应用数学和力学编委会编
重 庆 出 版 社 出 版

文章编号: 1000_0887(2001) 05_0458_07

一个新的 Liouville 可积系统及其 Lax 表示, Bi_Hamilton 结构*

范恩贵[1], 张鸿庆[2]

(1. 复旦大学 数学研究所, 上海 200433; 2. 大连理工大学 应用数学系, 大连 116024)

(我刊编委张鸿庆来稿)

摘要: 从一个特征值问题出发, 首先推导一族非线性发展方程, 其中包括著名 MKdV 方程做为特殊约化, 进一步证明这族方程在 Liounille 意义下可积并具有 Bi_Hamilton 结构• 而在位势函数和特征函数之间的一定约束下,特征值问题被非线性化为一完全可积的有限维 Hamilton 系统•

关 键 词: 可积系统; Lax 表示; Bi_Hamilton 结构
中图分类号: O175 文献标识码: A

引 言

寻找新的有限维和无穷维可积系统是孤立子理论研究的重要课题, 近年来, 二种分别产生有限维和无穷维可积 Hamilton 系统的有效途径被提出, 首先由屠规彰教授提出的迹恒等式是构造无穷维可积 Hamilton 系统的十分有效的代数方法, 从一个适当的特征值问题出发, 许多可积系统(如 AKNS, TC, TA, BPT, Yang 族等) 及其 Hamilton 结构被获得[1~3]; 其次曹策问教授提出非线性化方法是从孤立族中产生新的有限维可积 Hamilton 系统的一种有效途径, 在位势函数和特征函数之间的 Bargmann 或 Neumann 约束下, 特征值问题被非线性化为有限维完全可积 Hamilton 系统[4~9]• 在本文同时应用这二种方法研究如下特征值问题•

$$\phi_x = \boldsymbol{U}\phi = \begin{pmatrix} u & v+\lambda \\ v-\lambda & -u \end{pmatrix}\phi, \tag{1}$$

其中 u, v 表示位势, λ 为特征参数• 在第 1 节从问题(1) 出发, 首先推导一族新的非线性发展方程, 其中两个典型方程为

$$u_t = \frac{\beta}{2}[-v_{xx} + 2u^2v + 2v^3], \quad v_t = \frac{\beta}{2}[-u_{xx} - 2uv^2 - 2u^3]• \tag{2}$$

$$u_t = \frac{\beta}{4}[-u_{xxx} + 6(u^2 + v^2)u_x], \quad v_t = \frac{\beta}{4}[-v_{xxx} + 6(u^2 + v^2)v_x]• \tag{3}$$

当 $\beta = 4$, $v = 0$ 或 $\beta = 4$, $u = 0$ 时, 方程组(3) 化为 MKdV 方程• 进一步我们证明这族方程在 Liouville 意义下可积并具有 Bi_Hamilton 结构• 第 2 节我们将给出这族方程的 Lax 表示• 第 3

* 收稿日期: 2000_03_21; 修订日期: 2001_01_05
基金项目: 中国博士后基金资助课题; 国家基础研究重大课题“数学机械化及自动推理平台”资助课题(G1998030600)
作者简介: 范恩贵(1962—), 男, 河北唐山人, 副教授, 博士.

节通过对特征问题(1)的非线性倾,在位势函数和特征函数之间的一定约束下,得到了一个Bargmann系统,并进一步证明这个Bargmann系统是一个完全可积的有限维Hamilton系统•

1 非线性方程族及其Bi_Hamilton结构

取loop代数 $\boldsymbol{A} = A_1 \otimes \boldsymbol{C}[\lambda, \lambda^{-1}]$ 的一组基

$$\boldsymbol{h}(n) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^n, \quad \boldsymbol{e}(n) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \lambda^n, \quad \boldsymbol{f}(n) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \lambda^n•$$

容易验证它们之间具有如下运算关系:

$$[\boldsymbol{h}(m), \boldsymbol{e}(n)] = 2\boldsymbol{e}(m+n), \quad [\boldsymbol{h}(m), \boldsymbol{f}(n)] = -2\boldsymbol{f}(m+n),$$

$$[\boldsymbol{e}(m), \boldsymbol{f}(n)] = \boldsymbol{h}(m+n)•$$

将谱问题(1)中的 $\boldsymbol{U}$ 可表示为

$$\boldsymbol{U} = \boldsymbol{e}(1) - \boldsymbol{f}(1) + u\boldsymbol{h}(0) + v(\boldsymbol{e}(0) + \boldsymbol{f}(0))•$$

