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Newton_Riemann 时空中的动力学(Ⅳ)*

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(钱伟长推荐)

摘要: 讨论了 Newton_Riemann 时空中的 Lagrange 力学及其与 N_R 时空中的 Newton 力学及 Hamilton 力学的关系•

关 键 词: Riemann 流形; 切丛; 余切丛; 纤维丛; 纤维; 向量场; 形式场; 外微分; 绝对微分; Lie 导数; 泛函; 变分

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引 言

Lagrange 力学是分析力学的重要组成部分,它是 Newton 力学的推广•

Newton 力学建立在运动系统的构形空间——Riemann 流形上, Lagrange 力学建立在它的切丛上, Hamilton 力学建立在它的余切丛或辛流形上• 为了用 Lagrange 力学的理论处理时间相关的力学问题或非平稳的物理过程则不能在上述的切丛上考虑• 如构形空间是 Riemann 流形 M^n, TM 是切丛, $T^* M$ 是余切丛, 则一般作者在 $TM \times R$ 上建立 Lagrange 力学, 在 $T^* M \times R$ 上建立 Hamilton 力学• 我们以自然的方式在 N_R 时空 $N = R^1 \times M^n$ 及其切丛 TN 和余切丛 $T^* N$ 上考虑上述相应的问题, 推广到更一般的情形得到更一般广泛深入的结果, 进一步可沟通相对论力学• 这样 Riemann 流形 M^n 上的动力学是 N_R 时空 $N = R^1 \times M^n$ 中的动力学的特殊情形•

本文主要研究 TN 上的 Lagrange 力学及其与 N 上的 Newton 力学和 $T^* N$ 上的 Hamilton 力学的关系,并给出分析力学的现代数学方法• 可参阅[1~5]•

1 几何结构

时空 $N = R^1 \times M^n$ 的几何结构如(Ⅰ)、(Ⅱ)(见: 应用数学和力学, 2000 年 21 卷 7 期, 746—754 页; 2001 年 22 卷 4 期, 373—381 页) 中所述, 为便于后面的叙述, 再简述如下•

设 M^n 是具有度量 g 及连络 D 的 n_维 Riemann 流形, 代表空间• R^1 是具有度量 g_0 的 1_维 Riemann 流形, 代表 Newton 观点的时间• Newton_Riemann 时空是积流形 $N = R^1 \times M^n$ ($M^3 = R^3$, 则 $N = R^1 \times R^3$ 是通常的 Newton_Galile 时空) 是 $n+1$ 维 Riemann 流形• 在局部坐标系 $(I \times U, \phi = id \times \varphi)$ 下,

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$$\forall(t, \boldsymbol{x}) \in I \times U \subset N, \quad \phi:(t, \boldsymbol{x}) \mapsto (t, x^1, \cdots, x^n), \tag{1}$$

则 N 上的时空度量, 在 $(I \times U, \phi)$ 下为

$$\boldsymbol{g} = \boldsymbol{g}_0 + \boldsymbol{g} = \mathrm{d}t \otimes \mathrm{d}t + g_{jk}\mathrm{d}x^j \otimes \mathrm{d}x^k, \tag{2}$$

$\forall(t, \boldsymbol{x}) \in N$, 切空间

$$T_{(t,\boldsymbol{x})}N \approx T_tR^1 \oplus T_{\boldsymbol{x}}M; \tag{3}$$

余切空间

$$T^*_{(t,\boldsymbol{x})}N \approx T^*_tR^1 \oplus T^*_{\boldsymbol{x}}M; \tag{4}$$

切丛

$$TN = \bigcup_{(t,\boldsymbol{x})\in N} T_{(t,\boldsymbol{x})}N; \tag{5}$$

余切丛

$$T^*N = \bigcup_{(t,\boldsymbol{x})\in N} T^*_{(t,\boldsymbol{x})}N. \tag{6}$$

设 $\left\{\partial/\partial t, \partial/\partial x^j\right\}_{j=1}^n$ 为 TN 在 $I \times U$ 上的标架场, 则每一向量场 $\boldsymbol{V} \in \mathscr{X}^r(N)$ 可表为

$$\boldsymbol{V} = \partial/\partial t + v^j(t, \boldsymbol{x})\partial/\partial x^j;$$

$\left\{\mathrm{d}t, \mathrm{d}x^j\right\}_{j=1}^n$ 为 T^*N 在 $(I \times U)$ 上的标架场, 则每一形式场 $\omega \in \mathscr{F}^1(N)$ 在 $I \times U$ 上可表为

$$\omega = \alpha\mathrm{d}t + \beta_j\mathrm{d}x^j;$$

$(t, \boldsymbol{x})$ 是 N 的点, 则 $(t, \boldsymbol{x}, 1, \boldsymbol{v})$ 是 TN 的点, $(t, \boldsymbol{x}, 1, \boldsymbol{p})$ 是 T^*N 的点.

设 $\Pi: TN \to N, \ (t, \boldsymbol{x}, 1, \boldsymbol{v}) \mapsto (t, \boldsymbol{x})$,

$\Pi^*: T^*N \to N, \ (t, \boldsymbol{x}, 1, \boldsymbol{p}) \mapsto (t, \boldsymbol{x})$

是丛投影, $(\Pi^{-1}(I \times U), \phi)$ 是 TN 的局部坐标系,

$$\phi:(t, \boldsymbol{x}, 1, \boldsymbol{v}) \in \Pi^{-1}(I \times U) \mapsto (t, x^1, \cdots, x^n, 1, v^1, \cdots, v^n) \in R^{n+1} \times R^{n+1},$$

$(\Pi^{*-1}(I \times U), \phi^*)$ 是 T^*N 的局部坐标系,

$$\phi^*:(t, \boldsymbol{x}, 1, \boldsymbol{p}) \in \Pi^{*-1}(I \times U) \mapsto (t, x^1, \cdots, x^n, 1, p_1, \cdots, p_n) \in R^{n+1} \times R^{n+1},$$

$N = R^1 \times M^n$ 是纤维丛: $N \xrightarrow{\Pi} R^1, \ (t, \boldsymbol{x}) \overset{\Pi}{\mapsto} t$,

$\forall t \in R^1$, 纤维 $\Pi_1^{-1}(t) = N_t$, (也记作 M^n_t) 是 n_维 Riemann 流形 M^n_t.

2 N_R 时空中的 Lagrange 力学

设 $I = [t_0, t_1] \subset R^1$,

$$\gamma: I \to N, \qquad t \mapsto (t, \boldsymbol{x}(t)) \in N \tag{7}$$

是时空 N 中的曲线, 称世界线, 其全体记作 $C^k(I, N)$, k 是 k 次连续可微.

