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纤维混凝土中应力传递机制的三维弹性理论分析*

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(陈山林推荐)

摘要: 对纤维混凝土中应力传递机制问题作了分析研究· 将应力和位移看作一组"初等解"和一组"修正解"的叠加;"初等解"即为一般的二维理论所得的解答, 而"修正解"应用拉甫(Love)位移函数法求得· 通过计算实例表明, 解的收敛性良好· 将三维弹性理论解与剪滞法的解比较后, 可见两种解存在较明显的差别·

关 键 词: 纤维混凝土; 应力传递; 弹性理论

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引 言

纤维混凝土是一种以水泥砂浆或混凝土为基材, 以钢纤维为增强材料的典型水泥基复合材料, 近年来, 在各工程中得到了广泛的应用· 纤维混凝土的力学性质很大程度上受纤维与基体间应力传递的控制, 界面的应力传递机制是复合材料界面力学性能研究的重要方面· 纤维混凝土的模量和强度如何依赖于组分材料的性能及其含量是其宏观力学性能预报的基本问题, 这时问题的关键在于确定应力是怎样在纤维与基体之间进行传递的· 当荷载作用在纤维混凝土上时, 纤维并不直接受力, 首先是纤维周围的基体受载, 然后通过纤维和基体的界面传递到纤维上· 在纤维混凝土的情况下, 纤维末端附近的应力传递机制的重要性变得显著· 以往的研究者曾用剪滞法[1]、有限元法[2]作过分析, 利用基于连续介质理论的有限元法不能有效地直接求解这种非均质问题, 因为描述材料的非均匀分布需要很密的网格和大量的未知数·

1 理论模型、基本方程及求解方法

设作用在短纤维复合材料上的作用力与纤维平行· 模型中有一个圆柱形的纤维, 其外围是一个圆柱形的基体套层, 其余的外围材料具有该复合材料的平均性能, 并作如下假定:

1) 在纤维基体之间的界面粘接完好;
2) 最近的相邻纤维对沿基体套层的应力场无显著干扰;
3) 纤维与基体材料是线弹性的·

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根据该模型,纤维与基体中的应力、应变分布具有轴对称性.

在柱坐标下,轴对称问题的位移分量可表示为:

$$u = u(r, z), \quad w = w(r, z).$$

位移分量满足如下的平衡方程

$$\begin{cases} \nabla^2 u + \dfrac{1}{1-2\nu}\dfrac{\partial e}{\partial r} - \dfrac{u}{r^2} = 0, \\ \nabla^2 w + \dfrac{1}{1-2\nu}\dfrac{\partial e}{\partial z} = 0; \end{cases} \tag{1}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (\text{拉普拉斯算子}),$$

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}.$$

应用 Love 位移函数 $\phi(r, z)$, 使 $\nabla^2\nabla^2\phi = 0$, 用

$$u = -\frac{\partial^2\phi}{\partial r\partial z}, \quad w = 2(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}, \tag{2}$$

则位移平衡方程得到满足,经过计算,应力分量为:

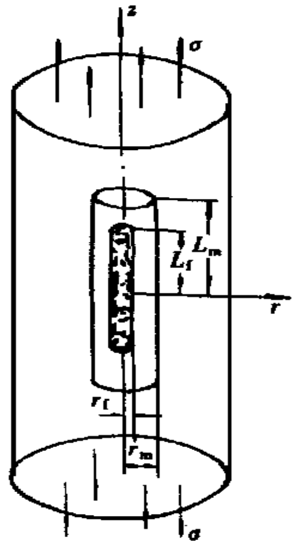

图 1 计算模型

$$\left.\begin{aligned} \sigma_r &= 2\mu\frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right), \\ \sigma_\theta &= 2\mu\frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right), \\ \sigma_z &= 2\mu\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right], \\ \tau_{rz} &= 2\mu\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]. \end{aligned}\right\} \tag{3}$$

显然在纤维与基体之间,将相互作用着正应力

$$\sigma_r = -p^0 + \sum_{n=1}^{\infty}\sigma_n\cos\frac{n\pi}{L_f}z \tag{4}$$

与剪应力

$$\tau_{rz} = \sum_{n=1}^{\infty}\tau_n\sin\frac{n\pi}{L_f}z, \tag{5}$$

其中待定常数 p^0, σ_n, τ_n 可根据纤维与基体界面 $r = r_f$ 处应力与位移的连续条件确定.

一般说来,因为纤维末端附近的基体由于高的应力集中而使纤维末端与基体脱胶,因而对纤维与基体套层,均有:

$$\text{当} \quad z = L_f, \quad \sigma_z = 0, \quad \tau_{rz} = 0. \tag{6}$$

在基体套层外侧处,将受到复合材料的纵向均匀拉应变,即

$$r = r_m, \quad \sigma_z = \sigma = E_c\varepsilon = E_c\frac{dw}{dz}, \quad \sigma_r = 0. \tag{7}$$

σ 为已知的纵向拉应力,于是有:

$$\text{当}\ r = r_m, \quad w = \frac{\sigma}{E_c}z, \quad \sigma_r = 0. \tag{8}$$

将柱坐标与位移分量进行无量纲化

$$V_f = \frac{r_f^2 L_f}{r_m^2 L_m}, \quad \rho = \frac{r}{r_f}, \quad \zeta = \frac{z}{r_f}, \quad U = \frac{u}{r_f}, \quad W = \frac{w}{r_f}, \quad \beta = \frac{r_m}{r_f}, \quad d = \frac{L_f}{r_f}. \tag{9}$$

用无量纲位移表示的平衡方程化为:

$$\nabla^2 U + \frac{1}{1-2\nu}\frac{\partial e}{\partial \rho} - \frac{U}{\rho^2} = 0, \quad \nabla^2 W + \frac{1}{1-2\nu}\frac{\partial e}{\partial \zeta} = 0, \tag{10}$$

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \zeta^2}.$$

纤维与基体套层中的应力解均可看作是一组“初等解”与“修正解”的叠加. 纤维中应力的“初等解”满足如下的边界条件:

$$\rho = 1, \quad \sigma_r^{(1)} = -p^0, \quad \tau_{rz}^{(1)} = 0, \quad \zeta = d, \quad \sigma_z^{(1)} = 0, \quad \tau_{rz}^{(1)} = 0. \tag{11}$$

