解

$$X_1 = X_{11}^n e^{i\Omega T} + \text{c.c.}, \quad Y_1 = Y_{11}^n e^{i\Omega T} + \text{c.c.}, \tag{68}$$

其中上标 n 表示自然模式响应, 显然在该阶上不发生共振• 在 $O(\nu^2)$ 上, 摄动方程为

$$\frac{\partial X_2}{\partial T} - (1+\tau) Y_2 = -\gamma_1 Y_1 \cos T, \tag{69}$$

$$\frac{\partial Y_2}{\partial T} - (1-\tau) X_2 = \gamma_1 X_1 \cos T, \tag{70}$$

因此二阶解为

$$X_2 = X_{21}^n e^{i\Omega T} + X_{21}^f e^{i(\Omega-1)T} + X_{22}^f e^{i(\Omega+1)T} + \text{c.c.}, \tag{71}$$

$$Y_2 = Y_{21}^n e^{i\Omega T} + Y_{21}^f e^{i(\Omega-1)T} + Y_{22}^f e^{i(\Omega+1)T} + \text{c.c.}, \tag{72}$$

其中上标 f 表示强迫模式响应• 在该阶上, 当 $\Omega = 1/2$ 时发生共振• 在 $O(\nu^3)$ 上, 摄动方程为

$$\frac{\partial X_3}{\partial T} - (1+\tau) Y_3 = -Y_1(X_1^2 + Y_1^2) + \gamma_1 Y_2 \cos T - \frac{\partial X_1}{\partial T_2}, \tag{73}$$

$$\frac{\partial Y_3}{\partial T} - (1-\tau) X_3 = X_1(X_1^2 + Y_1^2) - \gamma_1 X_2 \cos T - \frac{\partial Y_1}{\partial T_2}, \tag{74}$$

其中 $T_2 = \varepsilon^2 T$• 在该阶上, 当 $\Omega = 1/2, 1$ 时发生共振, 在中心 C^+ 和 C^- 处, 可做类似的摄动分析, 则由此可知: 在 $O(\nu)$ 上, 当 $\Omega = 1$ 时将发生瞬态共振; 在 $O(\nu^2)$ 上, 当 $\Omega = 1/2, 2$ 时发生瞬态共振; 在 $O(\nu^3)$ 上, 当 $\Omega = 1/3, 3$ 时发生瞬态共振•

当 $\gamma > 0$ 时, (60) 和(61) 构成了一个依赖于时间的 Hamilton 系统:

$$H(X, Y, T) = \frac{1}{2}\tau(X^2 + Y^2) - \frac{1}{4}(X^2 + Y^2)^2 + \frac{1}{2}\tau(Y^2 - X^2) + \frac{1}{2}\gamma(X^2 + Y^2)\cos T, \tag{75}$$

则该系统不再是可积的, 这就预示着混沌将要出现[7]• 混沌可作为物理参数的函数, 它的演变过程及其物理意义, 在 Sammarco 等人的工作中得到了揭示[5]•

4 结 果 与 讨 论

在非线性参数 ε 与调制参数 δ 属于同一大小量级的假定下, 本文运用 WKBJ 型摄动方法, 推导出自由表面流和沙纹海底次谐波共振而产生的弱非线性 Stokes 波的三阶演化方程• 针对定常流自治动力系统, 我们给出了各种稳态解及其分叉图, 对于弱振荡流非自治动力系统, 我们指出将要引起混沌响应•

本文的工作可进一步推广到能够包含粘度和湍流效应, 显然, 由此产生的混沌动力系统将

更为复杂•

附录: 在(45)中, $\mu_j\,(j = 1 \sim 4)$ 的式详细表达式

$$\mu_1 = -\frac{1}{k_0}\frac{\partial q}{\partial x_1}\frac{q}{\mathrm{ch}^2 q} + \frac{1}{k_0^2}\frac{\partial k_0}{\partial x_1}\left(\frac{1}{2}q^2 + q\,\mathrm{th}q + \mathrm{sh}^2 q\right) + \frac{\partial h_0}{\partial x_1}\mathrm{th}q;$$

$$\mu_2 = \frac{U_0}{k_0}\left[\frac{1}{4}\left(1 + \frac{2q}{\mathrm{sh}2q}\right) - q\,\mathrm{th}q\right];$$

$$\mu_3 = k_0 U_0\left\{\frac{1}{k_0^2}\frac{\partial q}{\partial x_1}\left[-\frac{2q}{\mathrm{ch}^2 q} + \frac{1}{2} + \frac{1}{4\mathrm{sh}2q} + \mathrm{th}q\left(\frac{2}{\mathrm{sh}2q} - 1\right)\right] + \frac{1}{k_0^3}\frac{\partial k_0}{\partial x_1}\left[q^3\left(-\frac{1}{4\mathrm{sh}2q}\right) + q^2\left(\frac{3}{8} - \frac{1}{8\mathrm{sh}2q}\right) + \frac{11}{8}q\,\mathrm{th}q + \frac{1}{16}\mathrm{th}q - \frac{1}{8}\mathrm{sh}^2 q\right] + \frac{1}{k_0\mathrm{sh}2q}\frac{\partial h_0}{\partial x_1}\right\},$$

