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一类共振条件下两个自由度系统的同宿轨道*

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摘要: 研究一类在参数和强迫激励下发生共振时的两个自由度系统, 利用多重尺度法证明存在锁频于 Ω 的周期解· 在一定条件下可变换成 Wiggins 的系统, 给出了判断这类系统同宿轨道存在的计算公式·

关 键 词: 两个自由度系统; 多重尺度法; 周期解; 同宿轨道; 混沌
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引 言

带有立方非线性的两个自由度系统在物理、力学中应用很广, 例如伸长量比较大的弦、梁、膜和板的振动、动力隔振系统、动力消振器、球摆、向心摆和双摆的运动以及非线性弹簧相连的质量的运动等许多问题都可归结为带立方非线性项的两个自由度系统· Nayfeh Mook 主要用奇摄动方法讨论了这类系统的周期解[1]· Holmes 等人研究了一类特殊的两个自由度的 Hamilton 系统的混沌运动, 但是这类问题的研究主要是数值方法·

我们在本文中研究如下一类在参数和强迫激励下的两个自由度系统:

$$\left.\begin{aligned}\ddot{u}_1 + w_1^2 u_1(1+p_1(t)) &= -2\varepsilon^2\mu_1\dot{u}_1 + \alpha_1 u_1^3 + \alpha_3 u_1 u_2^2 + \varepsilon^2 f_1\cos\Omega t,\\ \ddot{u}_2 + w_2^2 u_2(1+p_2(t)) &= -2\varepsilon^2\mu_2\dot{u}_2 + \alpha_6 u_1^2 u_2 + \alpha_8 u_2^3 + \varepsilon^2 f_2\cos\Omega t,\end{aligned}\right\} \tag{1}$$

其中

$$p_1 = \varepsilon^2\beta_1\cos 2\Omega t,\ p_2 = \varepsilon^2\beta_2\cos 2\Omega t,\ \Omega = w_1 + \varepsilon^2\sigma_1,\ \Omega = w_2 + \varepsilon^2\sigma_2·$$

在第 1 节中我们用多重尺度法得到锁频于 Ω 周期解· 在第 2 节中令 $u_1 = \varepsilon B_1\exp[\mathrm{i}\Omega T] + \mathrm{cc.}$, $u_2 = \varepsilon B_2\exp[\mathrm{i}\Omega T] + \mathrm{cc.}$ 可转化为复常微分方程组, 这是类似于[2] 的方程组· 在第 3、第 4 节中研究了这类系统, 得到了判断同宿轨道存在的公式· 这种同宿轨道是导致系统混沌的基本原因·

1 锁频于 Ω 的周期解

令

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$$\left.\begin{aligned} u_1 &= \varepsilon u_{11}(T_0, T_2) + \varepsilon^3 u_{13}(T_0, T_2) + \cdots, \\ u_2 &= \varepsilon u_{21}(T_0, T_2) + \varepsilon^3 u_{23}(T_0, T_2) + \cdots, \end{aligned}\right\} \tag{2}$$

其中 $T_n = \varepsilon^n t$, $D_n = \dfrac{\partial}{\partial T_n}(n = 1, 2)$, 代入(1) 可得

$$\left.\begin{aligned} D_0^2 u_{11} + w_1^2 u_{11} &= 0, \\ D_0^2 u_{21} + w_2^2 u_{21} &= 0. \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} & D_0^2 u_{13} + w_1^2 u_{13} + w_1^2 \beta_1 \cos 2\Omega t \cdot u_{11} = -2D_0(D_2 u_{11} + \mu_1 u_{11}) + \\ & \qquad \alpha_1 u_{11}^3 + \alpha_3 u_{11} u_{21}^2 + f_1 \cos \Omega T_0, \\ & D_0^2 u_{23} + w_2^2 u_{23} + w_2^2 \beta_2 \cos 2\Omega t \cdot u_{21} = -2D_0(D_2 u_{21} + \mu_2 u_{21}) + \\ & \qquad \alpha_6 u_1^2 u_2 + \alpha_8 u_2^3 + f_2 \cos \Omega T_0. \end{aligned}\right\} \tag{4}$$

(3) 的解为

$$\left.\begin{aligned} u_{11} &= A_1(T_2) \exp(\mathrm{i} w_1 T_0) + \mathrm{cc.}, \\ u_{21} &= A_2(T_2) \exp(\mathrm{i} w_2 T_0) + \mathrm{cc.}, \end{aligned}\right\} \tag{5}$$

其中 cc. 表示共轭项. 代入(4), 由消去长期项可得

$$\left.\begin{aligned} & -2\mathrm{i} w_1(A_1' + \mu_1 A_1) - w_1^2 \beta_1 A_1 \exp[2\mathrm{i}\sigma_1 T_2] + 3\alpha_1 A_1^2 A_1 + \alpha_3 A_1 A_2^2 \exp[2(\sigma_1 - \sigma_2) T_2 \mathrm{i}] + \\ & \qquad 2\alpha_3 A_2 A_2 A_1 + \frac{1}{2} f_1 \exp[\mathrm{i}\sigma_1 T_2] = 0, \\ & -2\mathrm{i} w_2(A_2' + \mu_2 A_2) - w_2^2 \beta_2 A_2 \exp[2\mathrm{i}\sigma_2 T_2] + 3\alpha_8 A_2^2 A_2 + 2\alpha_6 A_1 A_1 A_2 + \\ & \qquad \alpha_6 A_2 A_1^2 \exp[2(\sigma_1 - \sigma_2) T_2 \mathrm{i}] + \frac{1}{2} f_2 \exp[\mathrm{i}\sigma_2 T_2] = 0. \end{aligned}\right\} \tag{6}$$