设 $\boldsymbol{V} = a\boldsymbol{h}(0) + b\boldsymbol{e}(0) + c\boldsymbol{f}(0)$,解伴随方程

$$\boldsymbol{V}_x = [\boldsymbol{U}, \boldsymbol{V}],$$

可得到

$$a_x = v(c-b) + \lambda(b+c),$$

$$b_x = -2ub - 2va - 2\lambda a,$$

$$c_x = 2va - 2uc - 2\lambda a•$$

将 $a = \sum_{m \geqslant 0} a_m \lambda^{-m}$, $b = \sum_{m \geqslant 0} b_m \lambda^{-m}$, $c = \sum_{m \geqslant 0} c_m \lambda^{-m}$ 代入上述方程,进一步得到

$$a_{mx} = v(c_m - b_m) + (b_m + c_m),$$

$$b_{mx} = 2ub_m - 2va_m - 2a_{m+1},$$

$$c_{mx} = 2va_m - 2uc_m - 2a_{m+1}•$$

由此可解得前几个为

$$a_0 = 0, \quad b_0 = \beta, \quad c_0 = -\beta,$$

$$a_1 = \beta u, \quad b_1 + c_1 = 2\beta v, \quad b_1 - c_1 = 0,$$

$$a_2 = -\frac{\beta}{2} v_x, \quad b_2 + c_2 = \beta u_x, \quad b_2 - c_2 = \beta(u^2 + v^2),$$

$$a_3 = \frac{\beta}{4}(-u_{xx} + 2uv^2 + 2u^3), \quad b_3 + c_3 = \frac{\beta}{2}(-v_{xx} + 2vu^2 + 2v^3),$$

$$b_3 - c_3 = \beta(vu_x - uv_x), \quad a_4 = \frac{\beta}{8}[v_{xxx} - 6(u^2 + v^2)v_x],$$

$$b_4 + c_4 = \frac{\beta}{4}[-u_{xxx} + 6(u^2 + v^2)u_x],$$

$$b_4 - c_4 = \frac{\beta}{4}[-2(uu_{xx} + vv_{xx}) + (u_x^2 + v_x^2 + 3u^4 + 3v^4 + 6u^2v^2)]•$$

及

$$\begin{pmatrix} 2a_{m+1} \\ b_{m+1} + c_{m+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} 2a_m \\ b_m + c_m \end{pmatrix}, \tag{4}$$

其中 $b_m - c_m = 2\partial^{-1}u(b_m + c_m) - 4\partial^{-1}va_m$,

$$\boldsymbol{M} = \begin{pmatrix} -2u\partial^{-1}v & \frac{1}{2}\partial + 2u\partial^{-1}u \\ \frac{1}{2}\partial - 2v\partial^{-1}v & 2v\partial^{-1}u \end{pmatrix}, \quad \partial = \frac{\partial}{\partial x}, \quad \partial\partial^{-1} = \partial^{-1}\partial = 1•$$

下文中, 如果 $g = g(\lambda)$, 我们简记 $g' = g(\mu)$, 通过计算可得

$$[\mu(U(\lambda) - U(\mu))/(\lambda - \mu), V(\mu)] = [\mu(e(0) - f(0)), a'h(0) + b'e(0) + c'f(0)] = (\mu b' + \mu c')h(0) - 2\mu a'(f(0) + e(0)) .$$

于是我们获得一族非线性发展方程

$$u_t = b_{m+1} + c_{m+1}, \quad v_t = -2a_{m+1}, \tag{5}$$

在方程族(5) 中, 取 $m = 2$, $m = 3$ 时, 即得方程(2) 和(3)•

将方程族(5) 等价地表示为

$$(u_t, v_t)^{\mathrm{T}} = \boldsymbol{J}\boldsymbol{G}_m = \boldsymbol{K}\boldsymbol{G}_{m-1}, \tag{6}$$

其中

$$\boldsymbol{G}_m = \begin{pmatrix} G_m^{(1)} \\ G_m^{(2)} \end{pmatrix} = \begin{pmatrix} 2a_{m+1} \\ b_{m+1} + c_{m+1} \end{pmatrix}, \quad \boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\boldsymbol{K} = \boldsymbol{J}\boldsymbol{M} = \begin{pmatrix} \frac{1}{2}\partial - 2v\partial^{-1}v & 2v\partial^{-1}u \\ 2u\partial^{-1}v & \frac{1}{2}\partial - 2u\partial^{-1}u \end{pmatrix}.$$