γ 在 TN 中的提升:

$$\dot{\gamma}: I \to TN, \ t \mapsto \dot{\gamma}(t) = (t, \boldsymbol{x}(t), 1, \dot{\boldsymbol{x}}(t)); \tag{8}$$

在 T^*N 中的提升:

$$\tilde{\gamma}: I \to T^*N, \ t \mapsto \tilde{\gamma}(t) = (t, \boldsymbol{x}(t), 1, \boldsymbol{p}(t)); \tag{9}$$

映射

$$\mathscr{L}: TN \to T^*N, (t, \boldsymbol{x}, 1, \boldsymbol{v}) \mapsto (t, \boldsymbol{x}, 1, \boldsymbol{p}) \tag{10}$$

是丛同构, 且

$$\tilde{\gamma} = \mathscr{L}(\dot{\gamma}). \tag{11}$$

在局部坐标系 $(I \times U, \phi)$ 下,

$$\gamma(t) = (t, \boldsymbol{x}(t)) = (t, x^1(t), \cdots, x^n(t)), \tag{12}$$

故 $$V = \dot{\gamma} = (1, \dot{x}(t)) = (1, \dot{x}^1(t), \cdots, \dot{x}^n(t)) = \partial/\partial t + \dot{x}^j(t)\partial/\partial x^j \in T_{\gamma(t)}N, \tag{13}$$

$$V^\flat = \mathrm{d}t + g_{jk}\dot{x}^j\mathrm{d}x^k = \mathrm{d}t + p_k\mathrm{d}x^k \in T^*_{\gamma(t)}N, \tag{14}$$

且 $$V^\flat = \mathscr{L}(V), \quad p_k = g_{jk}\dot{x}^j \tag{15}$$

又

$$\gamma(t) = (t, x^1(t), \cdots, x^n(t), 1, \dot{x}^1(t), \cdots, \dot{x}^n(t)) = (t, x^j(t), 1, \dot{x}^j(t)), \tag{16}$$

$$V = \dot{\gamma} = (1, \dot{x}^j, 0, \ddot{x}^j) = \partial/\partial t + \dot{x}^j\partial/\partial x^j + \ddot{x}^j\partial/\partial\dot{x}^j \in T_{\gamma(t)}(TN), \tag{17}$$

$$\gamma(t) = (t, x^1(t), \cdots, x^n(t), 1, p_1(t), \cdots, p_n(t)) = (t, x^k(t), 1, p_k(t)), \tag{18}$$

$$X = \dot{\gamma} = (1, \dot{x}^k, 0, \dot{p}_k) = \partial/\partial t + \dot{x}^k\partial/\partial x^k + \dot{p}_k\partial/\partial p_k \in T_{\gamma(t)}(T^*N); \tag{19}$$

$$\begin{cases} \gamma^* t = t \circ \gamma = t, \quad \gamma^*\mathrm{d}t = \mathrm{d}\gamma^* t = \mathrm{d}t, \\ \gamma^* \boldsymbol{x} = \boldsymbol{x} \circ \gamma = \boldsymbol{x}(t), \quad \gamma^*\mathrm{d}\boldsymbol{x} = \mathrm{d}\gamma^*\boldsymbol{x} = \mathrm{d}\boldsymbol{x} \circ \gamma, \\ \gamma^* \boldsymbol{v} = \boldsymbol{v} \circ \gamma = \dot{\boldsymbol{x}}(t), \\ \gamma^* t = t \circ \gamma = t, \quad \gamma^*\mathrm{d}t = \mathrm{d}\gamma^* t = \mathrm{d}t, \\ \gamma^* \boldsymbol{x} = \boldsymbol{x} \circ \gamma = \boldsymbol{x}(t), \quad \gamma^*\mathrm{d}\boldsymbol{x} = \mathrm{d}\gamma^*\boldsymbol{x} = \mathrm{d}\boldsymbol{x} \circ \gamma, \\ \gamma^* \boldsymbol{p} = \boldsymbol{p} \circ \gamma = \boldsymbol{p}(t), \quad \gamma^*\mathrm{d}\boldsymbol{p} = \mathrm{d}\gamma^*\boldsymbol{p} = \mathrm{d}\boldsymbol{p} \circ \gamma, \\ \mathscr{L}^* t = t \circ \mathscr{L} = t, \quad \mathscr{L}^* 1 = 1 \circ \mathscr{L} = 1, \\ \mathscr{L}^* \boldsymbol{x} = \boldsymbol{x} \circ \mathscr{L} = \boldsymbol{x}, \quad \mathscr{L}^* p_k = p_k \circ \mathscr{L} = g_{jk}\dot{x}^j; \end{cases} \tag{20}$$

$$\gamma(t) = \mathscr{L} \circ \gamma(t) = (t, x^k(t), 1, g_{jk}\dot{x}^j(t)) = (t, x^k(t), 1, p_k(t)). \tag{21}$$

单位质点在 M^n 中运动的动能

$$T = \frac{\langle V^\flat, V\rangle}{2} = \frac{1}{2}g(V, V) = \frac{1}{2}g_{jk}v^j v^k; \tag{22}$$

在 N 中运动的动能:

$$T = g(V, V)/2 = \frac{\langle V^\flat, V\rangle}{2} = T + \frac{1}{2}; \tag{23}$$

势能 $U = U(t, \boldsymbol{x})$, 则总机械能 E 与 E:

$$E = T + U = \left(T + U + \frac{1}{2}\right) - \frac{1}{2} = E - \frac{1}{2} = T + U - \frac{1}{2}, \tag{24}$$

二者等效. 由

$$E = \frac{\langle V^\flat, V\rangle}{2} + U = \langle V^\flat, V\rangle - \left(\frac{\langle V^\flat, V\rangle}{2} - U\right) =$$
$$\langle V^\flat, V\rangle - L, \quad L = T - U,$$

于是

$$E \circ \mathscr{L}^{-1}(t, \boldsymbol{x}, 1, \boldsymbol{p}) = (\langle V^\flat, V\rangle - L) \circ \mathscr{L}^{-1}(t, \boldsymbol{x}, 1, \boldsymbol{p}) =$$
$$g^{lk}p_k p_l - L \circ \mathscr{L}^{-1}(t, \boldsymbol{x}, 1, \boldsymbol{p}) = H(t, \boldsymbol{x}, 1, \boldsymbol{p}) \in \mathscr{F}^0(T^*N),$$
$$L(t, \boldsymbol{x}, 1, \boldsymbol{v}) = g_{jk}v^j v^k - H \circ \mathscr{L}(t, \boldsymbol{x}, 1, \boldsymbol{v}) = (p_k v^k - H) \circ \mathscr{L}(t, \boldsymbol{x}, 1, \boldsymbol{v}), \tag{25}$$

故 $$L\mathrm{d}t = (p_k\mathrm{d}x^k - H\mathrm{d}t) \circ \mathscr{L} = \theta \circ \mathscr{L}, \tag{26}$$

其中 $L \in \mathscr{F}^0(TN), (\in C^k(TN, R)), \; L\mathrm{d}t \in \mathscr{F}^1(TN)$.