纤维中应力的“修正解”应满足边界条件:

$$\left.\begin{aligned} &\rho = 1, \quad \sigma_r^{(2)} = \sum_{n=1}^{\infty} \sigma_n \cos\frac{n\pi}{d}\zeta, \quad \tau_{rz}^{(2)} = \sum_{n=1}^{\infty} \tau_n \sin\frac{n\pi}{d}\zeta, \\ &\zeta = d, \quad \sigma_z^{(2)} = 0, \quad \tau_{rz}^{(2)} = 0. \end{aligned}\right\} \tag{12}$$

“修正解”可用拉甫(Love)位移函数法求得. 令无量纲拉甫位移函数为

$$\Phi = \frac{\phi}{r_{\mathrm{f}}^3},$$

则有
$$U^{(2)} = -\frac{\partial^2 \Phi}{\partial \rho \partial \zeta}, \quad W^{(2)} = 2(1-\nu_{\mathrm{f}})\nabla^2 \Phi - \frac{\partial^2 \Phi}{\partial \zeta^2}, \tag{13}$$

而修正应力为:

$$\left.\begin{aligned} &\sigma_r^{(2)} = 2\mu_{\mathrm{f}} \frac{\partial}{\partial \zeta}\left(\nu_{\mathrm{f}} \nabla^2 \Phi - \frac{\partial^2 \Phi}{\partial \rho^2}\right), \quad \sigma_\theta^{(2)} = 2\mu_{\mathrm{f}} \frac{\partial}{\partial \zeta}\left(\nu_{\mathrm{f}} \nabla^2 \Phi - \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}\right), \\ &\sigma_z^{(2)} = 2\mu_{\mathrm{f}} \frac{\partial}{\partial \zeta}\left((1-\nu_{\mathrm{f}}) \nabla^2 \Phi - \frac{\partial^2 \Phi}{\partial \zeta^2}\right), \quad \tau_{rz}^{(2)} = 2\mu_{\mathrm{f}} \frac{\partial}{\partial \rho}\left((1-\nu_{\mathrm{f}}) \nabla^2 \Phi - \frac{\partial^2 \Phi}{\partial \zeta^2}\right). \end{aligned}\right\} \tag{14}$$

基体套层应力的“初等解”满足边界条件

$$\left.\begin{aligned} &\rho = 1, \quad \sigma_r^{(1)} = -p^0, \quad \tau_{rz}^{(1)} = 0, \\ &\rho = \beta, \quad \sigma_r^{(1)} = 0, \quad W^{(1)} = -\frac{2\nu_{\mathrm{m}}}{\beta^2 - 1}\frac{p^0}{E_{\mathrm{m}}}\zeta, \\ &\zeta = d, \quad \sigma_z^{(1)} = 0, \quad \tau_{rz}^{(1)} = 0. \end{aligned}\right\} \tag{15}$$

基体套层应力的“修正解”应满足边界条件

$$\left.\begin{aligned} &\rho = 1, \quad \sigma_r^{(2)} = \sum_{n=1}^{\infty} \sigma_n \cos\frac{n\pi}{d}\zeta, \quad \tau_{rz}^{(2)} = \sum_{n=1}^{\infty} \tau_n \sin\frac{n\pi}{d}\zeta, \\ &\rho = \beta, \quad \sigma_r^{(2)} = 0, \quad W^{(2)} = \left(\frac{\sigma}{E_c} + \frac{2\nu_{\mathrm{m}}}{\beta^2 - 1}\frac{p^0}{E_{\mathrm{m}}}\right)\zeta, \\ &\zeta = d, \quad \sigma_z^{(2)} = \tau_{rz}^{(2)} = 0. \end{aligned}\right\} \tag{16}$$

基体套层的“修正解”亦用拉甫位移函数法求出. 根据以上所述纤维、基体套层的“初等解”与“修正解”所满足的边界条件, 可以看出, “初等解”与“修正解”之和已满足 $\rho = \beta$、ζ 处所有边界条件以及界面 $\rho = 1$ 处应力 σ_r、τ_{rz} 的连续条件, 再利用界面 $\rho = 1$ 处纤维与基体位移连续条件

$$U^{(1)} + U^{(2)} = U^{(1)} + U^{(2)}, \quad W^{(1)} + W^{(2)} = W^{(1)} + W^{(2)}, \tag{17}$$

即可确定前面的待定系数 p^0、σ_n、τ_n.

2 问题的解

2.1 纤维

1) 初等解

初等解极易求得为:

$$\left.\begin{aligned} &U^{(1)} = -\frac{1-\nu_f}{E_f}p^0\rho,\quad W^{(1)} = \frac{2\nu_f}{E_f}p^0\zeta,\\ &\sigma_r^{(1)} = -p^0,\quad \sigma_\theta^{(1)} = -p^0,\quad \sigma_z^{(1)} = 0\cdot \end{aligned}\right\} \tag{18}$$

2) 修正解

取单调和函数

$$\Phi_1 = \sum_{n=1}^{\infty}\frac{a_n^{(1)}}{k_n^2}\mathrm{I}_0(k_n\rho)\sin k_n\zeta + \sum_{m=1}^{\infty}\frac{C_m^{(1)}}{\alpha_m^2}\mathrm{J}_0(\alpha_m\rho)\,\mathrm{sh}\,\alpha_m\zeta \tag{19}$$

与双调和函数

$$\Phi_2 = \sum_{n=1}^{\infty}\frac{a_n^{(2)}}{k_n^3}k_n\rho\mathrm{I}_1(k_n\rho)\sin k_n\zeta + \sum_{m=1}^{\infty}\frac{C_m^{(2)}}{\alpha_m^3}\mathrm{J}_0(\alpha_m\rho)\,\alpha_m\zeta\,\mathrm{ch}\,\alpha_m\zeta \tag{20}$$

之和

$$\Phi = \Phi_1 + \Phi_2$$

为拉甫位移函数,其中 $k_n = n\pi/d$, 可求得纤维之修正解• 此处及以后要用到的贝塞尔函数:

$$\mathrm{J}_0(\alpha_m\rho),\ \mathrm{J}_1(\alpha_m\rho),\ \mathrm{Y}_0(\alpha_m\rho),\ \mathrm{Y}_1(\alpha_m\rho),$$
$$\mathrm{I}_0(k_n\rho),\ \mathrm{I}_1(k_n\rho),\ \mathrm{K}_0(k_n\rho),\ \mathrm{K}_1(k_n\rho)\cdot$$