令 $A_m = \dfrac{1}{2} a_m \exp(\mathrm{i}\theta_m)$, $(m = 1, 2)$, (6) 成为

$$\begin{aligned} & -2\mathrm{i} w_1\left(\frac{1}{2} a_1' \exp[\mathrm{i}\theta_1] + \frac{1}{2} a_1 \mathrm{i}\theta_1' \exp[\mathrm{i}\theta_1] + \frac{1}{2}\mu_1 a_1 \exp[\mathrm{i}\theta_1]\right) - \\ & \qquad w_1^2 \beta_1 \frac{1}{2} a_1 \exp[-\mathrm{i}\theta_1 + 2\mathrm{i}\sigma_1 T_2] + \\ & \qquad 3\alpha_1\left(\frac{1}{2} a_1 \exp[\mathrm{i}\theta_1]\right)^2\left(\frac{1}{2} a_1 \exp[-\mathrm{i}\theta_1]\right) + \\ & \qquad \alpha_3\left(\frac{1}{2} a_1 \exp[-\mathrm{i}\theta_1]\right)\left(\frac{1}{2} a_2 \exp[\mathrm{i}\theta_2]\right)^2 \exp[2(\sigma_1 - \sigma_2)\mathrm{i} T_2] + \\ & \qquad 2\alpha_3 \frac{1}{8} a_2^2 a_1 \exp[\mathrm{i}\theta_1] + \frac{1}{2} f_1 \exp[\mathrm{i}\sigma_1 T_2] = 0, \\ & -2\mathrm{i} w_2\left(\frac{1}{2} a_2' \exp[\mathrm{i}\theta_2] + \frac{1}{2} a_2 \mathrm{i} \exp[\mathrm{i}\theta_2]\theta_2' + \frac{1}{2}\mu_2 a_2 \exp[\mathrm{i}\theta_2]\right) - \\ & \qquad w_2^2 \beta_2 \frac{1}{2} a_2 \exp[-\mathrm{i}\theta_2 + 2\mathrm{i}\sigma_2 T_2] + \\ & \qquad 3\alpha_8\left(\frac{1}{2} a_2 \exp[\mathrm{i}\theta_2]\right)^2\left(\frac{1}{2} a_2 \exp[-\mathrm{i}\theta_2]\right) + \\ & \qquad 2\alpha_6\left(\frac{1}{2} a_1 \exp[-\mathrm{i}\theta_1]\right)\left(\frac{1}{2} a_1 \exp[\mathrm{i}\theta_1]\right)\left(\frac{1}{2} a_2 \exp[\mathrm{i}\theta_2]\right) + \end{aligned}$$

$$\alpha_6 \frac{1}{2} a_2 \exp[-\mathrm{i}\theta_2]\left(\frac{1}{2} a_1 \exp[\mathrm{i}\theta_1]\right)^2 \exp[2(\sigma_2-\sigma_1)T_2 \mathrm{i}] + \frac{1}{2} f_2 \exp[i\sigma_2 T_2] = 0.$$

分离实虚部得

$$\left.\begin{aligned}
&-w_1(a_1'+\mu_1 a_1)-w_1^2\beta_1 \frac{1}{2} a_1 \sin(2\sigma_1 T_2-2\theta_1)+ \\
&\quad \frac{\alpha_3}{8} a_1 a_2^2 \sin[2\theta_2-2\theta_1+2(\sigma_1-\sigma_2)T_2]+ \\
&\quad \frac{1}{2} f_1 \sin(\sigma_1 T_2-\theta_1)=0, \\
&w_1 a_1 \theta_1'-w_1^2\beta_1 \frac{1}{2} a_1 \cos(2\sigma_1 T_2-2\theta_1)+\frac{3}{8}\alpha_1\alpha_1^3+\frac{2}{8}\alpha_3 a_2^2 a_1+ \\
&\quad \frac{\alpha_3}{8} a_1 a_2^2 \cos[2\theta_2-\theta_1+2(\sigma_1-\sigma_2)T_2]+\frac{1}{2} f_1 \cos(\sigma_1 T_2-\theta_1)=0, \\
&-w_1(a_2'+\mu_2 a_2)-w_2^2\beta_2 \frac{1}{2} a_1 \sin(2\sigma_2 T_2-2\theta_2)+ \\
&\quad \frac{\alpha_6}{8} a_2 a_1^2 \sin[2\theta_1-2\theta_2+2(\sigma_2-\sigma_1)T_2]+ \\
&\quad \frac{1}{2} f_2 \sin(\sigma_2 T_2-\theta_2)=0, \\
&w_2 a_2 \theta_2'-w_2^2\beta_2 \frac{1}{2} a_2 \cos(2\sigma_2 T_2-2\theta_2)+\frac{3}{8}\alpha_8 a_2^3+\frac{2}{8}\alpha_6 a_1^2 a_2+ \\
&\quad \frac{\alpha_6}{8} a_2 a_1^2 \cos[2\theta_1-2\theta_2+2(\sigma_2-\sigma_1)T_2]+\frac{1}{2} f_2 \cos(\sigma_2 T_2-\theta_2)=0.
\end{aligned}\right\} \qquad (7)$$

令 $r_1 = 2\theta_2 - 2\theta_1 + 2(\sigma_1-\sigma_2)T_2$, $r_2 = \sigma_1 T_2 - \theta_1$, (7) 成为

$$\left.\begin{aligned}
&-w_1(a_1'+\mu_1 a_1)-w_1^2\beta_1 \frac{1}{2} a_1 \sin 2r_2+\frac{\alpha_3}{8} a_1 a_2^2 \sin r_1+\frac{1}{2} f_1 \sin r_2=0, \\
&r_1'=-2\left[-\frac{1}{2} w_2\beta_2 \cos(-r_1+2r_2)+\frac{3}{8w_2}\alpha_8 a_2^2+\frac{1}{4w_2}\alpha_6 a_1^2+\frac{\alpha_6 a_1^2}{8w_2}\cos r_1+\right. \\
&\quad \left.\frac{1}{2}\frac{f_2}{(a_2 w_2)}\cos\frac{1}{2}(2r_2-r_1)\right]+2\left[-\frac{1}{2} w_1\beta_1 \cos 2r_2+\frac{3}{8w_1}\alpha_1+\frac{1}{4w_1}\alpha_3 a_2^2+\right. \\
&\quad \left.\frac{\alpha_3}{8w_1} a_2^2 \cos r_1+\frac{1}{2w_1 a_1} f_1 \cos r_2\right]+2(\sigma_1-\sigma_2), \\
&-w_2(a_2'+\mu_2 a_2)-w_2^2\beta_2 \frac{1}{2} a_1 \sin(2r_2-r_1)+\frac{\alpha_6}{8} a_2 a_1^2 \sin(-r_1)+ \\
&\quad \frac{1}{2} f_2 \sin\frac{1}{2}(2r_2-r_1)=0, \\
&r_2'=\sigma_1+\frac{1}{w_1 a_1}\left[-\omega_1^2\beta_1 \frac{1}{2} a_1 \cos 2r_2+\frac{3}{8}\alpha_1 a_1^3+\frac{1}{4}\alpha_3 a_2^2 a_1+\right. \\
&\quad \left.\frac{\alpha_3}{8} a_1 a_2^2 \cos r_1+\frac{f_1}{2}\cos r_2\right].
\end{aligned}\right\}$$