又 $\gamma^* t = (\mathscr{L} \circ \gamma)^* t = \gamma^*\mathscr{L}^* t = \gamma^* t = t,$

故 $$L \circ \gamma\mathrm{d}t = \gamma^*(L\mathrm{d}t) = \gamma^*(\theta \circ \mathscr{L}) = \theta \circ \mathscr{L} \circ \gamma = \theta \circ \gamma. \tag{27}$$

如上, $\gamma \in C^k(I, N)$, 则 $\gamma \in C^{k-1}(I, TN)$, $\gamma = \mathscr{L} \circ \gamma \in C^r(I, T^*N)$.

泛函 $$J(\gamma) = \int_{t_0}^{t_1} L(t, \boldsymbol{x}(t), 1, \dot{\boldsymbol{x}}(t))\,\mathrm{d}t = \int_{t_0}^{t} L \circ \gamma\,\mathrm{d}t = \int_{t_0}^{t_1} \theta \circ \gamma, \tag{28}$$

即 $$J(\gamma) = \int_\gamma L\,\mathrm{d}t = \int_\gamma \theta \circ \mathscr{L} = \int_\gamma \theta。 \tag{29}$$

这样, J 是 $C^k(I, N)$ 上的 C^l 类泛函, 分别表为 1_形式 $L\,\mathrm{d}t \in \mathscr{F}^1(TN)$ 及 $\theta \in \mathscr{F}^1(T^* N)$ 在曲线 $\gamma \subset TN$ 及 $\gamma \subset T^* N$ 上的线积分。 设 J 的定义域为

$$\mathscr{D}(J) = \Big\{ \gamma \in C^k(I, N) \mid \gamma(t_0) = (t_0, x_0),\ \gamma(t_1) = (t_1, x_1),\ x_0 = \boldsymbol{x}(t_0),\ x_1 = \boldsymbol{x}(t_1),\ \text{固定} \Big\}。$$

下面考虑 J 的变分问题。 N 是 Riemann 流形, 不是特殊的 R^n, 不能象在 R^n 中那样, 简单地令 $\gamma_s = \gamma + sh$ 而考虑 J 的变分问题。 现考虑如下:

设曲线 $\gamma \in C^k(I, N)$ 全部位于坐标邻域 $I \times U$ 中, 且端点固定。 不然, 则取 γ 在 $I \times U$ 中的部分, 并固定这部分上的不同的两点为 $\gamma(t_0)$、$\gamma(t_1)$。 γ 在 $I \times U$ 中的这部分也是 $\mathscr{D}(J)$ 的元素。 设此曲线或其在 $I \times U$ 中的部分, 在任意向量场 $\boldsymbol{u} \neq \dot{c}$ 产生的单参数微分同胚群 $\{\sigma_s\}_{s \in I}$ 作用下, 微小变形, 端点固定, 从而得到 $\gamma_s(t) = \sigma_s \circ \gamma(t) = (t + s, \boldsymbol{x} \circ \sigma_s \circ \gamma(t)) \in \mathscr{D}(J)$, $t + s \in I_1 \subset I$, 则泛函 $J(\gamma)$ 变为

$$h(s) = J(\gamma_s)。 \tag{30}$$

对每一固定的 $t \in I$ 是 s 的函数。 由微积分知, 若它在 $s = 0$ 达到极值, 则必须

$$\frac{\mathrm{d}}{\mathrm{d}s} h(s)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} J(\gamma_s)\Big|_{s=0} = 0。 \tag{31}$$

于是

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} J(\gamma_s)\Big|_{s=0} = \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}s} L \circ \gamma_s(t)\Big|_{s=0} \mathrm{d}t, \tag{32}$$

而

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} L \circ \gamma_s(t) &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x^j} \frac{\partial x^j \circ \sigma_s(\gamma(t))}{\partial s} + \frac{\partial L}{\partial \dot{x}^j} \frac{\partial \dot{x}^j \circ \sigma_s(\gamma(t))}{\partial s} = \\ &\frac{\partial L}{\partial t} + u^j(\sigma_s(\gamma(t))) \frac{\partial L}{\partial x^j} + \frac{\mathrm{d}}{\mathrm{d}t} u^j(\sigma_s(\gamma(t))) \frac{\partial L}{\partial \dot{x}^j} = \\ &L_{\boldsymbol{u}(\gamma_s(t))} L \circ \gamma_s(t),\end{aligned} \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}s} L \circ \gamma_s(t)\Big|_{s=0} = L_{\boldsymbol{u}(\gamma(t))} L \circ \gamma(t), \tag{34}$$

其中

$$\boldsymbol{u} = \partial/\partial t + u^j \partial/\partial x^j + \frac{\mathrm{d}u^j}{\mathrm{d}t} \partial/\partial \dot{x}^j \in \mathscr{X}(TN), \tag{35}$$

$$\Pi_* \boldsymbol{u} = \boldsymbol{u} = \partial/\partial t + u^j \partial/\partial x^j \in \mathscr{X}(N), \tag{36}$$

$\boldsymbol{u}$ 是 σ_s 的无穷小生成元。 由(31)、(32)

$$\frac{\mathrm{d}}{\mathrm{d}s} J(\gamma_s)\Big|_{s=0} = \int_{t_0}^{t_1} L_{\boldsymbol{u}(\gamma(t))} L \circ \gamma(t)\,\mathrm{d}t = 0, \tag{37}$$

从而 $$L_{\boldsymbol{u}(\gamma(t))} L \circ \gamma(t) = 0。 \tag{38}$$

定理 2.1　$\gamma \in \mathscr{D}(J)$ 是泛函 J 的平稳点(极值点) 若在 γ 上对任意向量场 $\boldsymbol{u} \in \mathscr{X}^r(N)$, $\boldsymbol{u} \neq \dot{c}$, $L \circ \gamma$ 关于向量 $\boldsymbol{u} \in \mathscr{X}(TN)$ 的 Lie 导数为 0, c 是常数。

又 $$0 = \frac{\mathrm{d}}{\mathrm{d}s} J(\gamma_s)\Big|_{s=0} = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial t} + u^j(\gamma(t)) \frac{\partial L}{\partial x^j} + \frac{\mathrm{d}}{\mathrm{d}t} u^j(\gamma(t)) \frac{\partial L}{\partial \dot{x}^j} \right) \mathrm{d}t =$$

$$\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial t}+\left(\frac{\partial L}{\partial x^j}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^j}\right)u^j(\gamma(t))\right)\mathrm{d}t+\frac{\partial L}{\partial \dot{x}^j}u^j(\gamma(t))\Big|_{t_0}^{t_1},$$

$\boldsymbol{u}=\partial/\partial t+u^j\partial/\partial x^j\in\mathscr{X}^r(N)$ 任意, 且 $\boldsymbol{u}\neq c\dot{\gamma}$, γ 在 $\boldsymbol{u}$ 产生的变换群 $\{\sigma_s\}$ 作用下连续变形而端点不变, 故在端点 $\gamma(t_0)$、$\gamma(t_1)$、$\boldsymbol{u}(\gamma(t_0))=\boldsymbol{u}(\gamma(t_1))=0$. 上式第二项为0. 故有

$$\frac{\partial L\circ\gamma}{\partial t}=0,\quad \frac{\partial L\circ\gamma}{\partial x^j}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L\circ\gamma}{\partial x^j}=0. \tag{39}$$

$\partial L/\partial t=0$ 由时间均匀流逝引起. 它表明 L 不显含 t, 或显含 t 而视 t 不变. (39) 表明, 对每一固定的 $t\in I_1\subset I$, 在 N 的纤维 N_t(也记作 M_t^n) 的切丛 TM_t^n 上 γ 也就是 γ(在 TN 中的提升) 应满足 Lagrange 方程, 而要从 L 导出 Lagrange 方程只要视 L 中的 t 不变. 这对应着对每一瞬时 t 的 Newton 运动定律.