将拉甫函数代入(13)、(14),且令

$$A_m^* = C_m^{(1)} + \frac{C_m^{(2)}}{\alpha_m},\quad H_m^* = C_m^{(2)},\quad A_n = a_n^{(1)},\quad C_n = -a_n^{(2)}\cdot \tag{21}$$

实行求导运算,且利用贝塞尔函数的递推公式,可得修正解:

$$\left\{\begin{aligned} U^{(2)} =& \sum_{m=1}^{\infty}\mathrm{J}_1(\alpha_m\rho)[A_m^*\,\mathrm{ch}\,\alpha_m\zeta + H_m^*\zeta\,\mathrm{sh}\,\alpha_m\zeta] +\\ &\sum_{n=1}^{\infty}\cos k_n\zeta[-A_n\mathrm{I}_1(k_n\rho) + C_n\rho\mathrm{I}_0(k_n\rho)],\\ W^{(2)} =& \sum_{m=1}^{\infty}\mathrm{J}_0(\alpha_m\rho)\left\{\left[(3-4\nu_f)\frac{H_m^*}{\alpha_m} - A_m^*\right]\mathrm{sh}\,\alpha_m\zeta - H_m^*\zeta\,\mathrm{ch}\,\alpha_m\zeta\right\} +\\ &\sum_{n=1}^{\infty}\sin k_n\rho\left\{\mathrm{I}_0(k_n\rho)\left[A_n - 4(1-\nu_f)\frac{C_n}{k_n}\right] - C_n\rho\mathrm{I}_1(k_n\rho)\right\}; \end{aligned}\right. \tag{22}$$

$$\begin{aligned} \frac{\sigma_r^{(2)}}{2\mu_f} =& \sum_{m=1}^{\infty}\Big\{\mathrm{J}_0(\alpha_m\rho)[(A_m^*\alpha_m + 2\nu_f H_m^*)\,\mathrm{ch}\,\alpha_m\zeta + H_m^*\alpha_m\zeta\,\mathrm{sh}\,\alpha_m\zeta] -\\ &\frac{\mathrm{J}_1(\alpha_m\rho)}{\alpha_m\rho}[A_m^*\alpha_m\,\mathrm{ch}\,\alpha_m\zeta + H_m^*\alpha_m\zeta\,\mathrm{sh}\,\alpha_m\zeta]\Big\} + \sum_{n=1}^{\infty}\cos k_n\zeta\times\\ &\left\{\mathrm{I}_0(k_n\rho)[(1-2\nu_f)C_n - k_nA_n] + \mathrm{I}_1(k_n\rho)\left[\frac{A_n}{\rho} + C_nk_n\rho\right]\right\}, \end{aligned} \tag{23a}$$

$$\begin{aligned} \frac{\sigma_\theta^{(2)}}{2\mu_f} =& \sum_{m=1}^{\infty}\Big[\mathrm{J}_0(\alpha_m\rho)(2\nu_f H_m^*\,\mathrm{ch}\,\alpha_m\zeta) + \frac{\mathrm{J}_1(\alpha_m\rho)}{\alpha_m\rho}(A_m^*\alpha_m\,\mathrm{ch}\,\alpha_m\zeta +\\ &H_m^*\alpha_m\zeta\,\mathrm{sh}\,\alpha_m\zeta)\Big] + \sum_{n=1}^{\infty}\cos k_n\zeta\left[(1-2\nu_f)C_n\mathrm{I}_0(k_n\rho) - \frac{A_n}{\rho}\mathrm{I}_1(k_n\rho)\right], \end{aligned} \tag{23b}$$

$$\begin{aligned} \frac{\sigma_z^{(2)}}{2\mu_f} =& \sum_{m=1}^{\infty}\mathrm{J}_0(\alpha_m\rho)\Big\{[2(1-\nu_f)H_m^* - A_m^*\alpha_m]\,\mathrm{ch}\,\alpha_m\zeta - H_m^*\alpha_m\zeta\,\mathrm{sh}\,\alpha_m\zeta\Big\} +\\ &\sum_{n=1}^{\infty}\cos k_n\zeta\Big\{\mathrm{I}_0(k_n\rho)[k_nA_n - (4-2\nu_f)C_n] - C_nk_n\rho\mathrm{I}_1(k_n\rho)\Big\}, \end{aligned} \tag{23c}$$

$$\frac{\tau_{rz}^{(2)}}{2\mu_f} = \sum_{m=1}^{\infty} J_1(\alpha_m \rho)\left\{[A_m^* \alpha_m(1-2\nu_f)H_m^*]\operatorname{sh}\alpha_m\zeta + H_m^*\alpha_m\zeta\operatorname{ch}\alpha_m\zeta\right\} + \sum_{n=1}^{\infty}\sin k_n\zeta\left\{-C_n k_n\rho I_0(k_n\rho) + I_1(k_n\rho)[k_n A_n - 2(1-\nu_f)C_n]\right\}. \tag{23d}$$

为了确定以上各式中的待定系数A_m^*、H_m^*、A_n及C_n，并求得一个满足边界条件及纤维与基体间连续条件的解答，我们首先使$\tau_{rz}^{(2)}/2\mu_f$表达式中第一个级数在纤维表面$\rho=1$处及端部$\zeta=d$处为零，则有：

$$J_1(\alpha_m) = 0 \quad (\text{取正根}), \tag{24}$$

$$A_m^*\alpha_m = [(1-2\nu_f) - \alpha_m d\coth(\alpha_m d)]H_m^*. \tag{25}$$

于是α_m为$J_1(x)=0$的正根，即α_m取为$J_1(x)$的零点. (22)、(23)复可改写为：

$$\begin{cases} U^{(2)} = \sum_{m=1}^{\infty}\frac{H_m^*}{\alpha_m}J_1(\alpha_m\rho)\left\{[(1-2\nu_f) - \alpha_m d\coth(\alpha_m d)]\operatorname{ch}\alpha_m\zeta + \alpha_m\zeta\operatorname{sh}\alpha_m\zeta\right\} + \sum_{n=1}^{\infty}\cos k_n\zeta\left\{-A_n I_1(k_n\rho) + C_n\rho I_0(k_n\rho)\right\}, \\ W^{(2)} = \sum_{m=1}^{\infty}\frac{H_m^*}{\alpha_m}J_0(\alpha_m\rho)\left\{[2(1-2\nu_f) + \alpha_m d\coth(\alpha_m d)]\operatorname{sh}\alpha_m\zeta - \alpha_m\zeta\operatorname{ch}\alpha_m\zeta\right\} + \sum_{n=1}^{\infty}\frac{1}{k_n}\sin k_n\zeta\left\{A_n k_n I_0(k_n\rho) - C_n[4(1-\nu_f)I_0(k_n\rho) + k_n\rho I_1(k_n\rho)]\right\}; \end{cases} \tag{26}$$

