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Navier_Stokes 方程初值问题的形式解*

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摘要: 给出了 $R^3 \times R$ 上的 Navier_Stokes 方程初值问题存在形式解的充要条件, 同时给出了一个计算实例•

关 键 词: Navier_Stokes 方程; 形式解; 分层; 末方程

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在文献[1]中, 对于 $R^3 \times R$ 上的 Navier_Stokes 方程的不稳定性, 曾经以准确解的形式给出过若干例证, 构造这些准确解的根据则是其末方程 (equation secondaire)[1], 事实上这些准确解是按一定要求从 Navier_Stokes 方程的形式解中特别挑选出来的有限形式• 在一般情况下, 要求某个偏微分方程的形式解是非常困难的, 但也并非无法可想• 根据分层理论, 寻找 Navier_Stokes 方程的形式解问题, 在若干分析、计算之后是可能取得结果的: 首先, 必须对方程 D 构造它的本方程 D_* ; 然后确定流形 $E_{l,k}(D)$, $W_{l,k}(D)$; 最后通过分层[1] 确定末方程, 这时才可讨论形式解问题• 本文将根据关于 Navier_Stokes 方程的已知结果, 得到其形式解存在的充要条件•

为叙述方便, 罗列以下主要符号, 其定义请参看[1]•

$P_l(X, Y)$	C^∞ 流形 X 到 Y 的(C^ω 或 C^∞ 或 C^k) 嵌入空间(具有 C^∞ 或 C^k 拓扑);
$I_k(X, Y)$	$J^k(X, Y)$ 的 Cartan_Ehresmann 理想子代数;
e	Ehresmann 对应;
$e_i(f)$	复合对应 $p_2 \circ (j^1 f) \circ e$ 的第 i 分量, $f: J^k(R^n, R^m) \to R$, $e(f)$: $J^{k+1}(R^n, R^m) \xrightarrow{e} J^1(R^n, J^k(R^n, R^m)) \xrightarrow{j^1 f} J^1(R^n, R) = J^0(R^n, R) \times R^n \xrightarrow{p_2} R^n$;
D'_*, D_*	D 的准本方程和本方程;
$E_{l,k}(V, Z)$, $W_{l,k}(V, Z)$	V 到 Z 的(l, k) 阶 Shih_典则系统;
$E_{l,k}(D)$, $W_{l,k}(D)$	D 的(l, k) 阶 Shih_典则系统;

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$S^t_{l,k}(D)$	D 的(l,k)阶横截层;
$S^i_{l,k}(D)$	D 的(l,k)阶 i 层, 其纤维的维数 = i;
$T_{l,k}(D)$	D 的(l,k)阶陷阱;
$E(S^i_{l,k}(D))$	$S^i_{l,k}(D)$ 的末方程;
$V(f)$	对应$f: J^k(V,Z)\to R$ 的零点集, 并且 $V(f_1,f_2)=V(f_1)\cap V(f_2)$;
Δ_q	q 维标准单形·

1 已有的结论

设 $R^3\times R=V$, $R^3\times R_+=Z$, V 上的Navier_Stokes方程[2] D 看成Ehresmann空间 $J^2(V,Z)$ 的子集, 并使用 $J^2(V,Z)$ 的局部坐标, 可将其写成以下形式:

$$D:\begin{cases} f_1: \gamma[p^1_{11}+p^1_{22}+p^1_{33}]+\Phi_1=0,\\ f_2: \gamma[p^2_{11}+p^2_{22}+p^2_{33}]+\Phi_2=0,\\ f_3: \gamma[p^3_{11}+p^3_{22}+p^3_{33}]+\Phi_3=0,\\ f_4: p^1_1+p^2_2+p^3_3=0,\end{cases} \tag{*}$$

其中

$$(x,t)=(x_1,x_2,x_3,x_4)\in V,\ (u_1,u_2,u_3,p)=(u_1,u_2,u_3,u_4)\in Z,$$

$$\Phi_j=-\left[p^j_4+u_1p^j_1+u_2p^j_2+u_3p^j_3+\frac{1}{\rho}p^4_j\right]+F_j\qquad (j=1,2,3)· \tag{1}$$

设 $F_j\in C^\infty$, 并将(*)左侧各式依次记为f_1、f_2、f_3、f_4, 以下结果请参看[1]、[3]·

(A) D 的本方程 $D_*=\bigcup_l D_l$, $D_l\subseteq J^l(V,Z)$ $(l=-1,0,1,2,\cdots)$

$D_{-1}=J^{-1}(V,Z)=V=R^3\times R,$

$D_0=J^0(V,Z)=V\times Z,$

$D_1=V(f_4),$

$D_2=V(f_j,e_i(f_4),f,f_4),$

$D_3=V(e_{i_1}(f_j),e_{ii_1}(f_4),e_{i_1}(f),f_j,e_i(f_4),f,f_4),$

$\vdots$

$D_k=V(e_{i_1\cdots i_{k-2}}(f_j),e_{ii_1\cdots i_{k-2}}(f_4),e_{i_1\cdots i_{k-2}}(f),\cdots,f_j,e_i(f_4),f,f_4),$

$(i,i_1,\cdots,i_{k-2}=1,2,3,4;\ i\leqslant i_1\leqslant\cdots\leqslant i_{k-2};\ j=1,2,3;\ k\geqslant 3),$

其中f 代表下式的左侧

$$\begin{cases} f: \dfrac{1}{\rho}(p^4_{11}+p^4_{22}+p^4_{33})+\Phi_4=0,\\ \Phi_4=-\left[\dfrac{\partial F_1}{\partial x_1}+\dfrac{\partial F_2}{\partial x_2}+\dfrac{\partial F_3}{\partial x_3}\right]+(p^1_1)^2+(p^2_2)^2+(p^3_3)^2+\\ \qquad 2p^1_2p^2_1+2p^1_3p^3_1+2p^2_3p^3_2·\end{cases} \tag{2}$$