如所周知, L_方程的解(曲线)是泛函达到极值的 $\gamma\in\mathscr{D}(J)$. 于是有

定理 2.2 J 的平稳点 $\gamma\in\mathscr{D}(J)$ 是 L_方程的解.

定理 2.3 $\gamma\in\mathscr{D}(J)$ 是 J 的平稳点, 若且仅若它是 L_方程的解.

在力学上它表现为:

定理 2.4 $\gamma\in C^k(I,N)$ 是一质点的运动轨线, 若下述之一成立:

1. $\gamma\in\mathscr{D}(J)\subset C^k(I,N)$ 是 J 的平稳点;
2. 在 $\gamma\in\mathscr{D}(J)$ 上, $\forall\boldsymbol{u}\in\mathscr{X}^r(N)$, $\boldsymbol{u}\neq c\dot{\gamma}$, $c=$ const,
   $L_{\boldsymbol{u}(\gamma(t))}L\circ\gamma=0$;
3. γ 是 L_方程的解.

注意变换群 $\{\sigma_s\}$ 把 $\gamma\in C^k(I,N)$ 变为 $\gamma_s=\sigma_s(\gamma)\in C^l(I,N)$, $t+s\in I_1\subset I$. 对每一固定的 $t\in I$, $\sigma_s(\gamma(t))$ 是过点 $\gamma(t)\in N$ 以 s 为参数的曲线, 其切向量是 $\boldsymbol{u}=\partial/\partial t+u^j\circ\sigma_s(\gamma(t))\partial/\partial x^j\in T_{\gamma_s(t)}N$. 当 $s=0$ 时, $\boldsymbol{u}=\partial/\partial t+u^j(\gamma(t))\partial/\partial x^j\in T_{\gamma(t)}N$, 但不等于 $\dot{\gamma}$ 且 $\boldsymbol{u}\neq c\dot{\gamma}$, $c=$ const, 如前 $\dot{\gamma}(t)=(1,\dot{\boldsymbol{x}}(t))$ 是 $\gamma(t)=(t,\boldsymbol{x}(t))$ 的切向量.

Lagrange 方程的性质:

$$\forall L\in C^l(TN,R),\ \gamma\in C^k(I,N),\ \dot{\gamma}\in C^{k-1}(I,TN),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L\circ\gamma}{\partial\dot{x}^j}=\frac{\partial^2 L\circ\gamma}{\partial\dot{x}^j\partial t}+\frac{\partial^2 L\circ\gamma}{\partial\dot{x}^j\partial x^k}\frac{\mathrm{d}x^k}{\mathrm{d}t}+\frac{\partial^2 L\circ\gamma}{\partial\dot{x}^j\partial\dot{x}^k}\frac{\mathrm{d}\dot{x}^k}{\mathrm{d}t}.$$

L_方程(39)变为

$$\frac{\partial^2 L}{\partial\dot{x}^j\partial t}+\frac{\partial^2 L}{\partial x^j\partial x^k}\frac{\mathrm{d}x^k}{\mathrm{d}t}+\frac{\partial^2 L}{\partial\dot{x}^j\partial\dot{x}^k}\frac{\mathrm{d}\dot{x}^k}{\mathrm{d}t}=\frac{\partial L}{\partial x^j}, \tag{40}$$

它是关于 γ 的2阶微分方程. 若 $\partial^2 L/\partial\dot{x}^j\partial\dot{x}^k$ 在 $\gamma=(t,\boldsymbol{x},1,\dot{\boldsymbol{x}})$(在 TN 中) 的邻域可逆, 则命 $\mathrm{d}x^k/\mathrm{d}t=v^k$ 得到 TN 中的一阶方程组

$$\left.\begin{aligned}&\dot{t}=1,\quad \frac{\mathrm{d}x^k}{\mathrm{d}t}=v^k,\\&\frac{\mathrm{d}v^k}{\mathrm{d}t}=\left(\frac{\partial L}{\partial x^j}-\frac{\partial^2 L}{\partial\dot{x}^j\partial t}-\frac{\partial^2 L}{\partial\dot{x}^j\partial x^k}v^k\right)\Big/\frac{\partial^2 L}{\partial\dot{x}^j\partial\dot{x}^k},\end{aligned}\right\} \tag{41}$$

其解 $\gamma(t)=(t,\boldsymbol{x}(t),\boldsymbol{v}(t))$ 是 Lagrange 向量场

$$\boldsymbol{V}=\partial/\partial t+v^k\partial/\partial x^k+\left(\frac{\partial L}{\partial x^j}-\frac{\partial^2 L}{\partial\dot{x}^j\partial t}-\frac{\partial^2 L}{\partial\dot{x}^j\partial x^k}v^k\right)\Big/\frac{\partial^2 L}{\partial\dot{x}^j\partial\dot{x}^k}\partial/\partial\dot{x}^k\in\mathscr{X}(TN) \tag{42}$$

的积分曲线, 其投影 $\Pi(\gamma)=\gamma$ 是(40)的解.

定义 2.5 $L \in C^l(TN, R): \pi^{-1}(I \times U) \to R$ 是正则的(regular) 若 $l \geqslant 2$ 且对所有$(t, \boldsymbol{x}, 1, \boldsymbol{v}) \in \pi^{-1}(I \times U)$, $\partial^2 L/\partial v^j \partial v^k$ 是可逆的.

定理 2.6 对正则 L_函数$L \in C^2(TN, R)$, J 的极值曲线 γ 是C^2 类的, 给出任一$(t_0, x_0, 1, v_0) \in \pi^{-1}(I \times U) \subset TN$ 存在包含t_0的唯一的最大区间$I_0 \subset I$, 及唯一的 $\gamma: I_0 \to N$, 使得$\boldsymbol{x}(t_0) = x_0$, $\boldsymbol{v}(t_0, x_0) = v_0$.

证明 由常微分方程解的存在唯一性定理直接得到. 又 $\mathrm{d}x^j/\mathrm{d}t = v^j(t, \boldsymbol{x})$ 是 C^1 类的, 则 γ 是C^2 类的.