对于静态解 $a_n' = r_n' = 0 (n = 1, 2)$.

$$\left.\begin{aligned}
&-w_1\mu_1 a_1-\frac{1}{2}w_1^2\beta_1 a_1\sin 2r_2+\frac{\alpha_3}{8}a_1a_2^2\sin r_1+\frac{1}{2}f_1\sin r_2=0,\\
&w_2\beta_2\cos(-r_1+2r_2)-\frac{3\alpha_8}{4w_2}a_2^2-\frac{1}{2w_2}\alpha_6a_1^2-\frac{\alpha_6a_1^2}{4w_2}\cos r_1-\frac{f_2}{a_2w_2}\cos\left(r_2-\frac{r_1}{2}\right)-\\
&\qquad w_1\beta_1\cos 2r_2+\frac{3\alpha_1}{4w_1}a_1^2+\frac{1}{2w_1}\alpha_3a_2^2+\frac{\alpha_3}{8w_1}a_2^2\cos r_1+\\
&\qquad \frac{f_1}{a_1w_1}\cos r_2+2(\sigma_1-\sigma_2)=0,\\
&-w_2\mu_2a_2-\frac{1}{2}w_2^2\beta_2a_1\sin(2r_2-r_1)+\frac{\alpha_6}{8}a_2a_1^2\sin(-r_1)+\\
&\qquad \frac{1}{2}f_2\sin\frac{1}{2}(2r_2-r_1)=0,\\
&\sigma_1-\frac{1}{2}w_1\beta_1\cos 2r_2+\frac{3\alpha_1}{8w_1}a_1^2+\frac{\alpha_3}{4w_1}a_2^2+\frac{\alpha_3}{8w_1}a_2^2\cos r_1+\frac{1}{2w_1a_1}f_1\cos r_2=0.
\end{aligned}\right\}\quad(8)$$

从(8)中解出 $a_n, r_n, (n=1,2)$, 故有

$$\left.\begin{aligned}
u_1&=\frac{\varepsilon}{2}a_1\exp[-r_2\mathrm{i}]\exp[\mathrm{i}\Omega t]+O(\varepsilon^3)+\mathrm{cc}.\\
u_2&=\frac{\varepsilon}{2}a_2\exp\left[\frac{r}{2}\mathrm{i}\right]\exp[\mathrm{i}\Omega t]+O(\varepsilon^3)+\mathrm{cc}.
\end{aligned}\right\}$$

2　复常微分方程组

再令 $u_i=\varepsilon B_i\exp[\mathrm{i}\Omega T_0]+\mathrm{cc}.(i=1,2)$ 代入(1), 可得到

$$\left.\begin{aligned}
D_2B_1&=-\mu_1B_1+\frac{-\mathrm{i}}{2\Omega}\Big[2B_1\sigma_1w_1-w_1^2\beta_1B_1+3\alpha_1B_1^2\bar{B}_1+\\
&\qquad 2\alpha_3B_1B_2\bar{B}_2+\alpha_3\bar{B}_1B_2^2+\frac{f_1}{2}\Big],\\
D_2B_2&=-\mu_2B_2+\frac{-\mathrm{i}}{2\Omega}\Big[2B_2\sigma_2w_2-w_2^2\beta_2\bar{B}_2+3\alpha_8B_2^2\bar{B}_2+\\
&\qquad 2\alpha_6B_1\bar{B}_1B_2+\alpha_6B_1^2\bar{B}_2+\frac{f_2}{2}\Big].
\end{aligned}\right\}\quad(9)$$

设　$\Delta=\frac{w_1^2\beta_1}{2\Omega}=\frac{w_2^2\beta_2}{2\Omega},\ -\frac{3\alpha_1}{2\Omega}=-\frac{3\alpha_8}{2\Omega}=\pi_1,\ -\frac{2\alpha_3}{2\Omega}=-\frac{2\alpha_6}{2\Omega}=\pi_2,$

$\pi_3=\frac{1}{2}\pi_2,\ -\frac{\sigma_1w_1}{\Omega}=\sigma-\beta,\ -\frac{\sigma_2w_2}{\Omega}=\sigma+\beta,$

(9) 可化为类似于[2] 的形式, 但这是 f_1、f_2 不为 0, $\mu_1\neq\mu_2$ 的形式, 计算将更复杂. (9) 可写为

$$\left.\begin{aligned}
D_2B_1&=-\mu_1B_1+\mathrm{i}\Big[-(\sigma-\beta)B_1+\Delta\bar{B}_1+\pi_1B_1^2\bar{B}_1+\pi_2B_1B_2\bar{B}_2+\pi_3\bar{B}_1B_2^2-\frac{f_1}{2\Omega}\Big],\\
D_2B_2&=-\mu_2B_2+\mathrm{i}\Big[-(\sigma+\beta)B_2+\Delta\bar{B}_2+\pi_1B_2^2\bar{B}_2+\pi_2B_1\bar{B}_1B_2+\pi_3\bar{B}_2B_1^2-\frac{f_2}{2\Omega}\Big].
\end{aligned}\right\}$$

(10)