$$\frac{\sigma_r^{(2)}}{2\mu_f} = \sum_{m=1}^{\infty}H_m^*\left\{J_0(\alpha_m\rho)\left\{[1-\alpha_m d\coth(\alpha_m d)]\operatorname{ch}\alpha_m\zeta + \alpha_m\zeta\operatorname{sh}\alpha_m\zeta\right\} - \frac{J_1(\alpha_m\rho)}{\alpha_m\rho}\left\{[(1-2\nu_f) - \alpha_m d\coth(\alpha_m d)]\operatorname{ch}\alpha_m\zeta + \alpha_m\zeta\operatorname{sh}\alpha_m\zeta\right\}\right\} + \sum_{n=1}^{\infty}\cos k_n\zeta\left\{-A_n k_n\left[I_0(k_n\rho) - \frac{I_1(k_n\rho)}{k_n\rho}\right] + C_n[(1-2\nu_f)I_0(k_n\rho) + k_n\rho I_1(k_n\rho)]\right\}, \tag{27a}$$

$$\frac{\sigma_\theta^{(2)}}{2\mu_f} = \sum_{m=1}^{\infty}H_m^*\left\{J_0(\alpha_m\rho)(2\nu_f\operatorname{ch}\alpha_m\zeta) + \frac{J_1(\alpha_m\rho)}{\alpha_m\rho}\left\{[(1-2\nu_f) - \alpha_m d\coth(\alpha_m d)]\operatorname{ch}\alpha_m\zeta + \alpha_m\zeta\operatorname{sh}\alpha_m\zeta\right\}\right\} + \sum_{n=1}^{\infty}\cos k_n\zeta\left\{-A_n\frac{1}{\rho}I_1(k_n\rho) + C_n(1-2\nu_f)I_0(k_n\rho)\right\}, \tag{27b}$$

$$\frac{\sigma_z^{(2)}}{2\mu_f} = \sum_{m=1}^{\infty}H_m^* J_0(\alpha_m\rho)\left\{[1+\alpha_m d\coth(\alpha_m d)]\operatorname{ch}\alpha_m\zeta - \alpha_m\zeta\operatorname{sh}\alpha_m\zeta\right\} + \sum_{n=1}^{\infty}\cos k_n\zeta\left\{A_n k_n I_0(k_n\rho) + C_n[(4-2\nu_f)I_0(k_n\rho) + k_n\rho I_1(k_n\rho)]\right\}, \tag{27c}$$

$$\frac{\tau_{rz}^{(2)}}{2\mu_f} = \sum_{m=1}^{\infty}H_m^* J_1(\alpha_m\rho)[-\alpha_m d\coth(\alpha_m d)\operatorname{sh}\alpha_m\zeta + \alpha_m\zeta\operatorname{ch}\alpha_m\zeta] + \sum_{m=1}^{\infty}\sin k_n\zeta\left\{A_n k_n I_1(k_n\rho) - C_n[2(1-\nu_f)I_1(k_n\rho) + k_n\rho I_0(k_n\rho)]\right\}. \tag{27d}$$

由(27)可知，$\tau_{rz}^{(2)}/2\mu_f$已满足纤维端部条件：当$\zeta=d$，$\tau_{rz}^{(2)}=0$.

2.2 基体套层

1) *初等解*

极易求得初等解为:

$$
\left\{
\begin{aligned}
U^{(1)} &= (1-\nu_{\mathrm{m}})\frac{1}{\beta^2-1}\frac{p^0}{E_{\mathrm{m}}}\rho+\frac{\beta^2}{\beta^2-1}(1+\nu_{\mathrm{m}})\frac{p^0}{E_{\mathrm{m}}}\frac{1}{\rho},\\
W^{(1)} &= -2\nu_{\mathrm{m}}\frac{1}{\beta^2-1}\frac{p^0}{E_{\mathrm{m}}}\zeta;
\end{aligned}
\right.
\tag{28}
$$

$$
\left\{
\begin{aligned}
\sigma_r^{(1)} &= \frac{p^0}{\beta^2-1}-\frac{\beta^2}{\beta^2-1}p^0\frac{1}{\rho^2},\\
\sigma_\theta^{(1)} &= \frac{p^0}{\beta^2-1}+\frac{\beta^2}{\beta^2-1}p^0\frac{1}{\rho^2},\\
\sigma_z^{(1)} &= 0,\quad \tau_{rz}^{(1)} = 0\cdot
\end{aligned}
\right.
\tag{29}
$$

2) 修正解

通过与求纤维修正解相同的步骤, 有:

$$
\left.
\begin{aligned}
U^{(2)} = {} & \sum_{m=1}^{\infty}\frac{H_m^*}{\alpha_m}\Psi_1(\alpha_m\rho)\Big\{[(1-2\nu_{\mathrm{m}})-\alpha_m d\coth(\alpha_m d)]\mathrm{ch}\alpha_m\zeta+\alpha_m\zeta\mathrm{sh}\alpha_m\zeta\Big\}+\\
& \sum_{n=1}^{\infty}\cos k_n\zeta\Big\{-A_n\mathrm{I}_1(k_n\rho)+B_n\mathrm{K}_1(k_n\rho)+C_n\rho\mathrm{I}_0(k_n\rho)+D_n\rho\mathrm{K}_0(k_n\rho)\Big\},\\
W^{(2)} = {} & \sum_{m=1}^{\infty}\frac{H_m^*}{\alpha_m}\Psi_0(\alpha_m\rho)\Big\{[2(1-\nu_{\mathrm{m}})+\alpha_m d\coth(\alpha_m d)]\mathrm{sh}\alpha_m\zeta-\\
& \alpha_m\zeta\mathrm{ch}\alpha_m\zeta\Big\}+\sum_{n=1}^{\infty}\sin k_n\zeta\Big\{A_n\mathrm{I}_1(k_n\rho)+B_n\mathrm{K}_1(k_n\rho)-\\
& \frac{C_n}{k_n}[4(1-\nu_{\mathrm{m}})\mathrm{I}_0(k_n\rho)+k_n\rho\mathrm{I}_1(k_n\rho)]-\\
& \frac{D_n}{k_n}[4(1-\nu_{\mathrm{m}})\mathrm{K}_0(k_n\rho)-k_n\rho\mathrm{K}_1(k_n\rho)]\Big\};
\end{aligned}
\right\}
\tag{30}
$$