命题 1 $\boldsymbol{J}$ 和 $\boldsymbol{K}$ 均为斜对称算子, 即 $\boldsymbol{J}^* = -\boldsymbol{J}$, $\boldsymbol{J}\boldsymbol{M} = \boldsymbol{M}^*\boldsymbol{J}$•

证明

$$\boldsymbol{J}^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -J, \quad \boldsymbol{J}\boldsymbol{M} = \begin{pmatrix} \frac{1}{2}\partial - 2v\partial^{-1}v & 2v\partial^{-1}u \\ -2u\partial^{-1}v & \frac{1}{2}\partial - 2u\partial^{-1}u \end{pmatrix} = \boldsymbol{M}^*\boldsymbol{J}.$$

为把方程族(6) 写为 Hamilton 形式, 我们应用迹恒等式[1~3]

$$\frac{\delta}{\delta u_i}\left\langle \boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial \lambda}\right\rangle = \lambda^{-r}\frac{\partial}{\partial \lambda}\left(\lambda^{\gamma}\left\langle \boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial u_i}\right\rangle\right) \tag{7}$$

其中 $\langle x, y\rangle = \mathrm{tr}(xy)$, γ 为待定常数• 由于

$$\left\langle \boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial u}\right\rangle = 2a, \quad \left\langle \boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial v}\right\rangle = b + c, \quad \left\langle \boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial \lambda}\right\rangle = c - b,$$

则由(7) 得到

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}\right)^{\mathrm{T}}(c - b) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}(\lambda^{\gamma}(2a, b + c)^{\mathrm{T}}).$$

比较 λ^{-m} 的系数有

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}\right)^{\mathrm{T}}(c_{m+1} - b_{m+1}) = (-m + \gamma)(2a_m, b_m + c_m)^{\mathrm{T}}).$$

在上述方程中, 取 $m = 0$, 可得 $\gamma = 0$, 因此

$$\boldsymbol{G}_m = \left(\frac{\delta H_m}{\delta u}, \frac{\delta H_m}{\delta v}\right)^{\mathrm{T}}, \quad \boldsymbol{H}_m = \frac{b_{m+2} - c_{m+2}}{m + 1}. \tag{8}$$

由(6)、(8), 便得到方程族(6) 的 Bi_Hamilton 结构

$$(u_t, v_t)^{\mathrm{T}} = J\left(\frac{\delta H_m}{\delta u}, \frac{\delta H_m}{\delta v}\right)^{\mathrm{T}} = K\left(\frac{\delta H_m}{\delta u}, \frac{\delta H_m}{\delta v}\right)^{\mathrm{T}}. \tag{9}$$

概括起来, 我们得到如下结论

定理 1 i) 方程族(6) Liouville 意义下可积• ii) 函数 $\{H_m\}$ 构成方程族(6) 两两可换的守恒密度• iii) 方程族(6) 具有 Bi_Hamilton 结构形式(9)•

2 方程族(6)的 Lax 表示

命题 2 特征值问题(1)等价于

$$L\phi=\begin{pmatrix} v & -\partial-u \\ \partial-u & -v \end{pmatrix}\phi=\lambda\phi, \tag{10}$$

算子 L 的 Gateaux 导数为单同态映射

$$L'(\xi)=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}L(u+\varepsilon\xi_1,v+\varepsilon\xi_2)\mid_{\xi=0}=\begin{pmatrix} \xi_2 & -\xi_1 \\ -\xi_1 & -\xi_2 \end{pmatrix}.$$

考虑 $V=V_1+V_2\partial$ 和 $L=L_1+L_2\partial$ 的换位子 $[V,L]$, 其中

$$V_1=\begin{pmatrix} A & B \\ C & D \end{pmatrix},\quad V_2=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},\quad L_1=\begin{pmatrix} v & -u \\ -u & -v \end{pmatrix},\quad L_2=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