令 $B_1=x_1+\mathrm{i}y_1,\ B_2=x_2+\mathrm{i}y_2$, 则(10) 成为

$$\left.\begin{aligned}
x_1' &= (\sigma - \pi_1 E + \Delta) y_1 - 2\pi_3 L x_2 - \beta y_1 + c y_1 (x_2^2 + y_2^2) - \mu_1 x_1, \\
y_1' &= (-\sigma + \pi_1 E + \Delta) x_1 - 2\pi_3 L y_2 + \beta x_1 - c x_1 (x_2^2 + y_2^2) - \mu_1 y_1 - \frac{f_1}{2\Omega}, \\
x_2' &= (\sigma - \pi_1 E + \Delta) y_2 + 2\pi_3 L x_1 + \beta y_2 + c y_2 (x_1^2 + y_1^2) - \mu_2 x_2, \\
y_2' &= (-\sigma + \pi_1 E + \Delta) x_2 + 2\pi_3 L y_1 - \beta x_2 - c x_2 (x_1^2 + y_1^2) - \mu_2 y_2 - \frac{f_1}{2\Omega}.
\end{aligned}\right\} \tag{11}$$

设 $H = \frac{\sigma}{2}E - \frac{\pi_1}{4}E^2 + \pi_3 L^2 + \frac{\Delta}{2}(y_1^2 + y_2^2 - x_1^2 - x_2^2) + \frac{\beta}{2}(x_2^2 + y_2^2 - x_1^2 - y_1^2) + \frac{c}{2}(x_1^2 + y_1^2)(x_2^2 + y_2^2),$

其中

$$\left.\begin{aligned}
&E = x_1^2 + y_1^2 + x_2^2 + y_2^2,\ L = x_1 y_2 - x_2 y_1, \\
&c = \pi_1 - \pi_2 - \pi_3 = -\frac{3\alpha_1}{2\Omega} + \frac{3\alpha_6}{2\Omega}.
\end{aligned}\right\} \tag{12}$$

作变换

$$\left.\begin{aligned}
&x_1 = q_1 \cos q_2,\ x_2 = q_1 \sin q_2, \\
&y_1 = p_1 \cos q_2 - p_2 q_1^{-1} \sin q_2,\ y_2 = p_1 \sin q_2 + p_2 q_1^{-1} \cos q_2.
\end{aligned}\right\}$$

逆变换为

$$\left.\begin{aligned}
&q_1 = \sqrt{x_1^2 + x_2^2},\ p_1 = \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2}}, \\
&q_2 = \arctan x_2/x_1,\ p_2 = x_1 y_2 - x_2 y_1.
\end{aligned}\right\}$$

并引入小参数 ε_1, 即令 $\beta \to \varepsilon_1 \beta$, $c \to \varepsilon_1 c$, $\mu_i \to \mu_i \varepsilon_1$, $f_i \to \varepsilon_1 f_i\ (i = 1, 2)$ (12) 取形式为

$$H = H_0 + \varepsilon_1 H_1 = \frac{\sigma}{2}E - \frac{\pi_1}{4}E^2 + \pi_3 p_2^2 + \frac{\Delta}{2}(p_1^2 + p_2^2 q_1^{-2} - q_1^2) + \varepsilon_1\left[\frac{\beta}{2}D + \frac{c}{8}(E^2 - D^2)\right],$$

其中

$$E = x_1^2 + y_1^2 + x_2^2 + y_2^2 = q_1^2 + p_1^2 + (p_2 q_1^{-1})^2 = E,$$

$$D = x_2^2 + y_2^2 - x_1^2 - y_1^2 = 2p_2 p_1 q_1^{-1} \sin 2q_2 + (p_2^2 q_1^{-2} - q_1^2 - p_1^2)\cos 2q_2.$$

在新坐标下(11)成为

$$\left.\begin{aligned}
q_1' &= \frac{\partial H_0}{\partial p_1} + \varepsilon_1 \frac{\partial H_1}{\partial p_1} - \varepsilon_1 \mu_1 q_1 \cos^2 q_2 - \varepsilon_1 \mu_2 q_1 \sin^2 q_2, \\
p_1' &= -\frac{\partial H_0}{\partial q_1} - \varepsilon_1 \frac{\partial H_1}{\partial q_1} - \frac{q_1^{-1}}{2\Omega} f_1 \varepsilon_1 (p_1 \cos q_2 - p_2 q_1^{-1} \sin q_2) - \\
&\quad \frac{g_1^{-1}}{2\Omega} f_2 \varepsilon_1 (p_1 \sin q_2 + p_2 q_1^{-1} \cos q_2) - 2\varepsilon_1 \mu_1 q_1 \cos q_2 [p_1 \cos q_2 - p_1 q_1^{-1} \sin q_2] - \\
&\quad 2\varepsilon_1 \mu_2 q_1 \sin q_2 [p_1 \sin q_2 + p_2 q_1^{-1} \cos q_2], \\
q_2' &= \frac{\partial H_0}{\partial p_2} + \varepsilon_1 \frac{\partial H_1}{\partial p_2}, \\
p_2' &= -\varepsilon_1 \frac{\partial H_1}{\partial q_2} - \varepsilon_1 \mu_1 p_2 - \varepsilon_1 \mu_2 p_2.
\end{aligned}\right\} \tag{13}$$

当 $\varepsilon_1 = 0$ 时, (13) 是如下形式的可积 Hamilton 系统

$$\left.\begin{aligned}
q_1' &= \frac{\partial H_0}{\partial p_1} = (\sigma - \pi_1 E + \Delta) p_1 = [\sigma - \pi_1(q_1^2 + p_1^2 + (p_2 q_1^{-1})^2] p_1, \\
p_1' &= -\frac{\partial H_0}{\partial q_1} = -(\sigma - \pi_1 E)(q_1 - p_2^2 q_1^{-3}) + \Delta(p_2^2 q_1^{-3} + q_1), \\
q_2' &= \frac{\partial H_0}{\partial p_2} = p_2[(\sigma + \Delta) q_1^{-2} + E q_1^{-2} + 2\pi_3], \\
p_2' &= 0.
\end{aligned}\right\} \tag{14}$$

3 未摄动系统的结构

(14) 具有 2 维不变流形

$$\mu = \left\{ q_1 = q_{10}(p_2, \sigma), p_1 = p_{10}(p_2, \sigma), p_2 \in R, q_2 \in s^1 \right\}, \tag{15}$$