特别, $v^j = v^j(t, \boldsymbol{x})$ 明显依赖于 t 时,

$$\frac{\mathrm{d}v^k}{\mathrm{d}t} = \frac{\partial v^k}{\partial t} + \frac{\partial v^k}{\partial x^s} v^s.$$

(41) 是一阶偏微分方程组. 实质上归结为(41) 的后一方程.

特别, 当 $L = T - U = g_{jk} v^j v^k/2 - U$, $v^l = v^l(t, \boldsymbol{x})$, $U = U(t, \boldsymbol{x})$ 时, (41) 的后一方程也就是(40) 变为

$$\frac{\partial v^k}{\partial t} + v^s \frac{\partial v^k}{\partial x^s} + \Gamma^k_{rs} v^r v^s = - g^{jk} \frac{\partial U}{\partial x^j}, \tag{42$'$}$$

这是 Lagrange 方程, 也是 Newton 方程

$$\frac{\mathbf{D}\boldsymbol{V}}{\mathrm{d}t} = - \mathrm{d}U^{\#}. \tag{43}$$

对每一固定的 $t \in I$, 在纤维N_t 中的局部坐标表示. (见后第3节例). 命$\partial L \circ \gamma/\partial \dot{x}^j = p_j$, 由于$\partial^2 L \circ \gamma/\partial \dot{x}^j \partial \dot{x}^k$ 可逆, 由隐函数定理可局部地找到与$\partial L/\partial \dot{x}^j$ 同类的 h_k 使得从中解出$v^k = h_k(t, x^j, p_j)$, 令 $\mathrm{d}x^k/\mathrm{d}t = v^k$ 于是得到方程组

$$\left.\begin{aligned} &\dot{t} = 1, \quad \dot{x}^k = h_k(t, x^l, p_l), \\ &\dot{p}_k = \frac{\partial L}{\partial x^k}(t, x^j, 1, h_j(t, x^l, p_l)). \end{aligned}\right\} \tag{44}$$

这是与(41) 等价的在 T^*N 中关于$t, \boldsymbol{x}, \boldsymbol{p}$ 的一阶方程组. 它是定义在 $\pi^{*-1}(I \times U) \subset T^*N$ 中的Hamilton 方程组, 其解 γ 是Hamilton 向量场

$$\boldsymbol{X} = \partial/\partial t + h_k \partial/\partial x^k + \frac{\partial L}{\partial x^k} \partial/\partial p_k \tag{45}$$

的积分曲线且 $\dot{\gamma} = \boldsymbol{X}$. 因(44) 之右边是 C^1 类的其解 γ 是C^2 类的, $\pi^*(\gamma) = \gamma$ 是C^2 类的. 这样, 泛函 J 达到极值的曲线 γ 在TN 中的提升是L_向量场(42) 的积分曲线, 在 T^*N 中的提升 $\gamma = \mathscr{L}(\gamma)$ 是H_向量场(45) 的积分曲线.

我们一般地进一步考虑如下:

由(28), 考虑泛函

$$J(\gamma_s) = \int_{t_0}^{t_1} \theta \circ \mathscr{L} \circ \gamma_s \tag{46}$$

的极值的问题. 它在 $s = 0$ 达到极值的条件是

$$\frac{\mathrm{d}}{\mathrm{d}s} J(\gamma_s) \mid_{s=0} = \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}s} \theta \circ \mathscr{L} \circ \gamma_s \mid_{s=0} = 0, \tag{47}$$

而

$$\theta \circ \mathscr{L} \circ \gamma_s = g_{jk}(\boldsymbol{x} \circ \sigma_s(\gamma)) \dot{x}^j \circ \sigma_s(\gamma) \mathrm{d}x^k \circ \sigma_s(\gamma) -$$

$$
\begin{aligned}
&H(t\circ\sigma_s(\gamma), x^k\circ\sigma_s(\gamma), 1, g_{jk}(\boldsymbol{x}\circ\sigma_s(\gamma))\dot{x}^j\circ\sigma_s(\gamma))\mathrm{d}t\circ\sigma_s(\gamma),\\
&\frac{\mathrm{d}}{\mathrm{d}s}\theta\circ\mathscr{L}\circ\gamma_s = \frac{\partial g_{jk}(\boldsymbol{x}\circ\sigma_s(\gamma))}{\partial x^l}\frac{\mathrm{d}x^l\circ\sigma_s(\gamma)}{\mathrm{d}s}\dot{x}^j\circ\sigma_s(\gamma)\mathrm{d}x^k\circ\sigma_s(\gamma)+\\
&\quad g_{jk}(\boldsymbol{x}\circ\sigma_s(\gamma))\left[\frac{\mathrm{d}\dot{x}^j\circ\sigma_s(\gamma)}{\mathrm{d}s}\mathrm{d}x^k\circ\sigma_s(\gamma)+\dot{x}^j\circ\sigma_s(\gamma)\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{d}x^k\circ\sigma_s(\gamma)\right]-\\
&\quad\left(\frac{\partial H}{\partial t}+\frac{\partial H}{\partial x^l}\frac{\mathrm{d}x^l\circ\sigma_s(\gamma)}{\mathrm{d}s}+\frac{\partial H}{\partial p_l}\frac{\mathrm{d}p_l\circ\sigma_s(\gamma)}{\mathrm{d}s}\right)\mathrm{d}t\circ\sigma_s(\gamma)-H\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{d}t\circ\sigma_s(\gamma)=\\
&\quad\gamma_s^*\left\{\frac{\partial g_{jk}}{\partial x^l}u^l\dot{x}^j\mathrm{d}x^k+g_{jk}\left[\frac{\mathrm{d}}{\mathrm{d}t}u^j\mathrm{d}x^k+\dot{x}^j\mathrm{d}u^k\right]-\right.\\
&\quad\left.\left[\frac{\partial H}{\partial t}+\frac{\partial H}{\partial x^l}u^l+\frac{\partial H}{\partial p_l}\left(\frac{\partial g_{jl}}{\partial x^r}u^r\dot{x}^j+g_{jl}\frac{\mathrm{d}}{\mathrm{d}t}u^j\right)\right]\mathrm{d}t\right\}=\\
&\quad\gamma_s^*\left\{\left[\left(\frac{\mathrm{d}x^k}{\mathrm{d}t}-\frac{\partial H}{\partial p_k}\right)\frac{\partial g_{jk}}{\partial x^l}u^l\dot{x}^j+\left(\frac{\mathrm{d}x^k}{\mathrm{d}t}-\frac{\partial H}{\partial p_k}\right)g_{jk}\frac{\mathrm{d}u^j}{\mathrm{d}t}\right]\mathrm{d}t+\right.\\
&\quad\left.g_{jk}\dot{x}^j\mathrm{d}u^k-\left(\frac{\partial H}{\partial t}+\frac{\partial H}{\partial x^l}u^l\right)\mathrm{d}t\right\}=\\
&\quad\gamma_s^*\left\{\left(\frac{\mathrm{d}x^k}{\mathrm{d}t}-\frac{\partial H}{\partial p_k}\right)\left(\frac{\partial g_{jk}}{\partial x^l}u^l\dot{x}^j+g_{jk}\frac{\mathrm{d}u^j}{\mathrm{d}t}\right)+\left[\frac{\mathrm{d}}{\mathrm{d}t}(g_{jk}\dot{x}^ju^k)-\frac{\mathrm{d}}{\mathrm{d}t}(g_{jk}\dot{x}^j)u^k\right]-\right.\\
&\quad\left.\left(\frac{\partial H}{\partial x^k}u^k+\frac{\partial H}{\partial t}\right)\right\}\mathrm{d}t=\\
&\quad\gamma_s^*\left\{\left[\left(\frac{\mathrm{d}x^k}{\mathrm{d}t}-\frac{\partial H}{\partial p_k}\right)\frac{\partial p_k}{\partial s}-\left(\frac{\mathrm{d}}{\mathrm{d}t}p_k+\frac{\partial H}{\partial x^k}\right)u^k-\frac{\partial H}{\partial t}+\frac{\mathrm{d}}{\mathrm{d}t}(p_ku^k)\right]\mathrm{d}t\right\},
\end{aligned}
$$