A, B, C, D, E 为待定函数, 直接计算可知

$$\begin{aligned}[V,L]=&[V_1,L_1]-L_2V_{1x}+V_2L_{1x}+([V_1,L_2]-[L_1,V_2]+\\ &V_2L_{2x}-L_2V_{2x})\partial+[V_2,L_2]\partial^2=\\ &\begin{pmatrix} -(B-C)u-Eu_x+C_x & -2Bv-(A-D)u-Ev_x+D_x \\ 2Cv+(A-D)u-Eu_x+C_x & (B-C)u+Eu_x-B_x \end{pmatrix}+\\ &\begin{pmatrix} -2Eu+(B+C)-E_x & -2Ev-(A-D) \\ -2Ev-(A-D) & 2Eu-(B+c)-E_x \end{pmatrix}\partial.\end{aligned} \tag{11}$$

另一方面, 对任意给定 $G=(G^{(1)},G^{(2)})^{\mathrm{T}}$, 有

$$L'(KG)-L'(JG)L=\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}+\begin{pmatrix} -G^{(2)} & G^{(1)} \\ G^{(1)} & G^{(2)} \end{pmatrix}\partial, \tag{12}$$

其中

$$k_{11}=\frac{1}{2}G_x^{(2)}+2u\partial^{-1}(vG^{(1)}-uG^{(2)})+(vG^{(1)}-uG^{(2)}),$$

$$k_{12}=-\frac{1}{2}G_x^{(1)}+2v\partial^{-1}(vG^{(1)}-uG^{(2)})-(uG^{(1)}+vG^{(2)}),$$

$$k_{21}=-\frac{1}{2}G_x^{(1)}+2v\partial^{-1}(vG^{(1)}-uG^{(2)})+(uG^{(1)}+vG^{(2)}),$$

$$k_{22}=-\frac{1}{2}G_x^{(2)}-2u\partial^{-1}(vG^{(1)}-uG^{(2)})+(vG^{(1)}-uG^{(2)}).$$

我们希望有

$$[V,L]=L'(KG)-L'(JG)L. \tag{13}$$

将(11)、(12)代入(13), 比较可得

$$E=0,\quad A=\frac{1}{2}G^{(1)},\quad D=-\frac{1}{2}G^{(1)},$$

$$B=\frac{1}{2}G^{(2)}-\partial^{-1}(vG^{(1)}-uG^{(2)}),\quad C=\frac{1}{2}G^{(2)}+\partial^{-1}(vG^{(1)}-uG^{(2)}).$$

于是有

命题 3 设 $G^{(1)}, G^{(2)}$ 为二个任意光滑函数, $G=(G^{(1)}-G^{(2)})$, 取

$$V=V(G)=\frac{1}{2}\begin{pmatrix} G^{(1)} & G^{(2)}-2\partial^{-1}(vG^{(1)}-uG^{(2)}) \\ G^{(2)}+2\partial^{-1}(vG^{(1)}-uG^{(2)}) & -G^{(1)} \end{pmatrix},$$

则有

$$[V, L] = L'(KG) - L'(JG)L \cdot \tag{14}$$

命题 4　设 $G_j = (G_j^{(1)}, G_j^{(2)})^T$ 为(4)所定义的Lenard递推序列, 取 $V_j = V(G_j)$, $W_m = \sum_{j=0}^{m} V_{j-1} L^{m-j}$, 则

$$[W_m, L] = L'(JG_m) \cdot$$

证明　$[W_m, L] = \sum_{j=0}^{m} [V_j, L] L^{m-j} =$

$$\sum_{m=j=0} (L'(KG_{j-1})L^{m-j} - L'(JG_{j-1})L^{m-j+1}) = L'(JG_m) \cdot$$

定理 2　发展方程族(6)具有Lax表示

$$L_t = [W_m, L] \quad (m = 0, 1, \cdots),$$

即方程(6)为 $L\psi = \lambda\psi$, $\psi_t = W_m\psi$ 的相容条件•

3　方程(1)的非线性化和一个有限维可积系统

假设 $\lambda_1, \lambda_2, \cdots, \lambda_N$ 为方程(1) N 个不同的特征值, (q_j, p_j) 为相应的特征函数, 则 λ_j 的泛函梯度为

$$\nabla \lambda_j = \left(\frac{\delta \lambda_j}{\delta u}, \frac{\delta \lambda_j}{\delta u}\right)^T = (2q_j p_j, -q_j^2 + p_j^2)^T \cdot \tag{15}$$