其中 $q_{10}(p_2, \sigma)$ 和 $p_{10}(p_2, \sigma)$ 是(14) 的 $q_1 - p_1$ 分量的双曲不动点, R 是 p_2 的待定区间(σ 固定)• 我们希望 R 包含共振的双曲不动点, 即 $q_2'(q_{10}(p_2, \sigma)、p_{10}(p_2, \sigma), p_2) = 0$, (14) 的 $q_1 - p_1$ 分量不动点是

$$\left.\begin{aligned}
& p_1 = 0, \\
& -\sigma(q_1 - p_2^2 q_1^{-3}) + \pi_1(q_1^2 + p_2^2 q_1^{-2})(q_1 - p_2^2 q_1^{-3}) + \Delta(q_1 + p_2^2 q_1^{-3}) = 0,
\end{aligned}\right\} \tag{16}$$

或者

$$\left.\begin{aligned}
& \sigma - \pi_1(q_1^2 + p_1^2 + p_2^2 q_1^{-2}) + \Delta = 0, \\
& -\sigma(q_1 - p_2^2 q_1^{-3}) + \pi_1(q_1^2 + p_1^2 + p_2^2 q_1^{-2})(q_1 - p_2^2 q_1^{-3}) + \Delta(q_1 + p_2^2 q_1^{-3}) = 0,
\end{aligned}\right\} \tag{17}$$

的解经计算只有(16) 才能给出不动点• (16) 可重写为

$$g(z) = \pi_1 z^4 + (\Delta - \sigma) z^3 + p_2^2(\Delta - \sigma) z - \pi_1 p_2^4 = 0, \tag{18}$$

其中 $z = q_1^2$• 当 $\pi_1 < 0$ 时 $g(0) = -\pi_1 p_2^4 > 0$ 和 $g(\pm\infty) = -\infty$, 必存在一个或 3 个正解•

我们要求共振双曲不动点, 即必须满足

$$\left.\begin{aligned}
& p_1 = 0, \\
& -\sigma(q_1 - p_2^2 q_1^{-3}) + \pi_1(q_1^2 + p_2^2 q_1^{-2})(q_1 - p_2^2 q_1^{-3}) + \Delta(q_1 + p_2^2 q_1^{-3}) = 0. \\
& (\Delta + \sigma) q_1^{-2} - \pi_1(q_1^2 + p_2^2 q_1^{-2}) q_1^{-2} + 2\pi_3 = 0.
\end{aligned}\right\} \tag{19}$$

当 $\sigma < N < -1$ 时可从(19) 得解为

$$\left.\begin{aligned}
p_1^r &= 0, \\
q_1^r &= \left(-\frac{(N + \Delta)(\sigma - N)}{2N} \right)^{1/2}, \\
p_2^r &= \pm\left(\frac{(N + \Delta)\sqrt{\sigma^2 - N^2}}{2N} \right).
\end{aligned}\right\} \tag{20}$$

其中上标“r” 记共振, $N = \left(\frac{-\Delta - \Delta\pi_3}{\pi_3} \right)$•

将(20) 代入 Jacobian 表达式, 记为 J,

$$\det J = \frac{4\Delta(N^3 + \Delta\sigma^2)}{N^2}, N^3 + \Delta\sigma_2 < 0.$$

在 (N, σ) 平面上, 可定义($\Delta > 0$)

$$W = \left\{ N < -1,\ -\left(\frac{N^3}{-\Delta}\right)^{1/2} < \sigma < N \right\}.$$

这给出了包含共振双曲不动点的区域, 而且

$$p_2^r = \frac{(N+\Delta)\sqrt{\sigma^2 - N^2}}{2N} < \frac{1}{2}(\sigma^{2/3} - 1)^{3/2} \qquad (\sigma < N < -1).$$

为了利用 Melnikon 方法必须求出未摄动方程的同宿轨道. 为此引入以下坐标变换

$$\left.\begin{aligned} &E = x_1^2 + x_2^2 + y_1^2 + y_2^2,\ W = 2(x_1y_1 + x_2y_2), \\ &r = (x_1^2 + x_2^2)^{1/2},\ \varphi = \arctan x_2/x_1 \end{aligned}\right\} \tag{21}$$

得到

$$\left.\begin{aligned} &E' = 4\Delta x_1y_1 + 4\Delta x_2y_2, \\ &W' = 2(\pi_1E - \sigma)(x_1^2 + x_2^2 - y_1^2 - y_2^2) + 2\Delta E. \end{aligned}\right\} \tag{22}$$

由于 $H = \frac{\sigma}{2}E - \frac{\pi_1}{4}E_2 + \pi_3L^2 + \frac{\Delta}{2}(y_1^2 + y_2^2 - x_1^2 - x_2^2) = h$. (22) 成为

$$\left.\begin{aligned} &E' = 2\Delta w, \\ &W' = \frac{2(\pi_1E - \sigma)}{\Delta}\left(\sigma E - \frac{\pi_1}{2}E^2 + 2\pi_3p_2^2 - 2h\right) + 2\Delta E. \end{aligned}\right\} \tag{23}$$

于是 $\quad h = H(q_1^r, 0, p_2^r) = \frac{\sigma}{2}E_0 - \frac{\pi_1}{4}E_0^2 + \pi_3(p_2^r)^2 + \frac{\Delta}{2}[(p_2^r)^2(q_2^r)^{-2} - (q_1^r)^2].$

不动点 $(q_1^r, p_1^r = 0, p_2^r)$ 可映为 $E = E_0$, $W = 0$ 其中

$$E_0 = (p_2^r)^2(q_1^r)^{-2} + (q_1^r)^2 = -\frac{(N+\Delta)2\sigma}{2N}.$$

再令 $E = E_0 + V$, 可将不动点移到原点, (21) 成为

$$\left.\begin{aligned} &V' = 2\Delta w, \\ &W' = -\frac{V^3}{\Delta} + \frac{3\sigma}{N}V^2 - 2\left(\frac{\sigma^2\Delta}{N^2} + N\right)V. \end{aligned}\right\} \tag{24}$$

这里为容易计算取 $\pi_1 = -1$, (24) 的 Hamilton 函数为

$$\Delta W^2 + \frac{1}{4}\frac{V^4}{\Delta} - \frac{\sigma}{N}V^3 + \left(\frac{\sigma^2\Delta}{N^2} + N\right)V^2 = h,$$