$$
\begin{aligned}
\frac{\sigma_r^{(2)}}{2\mu_{\mathrm{m}}} = {} & \sum_{m=1}^{\infty}H_m^*\Big\{\Psi_0(\alpha_m\rho)[1-\alpha_m d\coth(\alpha_m d)]\mathrm{ch}\alpha_m\zeta+\alpha_m\zeta\mathrm{sh}\alpha_m\zeta\Big\}-\\
& \frac{\Psi_1(\alpha_m\rho)}{\alpha_m\rho}\Big\{[(1-2\nu_{\mathrm{m}})-\alpha_m d\coth(\alpha_m d)]\mathrm{ch}\alpha_m\zeta+\alpha_m\zeta\mathrm{sh}\alpha_m\zeta\Big\}+\\
& \sum_{n=1}^{\infty}\cos k_n\zeta\Big\{-A_n\Big[k_n\mathrm{I}_0(k_n\rho)-\frac{1}{\rho}\mathrm{I}_1(k_n\rho)\Big]-B_n\Big[k_n\mathrm{K}_0(k_n\rho)+\frac{1}{\rho}\mathrm{K}_1(k_n\rho)\Big]+\\
& C_n[(1-2\nu_{\mathrm{m}})\mathrm{I}_0(k_n\rho)+k_n\rho\mathrm{I}_1(k_n\rho)]+D_n[(1-2\nu_{\mathrm{m}})\mathrm{K}_0(k_n\rho)-k_n\rho\mathrm{K}_1(k_n\rho)]\Big\},
\end{aligned}
\tag{31a}
$$

$$
\begin{aligned}
\frac{\sigma_\theta^{(2)}}{2\mu_{\mathrm{m}}} = {} & \sum_{m=1}^{\infty}H_m^*\Big\{\Psi_0(\alpha_m\rho)2\nu_{\mathrm{m}}\mathrm{ch}\alpha_m\zeta+\frac{\Psi_1(\alpha_m\rho)}{\alpha_m\rho}\Big\{[(1-2\nu_{\mathrm{m}})-\\
& \alpha_m d\coth(\alpha_m d)]\mathrm{ch}\alpha_m\zeta+\alpha_m\zeta\mathrm{sh}\alpha_m\zeta\Big\}\Big\}+\sum_{n=1}^{\infty}\cos k_n\zeta\Big\{-A_n\frac{1}{\rho}\mathrm{I}_1(k_n\rho)+\\
& B_n\frac{1}{\rho}\mathrm{K}_1(k_n\rho)+C_n(1-2\nu_{\mathrm{m}})\mathrm{I}_0(k_n\rho)+D_n(1-2\nu_{\mathrm{m}})\mathrm{K}_0(k_n\rho)\Big\},
\end{aligned}
\tag{31b}
$$

$$
\begin{aligned}
\frac{\sigma_z^{(2)}}{2\mu_{\mathrm{m}}} = {} & \sum_{m=1}^{\infty}H_m^*\Psi_0(\alpha_m\rho)\Big\{[1+\alpha_m d\coth(\alpha_m d)]\mathrm{ch}\alpha_m\zeta-\alpha_m\zeta\mathrm{sh}\alpha_m\zeta\Big\}+\\
& \sum_{n=1}^{\infty}\cos k_n\zeta\Big\{A_n k_n\mathrm{I}_0(k_n\rho)+B_n k_n\mathrm{K}_0(k_n\rho)-C_n[(4-2\nu_{\mathrm{m}})\mathrm{I}_0(k_n\rho)+\\
& k_n\rho\mathrm{I}_1(k_n\rho)]-D_n[(4-2\nu_{\mathrm{m}})\mathrm{K}_0(k_n\rho)-k_n\rho\mathrm{K}_1(k_n\rho)]\Big\},
\end{aligned}
\tag{31c}
$$

$$\frac{\tau_{rz}^{(2)}}{2\mu_{\mathrm{m}}}=\sum_{m=1}^{\infty}H_m^*\Psi_0(\alpha_m\rho)[-\alpha_m d\coth(\alpha_m d)\operatorname{sh}\alpha_m\zeta+\alpha_m\zeta\operatorname{ch}\alpha_m\zeta]+\sum_{n=1}^{\infty}\sin k_n\zeta\Big\{A_nk_n\mathrm{I}_1(k_n\rho)-B_nk_n\mathrm{K}_1(k_n\rho)-C_n[2(1-\nu_{\mathrm{m}})\mathrm{I}_1(k_n\rho)+k_n\rho\mathrm{I}_0(k_n\rho)]+D_n[2(1-\nu_{\mathrm{m}})\mathrm{K}_1(k_n\rho)-k_n\rho\mathrm{K}_0(k_n\rho)]\Big\},\tag{31d}$$

其中 $\zeta_m=-\dfrac{\mathrm{J}_1(\alpha_m)}{\mathrm{Y}_1(\alpha_m)}=-\dfrac{\mathrm{J}_1(\alpha_m\beta)}{\mathrm{Y}_1(\alpha_m\beta)}$,

且 α_m 为特征方程

$$\mathrm{J}_1(x)\mathrm{Y}_1(\beta x)-\mathrm{J}_1(\beta x)\mathrm{Y}_1(x)=0\tag{32}$$

的根• 令

$$\Psi_i(\alpha_m\rho)=\mathrm{J}_i(\alpha_m\rho)+\zeta_m\mathrm{Y}_i(\alpha_m\rho)\qquad(i=0,1)•$$