命题 5　方程族(6)中的 K, J 为Lenard算子对, 即

$$K \nabla \lambda_j = \lambda_j J \nabla \lambda_j \cdot \tag{16}$$

证明　利用方程(1), 可得如下关系

$$(q_j p_j)_x = \lambda_j(-q_j^2 + p_j^2) + v(q_j^2 + p_j^2),$$

$$(-q_j^2 + p_j^2)_x = -4\lambda_j q_j p_j - 2u(q_j^2 + p_j^2),$$

$$q_j^2 + p_j^2 = 4\partial^{-1} v q_j p_j - 2\partial^{-1} u(-q_j^2 + p_j^2) \cdot$$

由此即得(16)•

考虑Bargmann约束 $G_0 = \sum_{j=0}^{N} \nabla \lambda_j$, 即

$$u = \langle q, p \rangle, \quad v = \frac{1}{2}(\langle p, p \rangle - \langle q, q \rangle), \tag{17}$$

其中 $q = (q_1, q_2, \cdots, q_N)^T$, $p = (p_1, p_2, \cdots, p_N)$, 而 $\langle \cdot, \cdot \rangle$ 表示 $\mathbf{R}^N$ 中的内积, 在Bargmann约束(17)之下, 特征值问题(1)被非线性化为

$$q_x = \langle q, p \rangle q + \frac{1}{2}(\langle p, p \rangle - \langle q, q \rangle) p + \Lambda p = -\frac{\partial H}{\partial p}, \tag{18a}$$

$$p_x = -\langle q, p \rangle p + \frac{1}{2}(\langle p, p \rangle - \langle q, q \rangle) q - \Lambda q = \frac{\partial H}{\partial q}, \tag{18b}$$

其中 $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$, Hamilton函数 H 为

$$H = \frac{1}{4}(\langle p, p \rangle - \langle q, q \rangle)\langle q, q \rangle - \frac{1}{4}\langle p, p \rangle^2 - \frac{1}{2}\langle q, p \rangle^2 - \frac{1}{2}\langle \Lambda p, p \rangle - \frac{1}{2}\langle \Lambda q, q \rangle,$$

在辛空间 $(\mathbf{R}^{2N}, dp \wedge dq)$ 上二个函数的Poisson括号定义为

$$(F, G) = \sum_{j=1}^{N} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right) = \left\langle \frac{\partial F}{\partial q}, \frac{\partial G}{\partial p} \right\rangle - \left\langle \frac{\partial F}{\partial p}, \frac{\partial G}{\partial q} \right\rangle,$$

可以证明它为斜对称、双线性且满足Jacobi恒等式, 特别如果 $(F, G) = 0$, 则称 F, G 可换•

假设 $\lambda_1 < \lambda_2 < \cdots < \lambda_N$, 并定义

$$\Gamma_k = \sum_{j=1, j\neq k}^{N} \frac{B_{kj}}{\lambda_k - \lambda_j}, \quad B_{kj} = p_k q_j - p_j q_k.$$

根据等式[5,6]

$$(\langle \boldsymbol{q}, \boldsymbol{p}\rangle, q_l p_l) = (\langle \boldsymbol{q}, \boldsymbol{p}\rangle, \Gamma_l) = (\Gamma_k, \Gamma_l) = 0,$$
$$(p_k^2, p_l^2) = (q_k^2, q_l^2) = (q_k p_k, q_l p_l) = 0,$$
$$(\langle \boldsymbol{q}, \boldsymbol{p}\rangle, p_l^2) = 2p_l^2, \quad (\langle \boldsymbol{q}, \boldsymbol{p}\rangle, q_l^2) = -2q_l^2,$$
$$(q_k, p_k, p_j^2) = 2p_k p_l \delta_{kl}, \quad (q_k^2, p_l^2) = 4q_k p_l \delta_{kl}, \quad (q_k^2, q_l p_l) = 2q_k q_l \delta_{kl},$$
$$(p_k^2, \langle \boldsymbol{q}, \boldsymbol{q}\rangle) = -4p_k q_k, \quad (q_k^2, \langle \boldsymbol{p}, \boldsymbol{p}\rangle) = 4p_k q_k,$$
$$(\langle \boldsymbol{q}, \boldsymbol{q}\rangle, \langle \boldsymbol{p}, \boldsymbol{p}\rangle) = 4\langle \boldsymbol{q}, \boldsymbol{p}\rangle, \quad (\langle \boldsymbol{q}, \boldsymbol{q}\rangle, \langle \boldsymbol{q}, \boldsymbol{p}\rangle) = 2\langle \boldsymbol{q}, \boldsymbol{q}\rangle,$$
$$(\Gamma_k, q_l^2) = \frac{4B_{kl}}{\lambda_k - \lambda_l} q_k, q_l, \quad (\Gamma_k, p_l^2) = \frac{4B_{kl}}{\lambda_k - \lambda_l} p_k, p_l, \quad (\Gamma_k, q_l p_l) = \frac{2B_{kl}}{\lambda_k - \lambda_l}(q_k p_l + p_k q_l)$$