其中

$$
\begin{aligned}
&p_k = g_{jk}\dot{x}^j,\quad \frac{\partial x^j\circ\sigma_s(\gamma)}{\partial s}=u^j\circ\sigma_s(\gamma),\\
&\frac{\partial g_{jk}}{\partial x^l}u^l\dot{x}^j+g_{jk}\frac{\mathrm{d}u^j}{\mathrm{d}t}=\frac{\partial g_{jk}}{\partial x^l}u^l\dot{x}^j+g_{jk}\frac{\partial\dot{x}^j}{\partial s}=\frac{\partial}{\partial s}(g_{jk}\dot{x}^j)=\frac{\partial p_k}{\partial s}.
\end{aligned}
$$

另方面,

$$
\begin{aligned}
&\gamma_s(t)=\mathscr{L}\circ\gamma_s(t)=(t+s, x^k\circ\sigma_s(\gamma(t)), 1, g_{jk}(\boldsymbol{x}\circ\sigma_s(\gamma(t)))\dot{x}^j\circ\sigma_s(\gamma(t))),\\
&\frac{\mathrm{d}}{\mathrm{d}s}\gamma_s=\partial/\partial t+\frac{\mathrm{d}x^k\circ\sigma_s(\gamma(t))}{\mathrm{d}s}\partial/\partial x^k+\frac{\mathrm{d}p_k\circ\sigma_s(\gamma(t))}{\mathrm{d}s}\partial/\partial p_k,\quad p_k=g_{jk}\dot{x}^j,\\
&\boldsymbol{u}=\left.\frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\right|_{s=0}=\partial/\partial t+u^k(\gamma(t))\partial/\partial x^k+\frac{\mathrm{d}p_k(\gamma(t))}{\mathrm{d}s}\partial/\partial p_k\in T_\gamma(T^*N),
\end{aligned}
$$

其中 $u^k=\mathrm{d}x^k/\mathrm{d}s$, 且 $\Pi_*\boldsymbol{u}=\boldsymbol{u}$.

又

$$
\begin{aligned}
&\theta=p_k\mathrm{d}x^k-H\mathrm{d}t,\\
&\boldsymbol{u}\lrcorner\theta=-H+p_ku^k,\\
&\mathrm{d}\theta=\mathrm{d}p_k\wedge\mathrm{d}x^k-\frac{\partial H}{\partial x^k}\mathrm{d}x^k\wedge\mathrm{d}t-\frac{\partial H}{\partial p_k}\mathrm{d}p_k\wedge\mathrm{d}t,\\
&\boldsymbol{u}\lrcorner\mathrm{d}\theta=\frac{\partial H}{\partial x^k}\mathrm{d}x^k+\frac{\partial H}{\partial p_k}\mathrm{d}p_k-u^kp_k+\frac{\partial p_k}{\partial s}\mathrm{d}x^k-u^k\frac{\partial H}{\partial x^k}\mathrm{d}t-\frac{\partial p_k}{\partial s}\frac{\partial H}{\partial p_k}\mathrm{d}t,\\
&\mathrm{d}(\boldsymbol{u}\lrcorner\theta)=-\left(\frac{\partial H}{\partial t}\mathrm{d}t+\frac{\partial H}{\partial x^k}\mathrm{d}x^k+\frac{\partial H}{\partial p_k}\mathrm{d}p_k\right)+\mathrm{d}(p_ku^k),\\
&\mathrm{L}_{\boldsymbol{u}}\theta=\boldsymbol{u}\lrcorner\mathrm{d}\theta+\mathrm{d}(\boldsymbol{u}\lrcorner\theta)=
\end{aligned}
$$

$$\left[-\left(\frac{\mathrm{d}p_k}{\mathrm{d}t}+\frac{\partial H}{\partial x^k}\right)u^k+\left(\frac{\mathrm{d}x^k}{\mathrm{d}t}-\frac{\partial H}{\partial p_k}\right)\frac{\partial p_k}{\partial s}-\frac{\partial H}{\partial t}\right]\mathrm{d}t+\mathrm{d}(p_k u^k).$$

所以

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta\circ\mathscr{L}\circ\gamma_s=\gamma_s^*(\mathrm{L}_{\boldsymbol{u}}\theta\circ\mathscr{L})=\gamma_s^*(\mathrm{L}_{\boldsymbol{u}}\theta)=\mathrm{L}_{\boldsymbol{u}(\gamma_s(t))}\theta\circ\gamma_s(t),\tag{48}$$

$$0=\frac{\mathrm{d}}{\mathrm{d}s}J(\gamma_s)\bigg|_{s=0}=\int_{t_0}^{t_1}\mathrm{L}_{\boldsymbol{u}(\gamma(t))}\theta\circ\gamma(t).\tag{49}$$

得到

$$\mathrm{L}_{\boldsymbol{u}(\gamma_s(t))}\theta\circ\gamma(t)=0.\tag{50}$$

定理2.7 $\gamma\in C^k(I,N)$ 使泛函

$$J(\gamma)=\int_\gamma L\,\mathrm{d}t=\int_\gamma\theta$$

达到极值. 若 $\gamma\in\mathscr{D}(J)$, 且 $\mathrm{L}_{\boldsymbol{u}(\gamma)}\theta\circ\gamma=0$, $\gamma=\mathscr{L}\circ\gamma$ 是 γ 在 T^*N 中的提升.