当 $h = 0$

$$\Delta W^2 = \frac{1}{4\Delta}V^2[-(V - V_1)(V - V_2)],$$

其中 $\quad V_{1,2} = \frac{2\sigma\Delta}{N} \pm 2\sqrt{-\Delta N},\ V_1 < 0,\ V_2 > 0.$

因此 $\quad V' = V[-(V - V_1)(V - V_2)]^{1/2}. \qquad (25)$

当 $V > 0$ 为同宿轨 A

$$V = \frac{-2V_1}{(k^2+1)\cosh(vT_2) + k^2 - 1},\ v = \sqrt{-V_1V_2},\ k^2 = -\frac{V_1}{V_2}. \tag{26}$$

当 $V < 0$ 为, 同宿轨 B

$$V = \frac{2V_1}{(k^2+1)\cosh(vT_2) + 1 - k^2}, \tag{27}$$

此时 $\quad q_1^2 = x_1^2 + x_2^2 = \frac{1}{4N}\left\{\frac{NV^2}{\Delta} + 2V(N - \sigma) - 2(N+\Delta)\sigma + 2N(N+\Delta)\right\}. \qquad (28)$

从这个表达式可计算出相移位, 当 V 和q_1 已知时可从 $q_1^2 + p_1^2 + p_2^2q_1^{-2} = E_0 + V$ 中得到p_1. 从

图中可画出 $N=-2,\ \sigma=-2.5,\ \Delta=1$ 时同宿轨道，对应于 $(q_1^r, p_1^r, p_2^r)=(\sqrt{0.125}, 0, 0.375)$•（见图 1）•

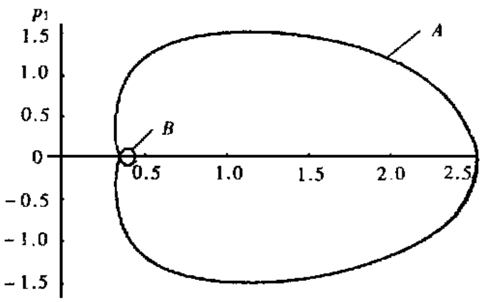

图 1

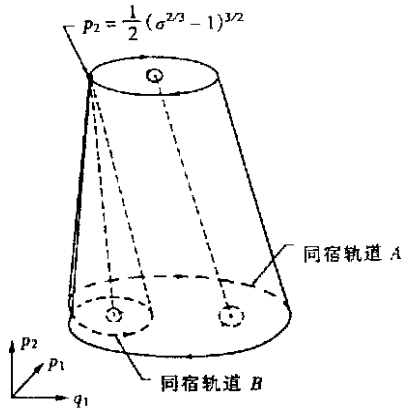

图 2

经计算得

$$p_1^2=-\frac{(N+\Delta)2\sigma}{2N}+V-\frac{1}{4N}\left\{\frac{NV^2}{\Delta}+2V(N-\sigma)-2(N+\Delta)(\sigma-N)\right\},$$

$$q'_2=-\left[\frac{2p_2^r}{N+\Delta}\right]\frac{NV^2-2N^2V-2\Delta\sigma V}{\frac{NV^2}{\Delta}+2V(N-\sigma)-2(N+\Delta)(\sigma-N)}\bullet$$

对于同宿轨道 A:

$$V=\frac{-2V_1}{(k^2+1)\cosh vT_2+k^2-1}=\frac{-2\left(\frac{\sigma^2\Delta^2}{N^2}+\Delta N\right)}{\sqrt{-\Delta N}\cosh vT_2-\frac{\sigma}{N}},$$

此时

$$q'_2=\left[\frac{-4p_2^r(\sigma^2\Delta^2+N^3\Delta)}{N+\Delta}\right]\frac{\sqrt{-N}(N^2+\Delta\sigma)\cosh vT_2+N(\Delta N-\sigma)}{a\cosh^2 vT_2+b\cosh vT_2+C},\tag{29}$$

其中

$$a=\Delta N^3(N+\Delta)(\sigma-N)<0,$$

$$b=2\sqrt{-N}(\Delta^2\sigma^2+\sigma N^2+\Delta N\sigma+\Delta N^3)(\sigma-N)>0,$$

$$c=\frac{2N^3}{\Delta}\left(\frac{\sigma^2\Delta^2}{N^2}+\Delta N\right)^2+2N\sigma\left(\frac{\sigma^2\Delta^2}{N^2}+\Delta N\right)(N-\sigma)-(N+\Delta)(\sigma-N)\sigma^2\bullet$$

对于同宿轨道 B:

$$q'_2=\left[\frac{4p_2^r(\sigma^2\Delta^2+\Delta N^3)}{N+\Delta}\right]\bullet\left[\frac{\sqrt{-N}(N^2+\Delta\sigma)\cosh vT_2-N(\Delta N-\sigma)}{a\cosh^2 vT_2-b\cosh vT_2+C}\right]\tag{30}$$

(29) 和 (30) 可简化为:

$$q'_2=k\lambda v\frac{\cosh vT_2+\rho}{(\cosh vT_2-\lambda)^2+k^2}\quad(\text{对同宿轨道 }A),\tag{31}$$

$$q'_2=-k\lambda v\frac{\cosh vT_2+\rho}{(\cosh vT_2+\lambda)^2+k^2}\quad(\text{对同宿轨道 }B),\tag{32}$$

其中

$$K=\frac{2}{bv}\left[\frac{4p_2^r(\sigma^2\Delta^2+\Delta N^3)}{N+\Delta}\right]\frac{\sqrt{-N}(N^2+\Delta\sigma)}{a}$$

$$\lambda = -\frac{b}{2a} > 0,\ k = \left(\frac{4ac - b^2}{4a^2}\right) > 0,\ \rho = \frac{N(\Delta N - \sigma)}{\sqrt{-N}(N^2 + \Delta\sigma)}$$

如定义 $I_0 = \lambda\int_{-\infty}^{+\infty}\frac{\cosh u + \rho}{(\cosh u - \lambda) + k^2}\mathrm{d}u,$

$$J_0 = -\lambda\int_{-\infty}^{+\infty}\frac{\cosh u - \rho}{(\cosh u + \lambda) + k^2}\mathrm{d}u.$$

则相移位表达式可简单写为

$$\delta q_2 = q_2(+\infty) - q_2(-\infty) = kI_0 \qquad (\text{对轨道 } A)$$

$$\delta q_2 = q_2(+\infty) - q_2(-\infty) = kJ_0 \qquad (\text{对轨道 } B)$$

未摄动系统的相空间几何结构可见图 2.