可证明如下结论

命题 6　定义

$$E_k = \frac{1}{4}(\langle \boldsymbol{p}, \boldsymbol{p}\rangle - \langle \boldsymbol{q}, \boldsymbol{q}\rangle) q_k^2 - \frac{1}{4}\langle \boldsymbol{p}, \boldsymbol{p}\rangle p_k^2 - \frac{1}{2}\langle \boldsymbol{q}, \boldsymbol{p}\rangle q_k p_k - \frac{1}{2}\lambda_k p_k^2 - \frac{1}{2}\lambda_k q_k^2 + \frac{1}{2}\Gamma_k$$

则 $E_1, E_2, \cdots, E_N$ 构成两两可换系统, 即$(E_k, E_l) = 0$.

在 $\mathbf{R}^N$ 上定义一个双线性函数

$$Q_z(\xi, \eta) = \langle (z - \Lambda)^{-1}\xi, \eta\rangle = \sum_{k=1}^{N} (z - \lambda_k)^{-1}\xi_k \eta_k = \sum_{m=0}^{\infty} z^{-m-1}\langle \Lambda^m \xi, \eta\rangle. \tag{19}$$

则可换系统$\{E_k\}$的生成函数为

$$F = \frac{1}{4}(\langle \boldsymbol{p}, \boldsymbol{p}\rangle - \langle \boldsymbol{q}, \boldsymbol{q}\rangle) Q_z(\boldsymbol{q}, \boldsymbol{q}) - \frac{1}{4}(\langle \boldsymbol{p}, \boldsymbol{p}\rangle Q_z(\boldsymbol{p}, \boldsymbol{p}) - \frac{1}{2}\langle \boldsymbol{q}, \boldsymbol{p}\rangle Q_z(\boldsymbol{q}, \boldsymbol{p}) - \frac{1}{2}Q_z(\Lambda \boldsymbol{p}, \boldsymbol{p}) - \frac{1}{2}Q_z(\Lambda \boldsymbol{q}, \boldsymbol{q}) + \frac{1}{2}\begin{vmatrix} Q_z(\boldsymbol{q}, \boldsymbol{q}) & Q_z(\boldsymbol{q}, \boldsymbol{p}) \\ Q_z(\boldsymbol{p}, \boldsymbol{q}) & Q_z(\boldsymbol{p}, \boldsymbol{p}) \end{vmatrix} = \sum_{k=1}^{N} \frac{E_k}{z - \lambda_k}. \tag{20}$$

命题 7　如下定义的函数两两可换, 且 $F_m = \sum_{k=1}^{N} \lambda_k^m E_k$, $m = 1, 2, \cdots$

$$F_0 = H = \frac{1}{4}(\langle \boldsymbol{p}, \boldsymbol{p}\rangle - \langle \boldsymbol{q}, \boldsymbol{q}\rangle)\langle \boldsymbol{q}, \boldsymbol{q}\rangle - \frac{1}{4}(\langle \boldsymbol{p}, \boldsymbol{p}\rangle^2 - \frac{1}{2}\langle \boldsymbol{q}, \boldsymbol{p}\rangle^2 - \frac{1}{2}(\langle \Lambda \boldsymbol{p}, \boldsymbol{p}\rangle - \frac{1}{2}(\Lambda \boldsymbol{q}, \boldsymbol{q}),$$