因 γ 在 $\boldsymbol{u}$ 产生的变换群 $\{\sigma_s\}$ 的作用下, 微小变形, 而端点 $\gamma(t_0)$、$\gamma(t_1)$ 固定, $\boldsymbol{u}(\gamma(t_0))=\boldsymbol{u}(\gamma(t_1))=0$. 再由 $\boldsymbol{u}$ 从而 $\boldsymbol{u}=\boldsymbol{u}+\frac{\mathrm{d}p_k}{\mathrm{d}s}\partial/\partial p_k$ 的任意性 $\frac{\mathrm{d}}{\mathrm{d}s}J(\gamma_s)\Big|_{s=0}=0$ 意味着

$$\frac{\partial H\circ\gamma}{\partial t}=0,\quad\frac{\mathrm{d}x^k}{\mathrm{d}t}=\frac{\partial H\circ\gamma}{\partial p_k},\quad\frac{\mathrm{d}p_k}{\mathrm{d}t}=-\frac{\partial H\circ\gamma}{\partial x^k}.\tag{51}$$

$\partial H/\partial t=0$ 由时间 t 均匀流逝产生. 且 $\partial H/\partial t=0$ 意味 H 不显含 t, 或显含 t 而视之不变. (51) 在每一固定的 $t\in I$ 所对应的纤维 N_t(即 M_t^n) 的余切丛 $T^*M_t^n$ 上成立, 即 $\gamma\subset T^*M_t^n$ 满足 (51) 而 $\gamma=\mathscr{L}^{-1}(\gamma)$, $\Pi(\gamma)=\Pi(\mathscr{L}^{-1}(\gamma))=\gamma$. $\Pi^*\gamma=\gamma$, $\Pi_*^*\boldsymbol{u}=\boldsymbol{u}$.

定理2.8 $\gamma\in C^k(I,N)$ 是泛函

$$J(\gamma)=\int_\gamma\theta$$

的平衡点, 则 $\gamma\in\mathscr{D}(J)$, 且 $\gamma=\mathscr{L}\circ\gamma\in C^l(I,T^*N)$ 是H_方程(51) 的解, 它是Hamilton 向量场

$$\boldsymbol{X}=\partial/\partial t+\frac{\partial H}{\partial p_k}\partial/\partial x^k-\frac{\partial H}{\partial x^k}\partial/\partial p_k\in\mathscr{X}(T^*N)\tag{52}$$

的积分曲线.

归纳之有下述

命题2.9 对 $L\in C^l(TN,R)$, $\gamma\in C^k(I,N)$ 是泛函

$$J(\gamma)=\int_{t_0}^{t_1}L\circ\gamma\,\mathrm{d}t=\int_{t_0}^{t_1}\theta\circ\gamma(t)=\int_\gamma L\,\mathrm{d}t=\int_\gamma\theta\tag{53}$$

的平稳点, 则 $\gamma\in\mathscr{D}(J)$, 且在任意可微向量场 $\boldsymbol{u}=\partial/\partial t+u^j\partial/\partial x^j\in\mathscr{X}^r(N)$, $\boldsymbol{u}\neq c\gg$, 产生的变换群 $\{\sigma_s\}$ 作用下, 连续变形使得 $t+s\in I$, $\gamma_s=\sigma_s(\gamma)\in\mathscr{D}(J)$, 并满足

$$\frac{\mathrm{d}}{\mathrm{d}s}J(\gamma_s)\bigg|_{s=0}=0,$$

从而

1) γ 在 TN 中的提升 γ 满足

$$\mathrm{L}_{\boldsymbol{u}(\gamma)}L\circ\gamma=0,$$

其中 $\boldsymbol{u}=\partial/\partial t+u^j\partial/\partial x^j+\frac{\mathrm{d}u^j}{\mathrm{d}t}\partial/\partial x^j\in\mathscr{X}(TN)$, $u^j(\gamma)=\frac{\partial x^j\circ\sigma_s(\gamma)}{\partial s}\bigg|_{s=0}$.

即 L 不显含 t, 或对每一固定的 $t \in I$, 在纤维 N_t 的切丛 TM_t^* 上满足 Lagrange 方程

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L\circ\gamma}{\partial \dot{x}^j} = \frac{\partial L\circ\gamma}{\partial x^j},$$

γ 是 Lagrange 向量场(41)的积分曲线•

2) γ 在 T^*N 中的提升 $\gamma = \mathscr{L}\circ\gamma$ 满足

$$\mathrm{L}_{\boldsymbol{u}(\gamma)}\theta\circ\gamma = 0,$$

其中 $\boldsymbol{u} = \partial/\partial t + u^k\partial/\partial x^k + \frac{\partial p_k}{\partial s}\partial/\partial p_k \in \mathscr{X}(T^*N)$,

即 H 不显含 t 或对每一固定的 $t \in I$ 在纤维 N_t 的余切丛 $T^*M_t^n$ 上满足 Hamilton 方程

$$\frac{\mathrm{d}x^k}{\mathrm{d}t} = \frac{\partial H\circ\gamma}{\partial p_k},\quad \frac{\mathrm{d}p_k}{\mathrm{d}t} = -\frac{\partial H\circ\gamma}{\partial x^k},$$

γ 是 Hamilton 向量场(52)的积分曲线• $\boldsymbol{u}$、$\boldsymbol{u}$ 由 $\boldsymbol{u}$ 决定, 且

$$\Pi_*\boldsymbol{u} = \boldsymbol{u},\quad \Pi^*_*\boldsymbol{u} = \boldsymbol{u},$$

Π, Π^* 分别是切丛及余切丛 TN 与 T^*N 的丛投影•

由此可见, 泛函 J 达到极值的曲线 γ 是力学系统——Lagrange 系统及 Hamilton 系统的运动轨线•

$\gamma \in \mathscr{D}(J)$ 是泛函 J 的平稳点, 则

$$\dot{\gamma}\lrcorner(\mathrm{d}L\wedge\mathrm{d}t) = 0, \tag{54}$$

$$\dot{\gamma}\lrcorner\mathrm{d}\theta = 0• \tag{55}$$

(55)表明 θ 是 $\dot{\gamma}$ 的相对积分不变量从而 $\dot{\gamma}$ 是 Hamilton 向量场而 γ 是此向量场的积分曲线•

这样, 我们把 Riemann 流形 M^n 上的平稳的动力学推广至 Newton_Riemann 时空 $N = R^1\times M^n$ 中的非平稳的动力学, 而前者是后者的特殊情况• 后者可直接应用于 Riemann 流形上的流体力学• 上述一系列文章提供分析力学的现代几何方法•

3 例

设单位质点在势场 $U(t, \boldsymbol{x})$ 下, 于 N 中以速度 $\boldsymbol{V} = \partial/\partial t + v^j(t, \boldsymbol{x})\partial/\partial x^j$ 运动则动能

$$T = \frac{\langle \boldsymbol{V}^\flat, \boldsymbol{V}\rangle}{2} = \frac{g_{jk}v^jv^k}{2} + \frac{1}{2};$$

Lagrange 函数

$$L = T - U = \frac{g_{jk}v^jv^k}{2} - U + \frac{1}{2} \in C^k(TN, R)• \tag{56}$$