4 摄动系统的动力学: Silnikov 轨道存在性

首先推导共振附近限制于 μ_ε 的向量场, 公式如下:

$$\left.\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}\tau} &= -\frac{\partial H_1}{\partial q_2}(q_1^r, p_1^r, p_2^r, r) - 2dp_2^r + O(\sqrt{\varepsilon}),\\ \frac{\mathrm{d}r}{\mathrm{d}\tau} &= \left(\frac{\partial^2 H_0}{\partial q_1 \partial p_2}\frac{\partial q_{10}}{\partial p_2} + \frac{\partial^2 H_0}{\partial p_1 \partial p_2}\frac{\partial p_{10}}{\partial p_2} + \frac{\partial^2 H_0}{\partial p_2^2}\right)(q_1^r, p_1^r, p_2^r)h + O(\sqrt{\varepsilon}).\end{aligned}\right\} \tag{33}$$

其中伸长时间 $\tau = \sqrt{\varepsilon_1}T_2$. 因此在不动点共振圆周附近在 μ_ε 上动力学行为可用以下方程描述:

$$\left.\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= -(\mu_1 + \mu_2)\left(\frac{(N + \Delta)\sqrt{\sigma^2 - N^2}}{2N}\right) + \frac{c}{4}(N + \Delta)^2\sin 4q_2 + \\ &\quad \frac{\beta}{2}(2N + 2\Delta)\sin 2q_2 + O(\sqrt{\varepsilon_1}) \triangleq \hat{d} - \hat{c}\sin 4r + \beta\sin 2r + O(\sqrt{\varepsilon_1}),\\ \frac{\mathrm{d}r}{\mathrm{d}\tau} &= \left[\frac{4(N^2 + \sigma\Delta)N\sqrt{-N}}{-3N\sigma^2 - 3N^2\sigma + 3N\sigma + 3N^2 - 4N^3 - 4\Delta\sigma^2} + \frac{-2N(\sigma + \Delta)}{(N + \Delta)(\sigma + N)} + \right.\\ &\quad \left.\frac{3(\sigma + N)}{\sigma - N} + 2\pi_3 + 1\right]h + O(\sqrt{\varepsilon_1}) \triangleq \hat{a}^2 h + O\sqrt{\varepsilon_1}).\end{aligned}\right\} \tag{34}$$

当 $\varepsilon_1 = 0$ 时, (34) 是 Hamilton 系统, 其 Hamilton 函数为

$$\mathscr{H} = \frac{\hat{a}^2}{2}h^2 - \hat{d}r + \frac{\hat{c}}{4}\cos 4r - \frac{\beta}{2}\cos 2r,$$

这是典型的"摆类型"系统, 在 μ_ε 上不动点为

$$\left.\begin{aligned}&h = 0,\\ &-\hat{d} - \hat{c}\sin 4r + \beta\sin 2r = 0.\end{aligned}\right\} \tag{35}$$

决定不动点稳定性的 Jacobi 矩阵为

$$\begin{bmatrix} 0 & -4\hat{c}\cos 4r + 2\beta\cos 2r \\ -\hat{a}^2 & 0 \end{bmatrix}$$

我们将用 r_c 记中心坐标, r_s 为鞍点的坐标. 在中心我们有

$$-4\hat{c}\cos 4r_c + 2\beta\cos 2r_c > 0.$$

在鞍点处我们有

$$-4\hat{c}\cos 4r_c + 2\beta\cos 2r_s < 0$$

可分为以三种情况研究:

情况 $\beta = 0$: 如果 $\beta = 0$, 当 $|\hat{c}| > \hat{d}$, 存在中心型不动点为

$$r_c=\begin{cases}-\dfrac{1}{4}\sin^{-1}\dfrac{\delta}{\hat{c}},\ \mathrm{mod}\,\dfrac{\pi}{2},\ \hat{c}<0,\\ -\dfrac{1}{4}\left(\pi-\arcsin\dfrac{\delta}{\hat{c}}\right),\ \mathrm{mod}\,\dfrac{\pi}{2},\ \hat{c}>0.\end{cases}$$

鞍点型不动点为

$$r_s=\begin{cases}\dfrac{\pi}{4}-r_c,\ \mathrm{mod}\,\dfrac{\pi}{2},\ \hat{c}<0,\\ -\dfrac{\pi}{4}-r_c,\ \mathrm{mod}\,\dfrac{\pi}{2},\ \hat{c}>0.\end{cases}$$

点 $(h,r)=(0,r)$ 处在由连接鞍点的同宿轨道所包围的区域中必须

$$r_n<r\,\mathrm{mod}\,\frac{\pi}{2}<r_s,$$

其中 r_n 由

$$-\frac{\delta}{\hat{c}}r_n+\frac{1}{4}\cos 4r_n=-\frac{\delta}{\hat{c}}r_s+\frac{1}{4}\cos 4r_s$$

解给出.

情况 $\hat{c}=0$: 如果 $\hat{c}=0$, 当 $|\beta|>\delta$, 存在中心型不动点为

$$r_c=\begin{cases}-\dfrac{1}{2}\left(\pi+\arcsin\dfrac{\delta}{\beta}\right),\ \mathrm{mod}\,\pi,\ \beta<0\\ \dfrac{1}{2}\arcsin\dfrac{\delta}{\beta},\ \mathrm{mod}\,\pi,\ \beta>0\end{cases}$$

鞍点型不动点为

$$r_s=\begin{cases}-\dfrac{\pi}{2}-r_c,\ \mathrm{mod}\,\pi,\ \beta<0\\ \dfrac{\pi}{2}-r_c,\ \mathrm{mod}\,\pi,\ \beta>0\end{cases}$$

点 (h,r) 处在连接鞍点的同宿轨道包围的区域中

必须 $r_n<r\,\mathrm{mod}\,\pi<r_s$

其中 r_n 由

$$\frac{\delta}{\beta}r_n+\frac{1}{2}\cos 2r_n=\frac{\delta}{\beta}r_s+\frac{1}{2}\cos 2r_s$$

解给出.