$$F_m = \frac{1}{4}(\langle \boldsymbol{p}, \boldsymbol{p}\rangle - \langle \boldsymbol{q}, \boldsymbol{q}\rangle)\langle \Lambda^m \boldsymbol{q}, \boldsymbol{q}\rangle - \frac{1}{4}\langle \boldsymbol{p}, \boldsymbol{p}\rangle\langle \Lambda^m \boldsymbol{p}, \boldsymbol{p}\rangle - \frac{1}{2}\langle \boldsymbol{q}, \boldsymbol{p}\rangle\langle \Lambda^m \boldsymbol{q}, \boldsymbol{p}\rangle - \frac{1}{2}\langle \Lambda^{m+1}\boldsymbol{p}, \boldsymbol{p}\rangle - \frac{1}{2}\langle \Lambda^{m+1}\boldsymbol{q}, \boldsymbol{q}\rangle + \frac{1}{2}\sum_{j=1}^{m}\begin{vmatrix} \langle \Lambda^{j-1}\boldsymbol{q}, \boldsymbol{q}\rangle & \langle \Lambda^{j-1}\boldsymbol{q}, \boldsymbol{p}\rangle \\ \langle \Lambda^{m-j}\boldsymbol{p}, \boldsymbol{q}\rangle & \langle \Lambda^{m-j}\boldsymbol{p}, \boldsymbol{q}\rangle \end{vmatrix}.$$

证明　将(19)中 Q_z 的 Laurent 展开及$(z - \lambda_k)^{-1} = \sum_{m=0}^{\infty} z^{-m-1}\lambda_k^m$ 代入(20)两边, 可得

$$F = \sum_{m=0}^{\infty} z^{-m-1} F_m = \sum_{m=0}^{\infty} z^{-m-1}\left(\sum_{k=1}^{N} \lambda_k^m E_k\right) = \sum_{k=1}^{N} \frac{E_k}{z - \lambda_k}.$$

定理 3　在 Bargmann 约束(17)之下, 由(18)定义的 Hamilton 系统$(\mathbf{R}^{2N}, \mathrm{d}\boldsymbol{p}\wedge\mathrm{d}\boldsymbol{q}, H = F_0)$

在Liouville意义下完全可积•

致谢 作者之一(范恩贵)感谢导师谷超豪院士、胡和生院士的热情指导和帮助•

[参 考 文 献]

[1] TU Gui_zhang. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems[J]. J Math Phys, 1989, **30**(2): 330—338.

[2] TU Gui_zhang. On Liouville integrability of zero_curvature equations and the Yang hierarchy[J]. J Phys A: Math Gen, 1989, **22**(13): 2375—2392.

[3] TU Gui_zhang. A trace identity and its applications to the theory of discrete integrable systems[J]. J Phys A: Math Gen, 1990, **23**(17): 3903—3922.

[4] CAO Ce_wen. AKNS族的Lax方程组的非线性化[J]. 中国科学 A, 1990, **33**(3): 528—536.

[5] CAO Ce_wen, GENG Xian_guo. Classical integrable systems generated through nonlinearization of eigenvalue problems[Z]. In: C H Gu Ed. Research Reports in Physics[R]. Berlin: Springer_Verlag, 1990, 68—78.

[6] CAO Ce_wen, GENG Xian_guo. C neumann and bargmann systems associated with the coupled KdV soliton hierarch[J]. J Phys A: Math Gen, 1990, **23**(18): 4117—4125.

[7] GENG Xiang_guo. A hierarchy of nonlinear evolution equation, its Hamiltonian structure and classical integrable systems[J]. Physica A, 1992, **180**(1～2): 241—251.

[8] QIAO Zhi_jun. A new completely integrable Liouville's system produced by the Kaup_Newell eigenvalue problem[J]. J Math Phys, 1993, **34**(7): 3100—3121.

[9] ZENG Yun_bo. An approach to the deduction of the finite_dimensional integrability form the infinite_dimensional integrability[J]. Phys Lett A, 1991, **160**(6): 541—547.

A New Completely Integrable Liouville's System, Its Lax Representation and Bi_Hamiltonian Structure

FAN En_gui[1], ZHANG Hong_qing[2]

(1. Institute of Mathematics, Fudan University, Shanghai 200433, P R China;
2. Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, P R China)

Abstract: A new isospectral problem and the corresponding hierarchy of nonlinear evolution equations is presented. As a reduction, the well_known MKdV equation is obtained. It is shown that the hierarchy of equations is integrable in Liouville's sense and possesses Bi_Hamiltonian structure. Under the constraint between the potentials and eigenfunctions, the eigenvalue problem can be nonlinearized as a finite dimensional completely integrable system.

Key words: integrable system; Lax representation; Bi_Hamiltonian structure