$$\begin{aligned}\mu_4 = k_0 U_0\Bigg\{&\frac{1}{k_0^2}\frac{\partial k_0}{\partial x_1}\frac{\partial h_0}{\partial x_1}\frac{1}{\mathrm{sh}2q}\left(\frac{1}{2}q^2 - 1\right) - \frac{1}{2k_0}\frac{\partial q}{\partial x_1}\frac{\partial h_0}{\partial x_1} + \frac{1}{k_0^3}\frac{\partial^2 k_0}{\partial x_1^2}\left[-\frac{q^3}{3\mathrm{sh}2q} + \frac{1}{4}q^2 + \right.\\ &\left. q\left(-\frac{3}{4\mathrm{sh}2q} + \frac{1}{2}\mathrm{th}q\right) + \frac{25}{32} + \frac{9}{16}\mathrm{sh}^2 q\right] + \frac{1}{k_0^2}\frac{\partial^2 q}{\partial x_1^2}\left[\frac{q^2}{2\mathrm{sh}2q} + \frac{q}{2\mathrm{ch}^2 q}(2\mathrm{sh}^2 q - 2\mathrm{sh}q + 1) - \right.\\ &\left.\frac{\mathrm{sh}2q}{8}\right] + \frac{1}{2k_0}\frac{\partial^2 h_0}{\partial x_1^2}\mathrm{th}q + \frac{1}{k_0^4}\left(\frac{\partial k_0}{\partial x_1}\right)^2\left[q^3\left(\frac{1}{4}\mathrm{cth}2q + \frac{4}{3\mathrm{sh}2q}\right) - q^2\left(\frac{3}{8}\mathrm{th}q + \frac{7}{8} - \frac{7}{16\mathrm{sh}2q}\right) - \right.\\ &\left. q\left(-\frac{7}{4\mathrm{sh}2q} + \frac{1}{16}\mathrm{th}q\left(\frac{19}{2} - 7\mathrm{sh}^2 q\right)\right) - \frac{229}{128} - \frac{19}{16}\mathrm{sh}^2 q\right] + \frac{1}{k_0^2}\left(\frac{\partial q}{\partial x_1}\right)^2\left[-\frac{3q^2}{2\mathrm{ch}^2 q} - \right.\\ &\left.\frac{1}{8}(7 + 2\mathrm{sh}^2 q) + \frac{q}{2\mathrm{sh}2q}\left(\frac{5}{2} + 3\mathrm{ch}^2 q + \frac{1}{\mathrm{ch}^2 q}\left(2 + \frac{3}{2}\mathrm{th}q\right)\right) - \frac{1}{8\mathrm{ch}^2 q}(3\mathrm{sh}^4 q + 5\mathrm{sh}^2 q + 8)\right] + \\ &\frac{1}{k_0^3}\frac{\partial k_0}{\partial x_1}\frac{\partial q}{\partial x_1}\left[\frac{13q^3}{24\mathrm{ch}^2 q} - \frac{59q^2}{16\mathrm{sh}2q} + q\left(-\frac{71}{32\mathrm{ch}q\,\mathrm{sh}2q} + \frac{1}{64\mathrm{ch}^2 q}(48\mathrm{sh}^4 q + 111\mathrm{sh}^2 q - \right.\right.\\ &\left.\left. 32\mathrm{sh}q + 72)\right) - \frac{1}{32\mathrm{sh}2q}(322\mathrm{sh}^4 q + 216\mathrm{sh}^2 q + 1)\right]\Bigg\}\end{aligned}$$

[参 考 文 献]

[1] Kennedy J F. The mechanics of dunes and antidunes in erodible_bed channels[J]. J Fluid Mech, 1963, **16**: 521—544.

[2] Heathershaw A D. Seabed_wave resonance and sand bar growth[J]. Nature, 1982, **296**(25): 343—345.

[3] Davies A G, Heathershaw A D. Surface_wave propagation over sinusoidally varying topography[J]. J Fluid Mech, 1984, **144**: 419—443.

[4] Begi S, Battjes J A. Numerical simulation of nonlinear wave propagation over a bar[J]. Coastal Engineering, 1994, **23**: 1—16.

[5] Sammarco P, Mei C C, Trulsen K. Nonlinear resonance of free surface waves in a current over a sinusoidal bottom: a numerical study[J]. J Fluid Mech, 1994, **279**: 377—405.

[6] Naciri M, Mei C C. Evolution of a short surface wave on a very long surface wave of finite amplitude[J]. J Fluid Mech, 1992, **235**: 415—452.

[7] Guckenheimer J, Holmes P. Nonlinear Oscillations Dynamical Systems and Bifurcations of Vector Fields[M]. New York: Springer_Verlag, 1983.

On the Resonant Generation of Weakly Nonlinear Stokes Waves in Regions With Fast Varying Topography and Free Surface Current

HUANG Hu[1,2], ZHOU Xi_reng[1]

(1. School of Civil Engineering, Tianjin University, Tianjin 300072, P R China;
2. Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, P R China)

Abstract: The effect of nonlinearity on the free surface wave resonated by an incident flow over rippled beds, which consist of fast varying topography superimposed on an otherwise slowly varying mean depth, is studied using a WKBJ_type perturbation approach. Synchronous, superharmonic and in particular subharmonic resonance were selectively excited over the fast varying topography with corresponding wavelengths. For a steady current the dynamical system is autonomous and the possible nonlinear steady states and their stability were investigated. When the current has a small oscillatory component the dynamical system becomes non_autonomous, chaos is now possible.

Key words: nonlinear resonance; weakly nonlinear Stokes waves; a free surface current; rippled beds; dynamical system