2.3 系数的确定

在前面已求得的“初等解”中有待定常数 p^0, “修正解” 中有待定常数 σ_n、τ_n、H_m^*、A_n、C_n、H_m^*、A_n、B_n、C_n、D_n• 这些常数可由边界条件、连续条件确定•

并令:

$$\frac{\mu_{\mathrm{f}}}{\mu_{\mathrm{m}}}=\frac{E_{\mathrm{f}}}{E_{\mathrm{m}}}\frac{1+\nu_{\mathrm{m}}}{1+\nu_{\mathrm{f}}}=b•$$

为了更便于计算, 使上机计算时不致产生溢出, 作如下变换

$$h_m=\frac{H_m^*\mathrm{e}^{\alpha_m d}}{\sigma/E_{\mathrm{c}}},\quad a_n=\frac{A_n}{\sigma/E_{\mathrm{c}}},\quad c_n=\frac{C_n}{\sigma/E_{\mathrm{c}}},$$

$$h_m=\frac{H_m^*\mathrm{e}^{\alpha_m d}}{\sigma/E_{\mathrm{c}}},\quad a_n=\frac{A_n}{\sigma/E_{\mathrm{c}}},\quad b_n=\frac{B_n}{\sigma/E_{\mathrm{c}}},$$

$$c_n=\frac{C_n}{\sigma/E_{\mathrm{c}}},\quad d_n=\frac{D_n}{\sigma/E_{\mathrm{c}}}•$$

由计算可知 $p^0=0$, 由此进一步可知“初等解” 为全零解• 确定待定常数 h_m、a_n、c_n、h_m、a_n、b_n、c_n、d_n 的线性代数方程组如下:

$$bk_n\mathrm{I}_1(k_n)a_n-b[2(1-\nu_{\mathrm{f}})\mathrm{I}_1(k_n)+k_n\mathrm{I}_0(k_n)]c_n-k_n\mathrm{I}_1(k_n)a_n+k_n\mathrm{K}_1(k_n)b_n+[2(1-\nu_{\mathrm{m}})\mathrm{I}_1(k_n)+k_n\mathrm{I}_0(k_n)]c_n-[2(1-\nu_{\mathrm{m}})\mathrm{K}_1(k_n)-k_n\mathrm{K}_0(k_n)]d_n=0\qquad(n=1,2,3,\cdots),\tag{33}$$

$$-b[k_n\mathrm{I}_0(k_n)-\mathrm{I}_1(k_n)]a_n+b[(1-2\nu_{\mathrm{f}})\mathrm{I}_0(k_n)+k_n\mathrm{I}_1(k_n)]c_n+[k_n\mathrm{I}_0(k_n)-\mathrm{I}_1(k_n)]a_n+[k_n\mathrm{K}_0(k_n)+\mathrm{K}_1(k_n)]b_n-[(1-2\nu_{\mathrm{m}})\mathrm{I}_0(k_n)+k_n\mathrm{I}_1(k_n)]c_n-[(1-2\nu_{\mathrm{m}})\mathrm{K}_0(k_n)-k_n\mathrm{K}_1(k_n)]d_n+b(-1)^n(n\pi)^2\times\sum_{m=1}^{\infty}\mathrm{J}_0(\alpha_m)\frac{2\alpha_m d(1-\mathrm{e}^{-2\alpha_m d})}{[(\alpha_m d)^2+(n\pi)^2]^2}h_m-(-1)^n(n\pi)^2\sum_{m=1}^{\infty}\Psi_0(\alpha_m)\times\frac{2\alpha_m d(1-\mathrm{e}^{-2\alpha_m d})}{[(\alpha_m d)^2+(n\pi)^2]^2}h_m=0\qquad(n=1,2,3,\cdots),\tag{34}$$

$$\sum_{n=1}^{\infty}\left\{(-1)^n\frac{k_n^2}{\alpha_m^2+k_n^2}\mathrm{I}_1(k_n)\cdot a_n-(-1)^n\left[(4-2\nu_{\mathrm{f}})\frac{k_n}{\alpha_m^2+k_n^2}\mathrm{I}_1(k_n)+\frac{k_n^2}{\alpha_m^2+k_n^2}\left\langle\mathrm{I}_0(k_n)-\frac{2k_n\mathrm{I}_1(k_n)}{\alpha_m^2+k_n^2}\right\rangle\right]c_n\right\}+\frac{1}{2}\mathrm{J}_0(\alpha_m)\left[\frac{1+\mathrm{e}^{-2\alpha_m d}}{2}+\frac{2\alpha_m d\,\mathrm{e}^{-2\alpha_m d}}{1-\mathrm{e}^{-2\alpha_m d}}\right]h_m=0\qquad(m=1,2,3,\cdots),\tag{35}$$