一般情况 $\beta\neq 0,\ \hat{c}\neq 0$ 时情况复杂以后再讨论.

现在 Melnikov 函数为

$$\begin{aligned}M=\int_{-\infty}^{+\infty}&\left\{\frac{\partial H_0}{\partial q_1}\left(\frac{\partial H_1}{\partial p_1}-\mu_1 q_1\cos^2 q_2-\mu_2 q_1\sin^2 q_2\right)-\frac{\partial H_0}{\partial p_1}\left[\frac{\partial H_1}{\partial q_1}+\right.\right.\\&\frac{q_1^{-1}}{2\Omega}f_1(p_1\cos q_2-p_2q_1^{-1}\sin q_2)+\frac{q_1^{-1}}{2\Omega}f_2(p_1\sin q_2+p_2q_1^{-1}\cos q_2)+\\&\left.2\mu_1 q_1 q_2(p_1\cos q_2-p_2q_1^{-1}\sin q_2)+\mu_2 q_1\sin q_2(p_1\sin q_2+p_2q_1^{-1}\cos q_2)\right]-\\&\left.\frac{\partial H_0}{\partial p_2}\left(\frac{\partial H_1}{\partial q_2}+\mu_1 p_2+\mu_2 p_2\right)\right\}\mathrm{d}T_2.\end{aligned}$$

注意在未摄动轨道上

$$\frac{\partial H_0}{\partial q_1}\cdot\frac{\partial H_1}{\partial p_1}-\frac{\partial H_0}{\partial p_1}\cdot\frac{\partial H_1}{\partial q_1}-\frac{\partial H_0}{\partial p_2}\cdot\frac{\partial H_1}{\partial q_2}=-\frac{\mathrm{d}H_1}{\mathrm{d}t},$$

则
$$M=-[H_1(T_2=+\infty)-H_1(T_2=-\infty)]+\int_{-\infty}^{+\infty}\Big\{(\mu_1 q_1\cos^2 q_2+\mu_2 q_1\sin^2 q_2)\mathrm{d}p_1-\Big[\frac{q_1^{-1}}{2\Omega}f_1(p_1\cos q_2-p_2q_1^{-1}\sin q_2)+\frac{q_1^{-1}}{2\Omega}f_2(p_1\sin q_2+p_2q_1^{-1}\cos q_2)+2\mu_1q_1\cos q_2(p_1\cos q_2-p_2q_1^{-1}\sin q_2)+\mu_2q_1\sin q_2(p_1\sin q_2+p_2q_1^{-1}\cos q_2)\Big]\mathrm{d}q_1-2(\mu_1p_2+\mu_2p_2)\mathrm{d}q_2\Big\}\cdot \tag{36}$$

一般情况下根据实际数据求(36)零点的数值解• 当 $\Delta=1,\mu_1=\mu_2=\mu,f_1=f_2=0$ 时,这归结为[2]的结果• 为完备起见,简单摘录如下:

从(36)同宿轨道 A 的 Melnikov 函数为
$$M_A=-\beta\Big\{[\cos(2\delta q_2)-1)]\cos 2r_c-\sin(2\delta q_2)\sin 2r_c\Big\}-\frac{\hat{c}}{4}\Big\{-[\cos(4\delta q_2)-1]\cos 4r_c+\sin(4\delta q_2)\sin 4r_c\Big\}+\partial X_A(\sigma,N) \tag{37}$$
其中
$$X_A(\sigma,N)=\left\{\sigma\left[\pi+2\arctan\left(-\frac{\sigma}{\sqrt{-(\sigma^2+N^3)}}\right)\right]+\frac{2}{N}\sqrt{-(\sigma^2+N^3)}\right\}\frac{N}{(N+1)\sqrt{\sigma^2-N^2}}$$

同宿轨道 B 的 Melnikov 函数为
$$M_B=-\beta\Big\{[\cos(2\delta q_2)-1)]\cos 2r_c+\sin(2\delta q_2)\sin 2r_c\Big\}-\frac{\hat{c}}{4}\Big\{-[\cos(4\delta q_2)-1]\cos 4r_c+\sin(4\delta q_2)\sin 4r_c\Big\}+\partial X_B(\sigma,N)$$
其中
$$X_B(\sigma,N)=\left\{-\sigma\left[\pi-2\arctan\left(-\frac{\sigma}{\sqrt{-(\sigma^2+N^3)}}\right)\right]+\frac{2}{N}\sqrt{-(\sigma^2+N^3)}\right\}\frac{N}{(N+1)\sqrt{\sigma^2-N^2}}\cdot \tag{38}$$

在 $\beta=0$ 情况, $M_A=0$ 可表为
$$\frac{\partial}{\hat{c}}=\frac{1-\cos 4\delta q_{2A}}{[(1-\cos 4\delta q_{2A})^2(\sin 4\delta q_{2A}+4X_A)^2]^{1/2}}\cdot \tag{39}$$
相移位满足
$$r_n<r_c+\delta q_{2A}\left(\mathrm{mod}\,\frac{\pi}{2}\right)<r_s \tag{40}$$
(39)与(40)可求出 Silnikov 类混沌存在的参数条件•

在 $\hat{c}=0$ 情况, $M_A=0$ 可表为
$$\frac{\partial}{\beta}=\frac{1-\cos\delta q_{2A}}{[(1-\cos 2\delta q_{2A})^2+(\sin 2\delta q_{2A}+X_A)^2]^{1/2}} \tag{41}$$
相移位满足
$$r_n<r_c+\delta q_{2A}(\mathrm{mod}\,\pi)<r_s\cdot \tag{42}$$
(41)与(42)可求出 Silnikov 类混沌存在的参数条件•

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On the Homoclinic Orbits in a Class of Two_Degree of Freedom Systems Under the Resonance Conditions

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Abstract: A class of two_degree_of_freedom systems in resonance with an external, parametric excitation is investigated, the existence of the periodic solutions locked to Ω is proved by the use of the method of multiple scales. This systems can be transformed into the systems of Wiggins, under some conditions. A calculating formula which determines the exsitence of homoclinic orbits of the systems is given.

Key words: a two_degree of freedom system; method of multiple scales; periodic solution; homoclinic orbit; chaos