那么

$$\frac{\partial L}{\partial x^l} = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^l}v^jv^k - \frac{\partial U}{\partial x^l},\qquad \frac{\partial L}{\partial \dot{x}^l} = g_{jl}v^j,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^l} = \frac{\partial g_{jl}}{\partial x^s}v^sv^j + g_{jl}\left(\frac{\partial v^j}{\partial t} + \frac{\partial v^j}{\partial x^s}v^s\right),$$

L_方程

$$g_{jl}\left(\frac{\partial v^j}{\partial t} + v^s\frac{\partial v^j}{\partial x^s}\right) + \frac{\partial g_{jl}}{\partial x^s}v^sv^j - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^l}v^jv^k = -\frac{\partial U}{\partial x^l},$$

即

$$g_{jl}\left(\frac{\partial v^j}{\partial t} + v^s\frac{\partial v^j}{\partial x^s}\right) + \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial x^s} + \frac{\partial g_{sl}}{\partial x^j} - \frac{\partial g_{js}}{\partial x^l}\right)v^jv^s = -\frac{\partial U}{\partial x^l}• \tag{57}$$

以 g^{lk} 乘方程两端得

$$\frac{\partial v^k}{\partial t} + v^s \frac{\partial v^k}{\partial x^s} + \Gamma^k_{js} v^j v^s = - g^{lk} \frac{\partial U}{\partial x^l}, \tag{58}$$

其中 $$\Gamma^k_{js} = \frac{1}{2} g^{lk} \left(\frac{\partial g_{jl}}{\partial x^s} + \frac{\partial g_{sl}}{\partial x^j} - \frac{\partial g_{js}}{\partial x^l} \right).$$

这是对每一固定的 $t \in I$ 的 Newton 方程

$$\frac{\mathbf{D} \boldsymbol{V}}{\mathrm{d} t} = - \mathrm{d} U^{\#} \tag{59}$$

在局部坐标系 $(I \times U, \psi)$ 中的坐标表示. 若运动是平稳的, $\partial v^j / \partial t = 0$, 质点在 Riemann 流形 M^n 中运动, (58) 可写成

$$\frac{\mathrm{d} \dot{x}^k}{\mathrm{d} t} + \Gamma^k_{js} \dot{x}^j \dot{x}^s = - g^{lk} \frac{\partial U}{\partial x^l}. \tag{59$'$}$$

若外作用力为 0, 则(59)变为

$$\frac{\mathbf{D} \boldsymbol{V}}{\mathrm{d} t} = 0. \tag{60}$$

相应地(58)、(59)$'$之右端为0, (59)$'$变为

$$\frac{\mathrm{d} \dot{x}^k}{\mathrm{d} t} + \Gamma^k_{js} \dot{x}^j \dot{x}^s = 0, \tag{61}$$

这时质点在 M^n 上沿测地线运动.

但若是非平稳的运动, 特别在流体力学中必须以 $v^j(t, \boldsymbol{x})$ 取代 $\dot{x}^j$. 上述表明对Riemann流形 M^n 或N_R时空 $N = R^1 \times M^n$ 中的平稳或非平稳运动, Lagrange 力学与 Newton 力学的一致性. 另方面,

$$H = \langle \boldsymbol{V}^\flat, \boldsymbol{V} \rangle - L = \frac{1}{2} \langle \boldsymbol{V}^\flat, \boldsymbol{V} \rangle + U = E$$

即 $$H = E \circ \mathscr{L}^{1}(t, \boldsymbol{x}, 1, \boldsymbol{p}) = \frac{1}{2} g^{jk} p_j p_k + U(t, \boldsymbol{x}), \tag{62}$$

$$\frac{\partial H}{\partial x^s} = \frac{1}{2} \frac{\partial g^{jk}}{\partial x^s} p_j p_k + \frac{\partial U}{\partial x^s}, \qquad \frac{\partial H}{\partial p_s} = g^{js} p_j.$$

Hamilton 向量场是

$$\boldsymbol{X} = \partial / \partial t + g^{js} p_j \partial / \partial x^s - \left(\frac{1}{2} \frac{\partial g^{jk}}{\partial x^s} p_j p_k + \frac{\partial U}{\partial x^s} \right) \partial / \partial p_s, \tag{63}$$

相应的 Hamilton 方程是

$$\begin{cases} \dot{x}^s = g^{js} p_j, \quad \dot{t} = 1, \\ \dot{p}_s = - \dfrac{1}{2} \dfrac{\partial g^{jk}}{\partial x^s} p_j p_k - \dfrac{\partial U}{\partial x^s}. \end{cases} \tag{64}$$

微分之

$$\begin{aligned} \frac{\mathrm{d} \dot{x}^s}{\mathrm{d} t} &= \frac{\partial g^{js}}{\partial x^l} v^l p_j + g^{js} \dot{p}_j = \\ & \frac{\partial g^{js}}{\partial x^l} g_{kj} v^l v^k + g^{js} \left(- \frac{1}{2} \frac{\partial g^{ar}}{\partial x^j} g_{ba} v^b g_{mj} v^m - \frac{\partial U}{\partial x^j} \right) = \\ & - \frac{\partial g_{kj}}{\partial x^l} g^{js} v^l v^k + g^{js} \left(\frac{1}{2} \frac{\partial g_{ra}}{\partial x^j} g^{ar} g_{mr} v^r v^m - \frac{\partial U}{\partial x^j} \right) = \\ & - g^{js} \left(\frac{\partial g_{kj}}{\partial x^l} v^l v^k - \frac{1}{2} \frac{\partial g_{ra}}{\partial x^j} v^r v^a + \frac{\partial U}{\partial x^j} \right) = \end{aligned}$$

$$- \frac{1}{2} g^{js} \left(\frac{\partial g_{kj}}{\partial x^l} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^j} \right) v^l v^k - g^{js} \frac{\partial U}{\partial x^j} =$$

$$- \Gamma^s_{kl} v^k v^l - g^{js} \frac{\partial U}{\partial x^j}, \tag{65}$$

其中 $\Gamma^s_{lk} = \Gamma^s_{kl} = \frac{1}{2} g^{js} \left(\frac{\partial g_{kj}}{\partial x^l} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^j} \right)$•

令 $\dot{x}^j = v^j(t, \boldsymbol{x})$, 由(65) 得

$$\frac{\partial v^s}{\partial t} + v^r \frac{\partial v^s}{\partial x^r} + \Gamma^s_{kl} v^l v^k = - g^{js} \frac{\partial U}{\partial x^j}, \tag{66}$$

此即(58)• 这表明 M^n 或 $N = R^1 \times M^n$ 中的 Hamilton 力学与 Newton 力学的一致性•

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Dynamics in Newtonian_Riemannian Space_Time(Ⅳ)

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Abstract: Lagrangian mechanics in Newtonian_Riemannian space_time and relationship between Lagrangian mechanics and Newtonian mechanics, and between Lagrangian mechanics and Hamiltonian mechanics in N_R space_time are discussed.

Key words: Riemannian manifold; tangent bundle; cotangent bundle; fiber bundle; fiber; vector field; form field; exterior differential; absolute differential; Lie derivative; functional; variation