$$
\begin{aligned}
& k_n \mathrm{I}_0(k_n\beta)\, a_n + k_n \mathrm{K}_0(k_n\beta)\, b_n - [4(1-\nu_{\mathrm{m}})\mathrm{I}_0(k_n\beta) + k_n\beta \mathrm{I}_1(k_n\beta)]\, c_n - \\
&\quad [4(1-\nu_{\mathrm{m}})\mathrm{K}_0(k_n\beta) - k_n\beta\mathrm{K}_1(k_n\beta)]\, d_n + \\
&\quad (-1)^{n+1}\sum_{m=1}^{\infty}\Psi_0(\alpha_m\beta)\,\frac{2k_n^2}{\alpha_m^2+k_n^2}\,\frac{1-\mathrm{e}^{-2\alpha_m d}}{\alpha_m d}\left[(1-\nu_{\mathrm{m}}) + \frac{\alpha_m^2}{\alpha_m^2+k_n^2}\right] h_m = 2(-1)^{n+1} \\
&\qquad\qquad\qquad\qquad (n = 1, 2, 3, \cdots),
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
& -k_n\left[\mathrm{I}_0(k_n\beta) - \frac{1}{k_n\beta}\mathrm{I}_1(k_n\beta)\right] a_n - k_n\left[\mathrm{K}_0(k_n\beta) + \frac{1}{k_n\beta}\mathrm{K}_1(k_n\beta)\right] b_n + \\
&\quad [(1-2\nu_{\mathrm{m}})\mathrm{I}_0(k_n\beta) + k_n\beta\mathrm{I}_1(k_n\beta)]\, c_n + [(1-2\nu_{\mathrm{m}})\mathrm{K}_0(k_n\beta) - \\
&\quad k_n\beta\mathrm{K}_1(k_n\beta)]\, d_n + (-1)^n (n\pi^2)\sum_{m=1}^{\infty}\Psi_0(\alpha_m\beta)\,\frac{2\alpha_m d(1-\mathrm{e}^{-2\alpha_m d})}{[(\alpha_m d)^2 + (n\pi)^2]^2} h_m = 0 \\
&\qquad\qquad\qquad\qquad (n = 1, 2, 3, \cdots),
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
& \frac{1}{2}[\beta^2\Psi_0^2(\alpha_m\beta) - \Psi_0^2(\alpha_m)]\left[\frac{1+\mathrm{e}^{-2\alpha_m d}}{2} + \frac{2\alpha_m d\,\mathrm{e}^{-2\alpha_m d}}{1-\mathrm{e}^{-2\alpha_m d}}\right] h_m + \\
&\quad \sum_{n=1}^{\infty}(-1)^n\frac{k_n^2}{\alpha_m^2+k_n^2}\Big\{[\beta\mathrm{I}_1(k_n\beta)\Psi_0(\alpha_m\beta) - \mathrm{I}_1(k_n)\Psi_0(\alpha_m)]\, a_n + \\
&\quad [\mathrm{K}_1(k_n)\Psi_0(\alpha_m) - \beta\mathrm{K}_1(k_n\beta)\Psi_0(\alpha_m\beta)]\, b_n - \\
&\quad \Big[(4-2\nu_{\mathrm{m}})\langle\beta\mathrm{I}_1(k_n\beta)\Psi_0(\alpha_m\beta) - \mathrm{I}_1(k_n)\Psi_0(\alpha_m)\rangle\frac{1}{k_n} + \\
&\quad \langle\beta^2\mathrm{I}_0(k_n\beta)\Psi_0(\alpha_m\beta) - \mathrm{I}_0(k_n)\Psi_0(\alpha_m)\rangle + \\
&\quad 2\langle\mathrm{I}_1(k_n)\Psi_0(\alpha_m) - \beta\mathrm{I}_1(k_n\beta)\Psi_0(\alpha_m\beta)\rangle\frac{k_n}{\alpha_m^2+k_n^2}\Big]\, c_n - \\
&\quad \Big[(4-2\nu_{\mathrm{m}})\frac{1}{k_n}\langle\mathrm{K}_1(k_n)\Psi_0(\alpha_m) - \beta\mathrm{K}_1(k_n\beta)\Psi_0(\alpha_m\beta)\rangle - \\
&\quad \langle\mathrm{K}_0(k_n)\Psi_0(\alpha_m) - \beta^2\mathrm{K}_0(k_n\beta)\Psi_0(\alpha_m\beta)\rangle - \\
&\quad \frac{2k_n}{\alpha_m^2+k_n^2}\langle\mathrm{K}_1(k_n)\Psi_0(\alpha_m) - \beta\mathrm{K}_1(k_n\beta)\Psi_0(\alpha_m\beta)\rangle\Big]\, d_n\Big\} = 0 \qquad (m = 1, 2, 3, \cdots),
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
& -a_n\mathrm{I}_1(k_n) + c_n\mathrm{I}_0(k_n) + a_n\mathrm{I}_1(k_n) - b_n\mathrm{K}_1(k_n) - c_n\mathrm{I}_0(k_n) - d_n\mathrm{K}_0(k_n) = 0 \\
&\qquad\qquad\qquad\qquad (n = 1, 2, 3, \cdots),
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
& 2(-1)^{n+1}k_n^2\sum_{m=1}^{\infty}\mathrm{J}_0(\alpha_m)\frac{1}{\alpha_m^2+k_n^2}\,\frac{1-\mathrm{e}^{-2\alpha_m d}}{\alpha_m d}\left[(1-\nu_{\mathrm{f}}) + \frac{\alpha_m^2}{\alpha_m^2+k_n^2}\right] h_m - \\
&\quad -2(-1)^{n+1}k_n^2\sum_{m=1}^{\infty}\Psi_0(\alpha_m)\frac{1}{\alpha_m^2+k_n^2}\,\frac{1-\mathrm{e}^{-2\alpha_m d}}{\alpha_m d}\left[(1-\nu_{\mathrm{m}}) + \frac{\alpha_m^2}{\alpha_m^2+k_n^2}\right] h_m + \\
&\quad k_n\mathrm{I}_0(k_n)\, a_n - [4(1-\nu_{\mathrm{f}})\mathrm{I}_0(k_n) + k_n\mathrm{I}_1(k_n)]\, c_n - k_n\mathrm{I}_0(k_n)\, a_n - \\
&\quad k_n\mathrm{K}_0(k_n)\, b_n + [4(1-\nu_{\mathrm{m}})\mathrm{I}_0(k_n) + k_n\mathrm{I}_1(k_n)]\, c_n + \\
&\quad [4(1-\nu_{\mathrm{m}})\mathrm{K}_0(k_n) - k_n\mathrm{K}_1(k_n)]\, d_n = 0 \qquad (n = 1, 2, 3, \cdots),
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
\frac{\sigma_z}{\sigma} = {} & \frac{2\mu_{\mathrm{f}}}{E_{\mathrm{c}}}\Big\{\sum_{m=1}^{\infty}h_m\mathrm{J}_0(\alpha_m\rho)\Big[\Big(1+\alpha_m d\,\frac{1+\mathrm{e}^{-2\alpha_m d}}{1-\mathrm{e}^{-2\alpha_m d}}\Big)\frac{\mathrm{e}^{-\alpha_m(d-\zeta)} + \mathrm{e}^{-\alpha_m(d+\zeta)}}{2} - \\
& \alpha_m\zeta\,\frac{\mathrm{e}^{-\alpha_m(d-\zeta)} - \mathrm{e}^{-\alpha_m(d+\zeta)}}{2}\Big] + \sum_{n=1}^{\infty}\cos k_n\zeta[a_n k_n\mathrm{I}_0(k_n\rho) - \\
& c_n\langle(4-2\nu_{\mathrm{f}})\mathrm{I}_0(k_n\rho) + k_n\rho\mathrm{I}_1(k_n\rho)\rangle]\Big\},
\end{aligned} \tag{41}
$$

$$\frac{\tau_{rz}}{\sigma} = \frac{2\mu_f}{E_c}\left\{\sum_{m=1}^{\infty} h_m J_1(\alpha_m\rho)\left[\left(-\alpha_m d\,\frac{1+e^{-2\alpha_m d}}{1-e^{-2\alpha_m d}}\right)\left\langle\frac{e^{-\alpha_m(d-\zeta)}-e^{-\alpha_m(d+\zeta)}}{2}\right\rangle + \alpha_m\zeta\left\langle\frac{e^{-\alpha_m(d-\zeta)}+e^{-\alpha_m(d+\zeta)}}{2}\right\rangle\right] + \sum_{n=1}^{\infty}\sin k_n\zeta\left[a_n k_n I_1(k_n\rho) - c_n\langle 2(1-\nu_f)I_1(k_n)+k_n I_0(k_n)\rangle\right]\right\} \cdot \quad (42)$$

至此，我们求得了确定待定系数 h_m、a_n、c_n、h_m、a_n、b_n、c_n、d_n 的线性代数方程组• 当截取 $m = M$、$n = N$，则有 $2M + 6N$ 个待定常数，同时也有 $2M + 6N$ 个方程•

3 数值计算及分析

为了检验理论分析的正确性及解的收敛性，在 IBM586 微机上进行了详细的计算• 为了求得 σ_f 和 τ_f 与 ζ、d 和 V_f 的关系，对 ζ、d 和 V_f 三个参数采用固定其中两个参数，让第三个参数变化的方法求得了 σ_f 与 ζ，τ_f 与 ζ，σ_f 与 d，τ_f 与 d，σ_f 与 V_f，τ_f 与 V_f 的变化曲线（见图 2～图 7）• 从图 2 中可以看出，最大纤维应力发生在纤维长度的中央，当 $\zeta = d$ 时，由于纤维末端与基体脱粘，所以 σ_f 为零；从图 3 中可以看出，最大剪应力发生在靠近纤维的末端处；从图 4 中可见，σ_f 随 d 的增大而增大，当 $d > 50$ 后，σ_f 很快趋近于它的最大值；从图 5 可见剪应力随长径比的增加而减小；图 6、7 表明，σ_f、τ_f 随纤维体积含量的增加而减小，这是因为当 V_f 增加时，有更多的纤维参与承受荷载•

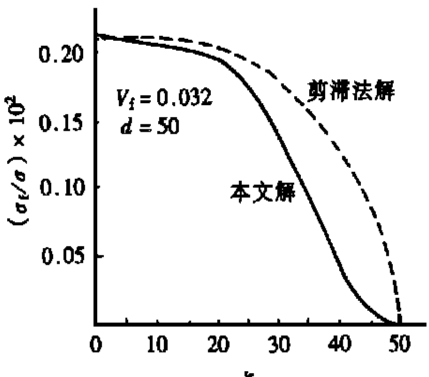

图 2 σ_f/σ 与 ζ 的关系曲线

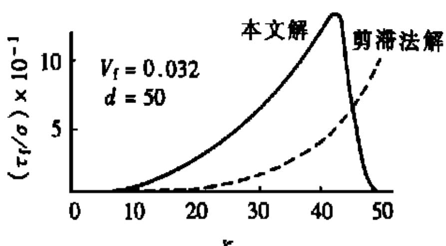

图 3 τ_f/σ 与 ζ 的关系曲线

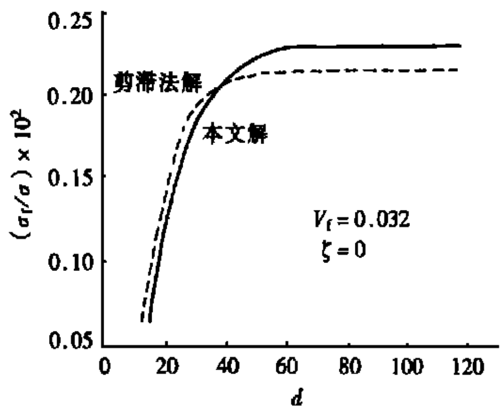

图 4 σ_f/σ 与 d 的关系曲线

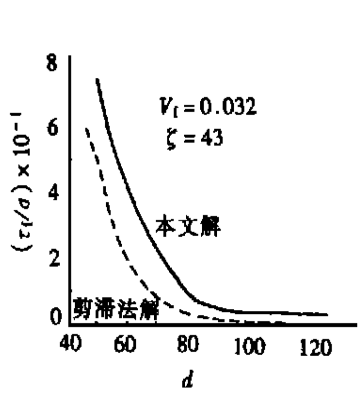

图 5 τ_f/σ 与 d 的关系曲线

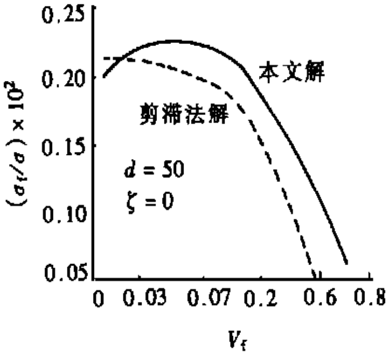

图6 σ_f/σ 与 V_f 的关系曲线

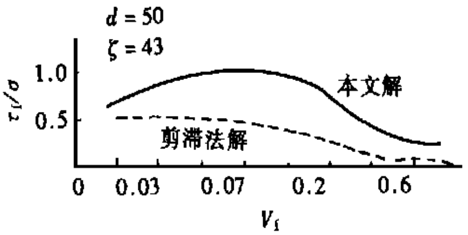

图7 τ_f/σ 与 V_f 的关系曲线

对于弹性理论解, 当项数 N 及 M 取到 12 时, 系数的收敛情况与最终应力计算结果的收敛情况都较好, 说明本文所得应力和位移的表达式是收敛的•

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Studies on Stress Transference Mechanism of Steel Fibre Reinforced Concrete

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Abstract: The stress transfer mechanism of steel fibre reinforced concrete is studied. The solutions for the stress and displacement were regarded as the superposition of "the elementary solutions" and "the improved solutions". The elementary solutions were found by using two dimensional elastic theory and the improved solutions were found by using the Love displacement function method. The calculated results indicate that the solutions possess good convergence. By comparing the three_dimensional solutions with the shear_lag solutions, evident difference may be found.

Key words: steel fiber reinforced concrete; stress transference; elastic theory