(B) 对任何 $k\geqslant 2$, $\rho_{3,k-1}: E_{3,k-1}(V,Z)\to W_{3,k-1}(V,Z)$ 的分层是

$W_{3,k-1}(V,Z)=W_{3,k-1}(D)\cup T_{3,k-1}(D)=(S^0_{3,k-1}(D)\cup S^4_{3,k-1}(D))\cup T_{3,k-1}(D),$

这里

$E_{3,k-1}(V,Z) \subseteq G_3^*(TJ^{k-1}(V,Z)) \times_{J^{k-1}(V,Z)} J^k(V,Z)$,

$W_{3,k-1}(V,Z) = \mathrm{Im}p_1(E_{l,k-1}(V,Z)) \subseteq G_3^*(TJ^{k-1}(V,Z))$•

纤维空间

$\rho^0_{3,k-1}: E^0_{3,k-1}(D) \to S^0_{3,k-1}(D)$,

$\rho^4_{3,k-1}: E^4_{3,k-1}(D) \to S^4_{3,k-1}(D)$,

其纤维的维数分别是 0 和 4•

对任何 $k \geqslant 2$, D 的$(3, k-1)$ 阶横截层是空集:

$S^t_{3,k-1}(D) = \varnothing$•

(C) Navier_Stokes 方程(*)的任何 $C^k(k \geqslant 2)$ 初始条件或混合条件都是不适定的•

2 初值问题存在形式解的条件

2.1 末方程 $E(S^0_{3,k-1}(D))$, $E(S^4_{3,k-1}(D))$

在 $W_{3,k-1}(V,Z) \subseteq G_3^*(TJ^{k-1}(V,Z))$ 的开覆盖$\{\Omega_i\}$, $(i = 1,2,3,4)$ 中, 考虑 Ω_3 和 Ω_4 两种情形(Ω_1, Ω_2 则和 Ω_3 有相似的结论)•

设 $\tau \in \Omega_4$, τ 由 $J^{k-1}(V,Z)$ 在点 $p(\tau)$ 的如下三个切向量生成:

$$\begin{cases} \eta_1 = (1,0,0,\alpha_1,\hat{u}_i(1),\hat{p}_j^{i}{}_\lambda(1)), \\ \eta_2 = (0,1,0,\alpha_2,\hat{u}_i(2),\hat{p}_j^{i}{}_\lambda(2)), \qquad (i = 1,2,3,4,\ |\lambda| \leqslant k-1)\text{•} \\ \eta_3 = (1,0,0,\alpha_3,\hat{u}_i(3),\hat{p}_j^{i}{}_\lambda(3)), \end{cases}$$

$p(\tau) = (x_i, u_i, p_j^{i}{}_\lambda) \in J^{k-1}(V,Z) \qquad (i,j = 1,2,3,4,\ |\lambda| \leqslant k-1)$,

则得到 $\tau \in \Omega_4 \cap S^0_{3,k-1}(D)$ 的充要条件是:

$$\begin{cases} \alpha = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0, \\ \varphi_{\alpha,k-2}(\tau) = \alpha_1\varphi^{(4)}_{1,k-2}(\tau) + \alpha_2\varphi^{(4)}_{2,k-2}(\tau) + \alpha_3\varphi^{(4)}_{3,k-2}(\tau) + \nu\alpha\varphi^{(4)}_{4,k-2}(\tau)\text{•} \end{cases} \tag{3}$$

而为使 $\tau \in \Omega_4 \cap S^4_{3,k-1}(D)$, 其充要条件是

$$\begin{cases} \alpha = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0, \\ (\varphi^{(4)}_{4,k-2}(\tau))^2 + \sum_{i=1}^{4}(\varphi^{(4)}_{i,k-2}(\tau))^2 = 0\text{•} \end{cases} \tag{4}$$

(3) 式和(4) 式即末方程 $E(S^0_{3,k-1}(D))$ 和 $E(S^4_{3,k-1}(D))$•

$\varphi^{(4)}_{i,k-2}(\tau)$ $(i = 1,2,3,4)$ 和 $\varphi^{(4)}_{4,k-2}(\tau)$ 的计算公式如下:

$$\begin{cases} \varphi^{(4)}_{i,k-2}(\tau) = -\Phi^{(4)}_{i,k-2}(\tau) + \nu[\alpha_1\hat{p}^{i}_{4^{k-1}}(1) + \alpha_2\hat{p}^{i}_{4^{k-1}}(2) + \alpha_3\hat{p}^{i}_{4^{k-1}}(3) - \\ \qquad \hat{p}^{i}_{14^{k-2}}(1) - \hat{p}^{i}_{24^{k-2}}(2) - \hat{p}^{i}_{34^{k-2}}(3)] \qquad (i = 1,2,3), \\ \varphi^{(4)}_{4,k-2}(\tau) = -[\hat{p}^{1}_{4^{k-1}}(1) + \hat{p}^{2}_{4^{k-1}}(2) + \hat{p}^{3}_{4^{k-1}}(3)], \\ \varphi^{(4)}_{4,k-2}(\tau) = -\Phi^{(4)}_{4,k-2}(\tau) + \dfrac{1}{\rho}[\alpha_1\hat{p}^{4}_{4^{k-1}}(1) + \alpha_2\hat{p}^{4}_{4^{k-1}}(2) + \alpha_3\hat{p}^{4}_{4^{k-1}}(3) - \\ \qquad \hat{p}^{4}_{14^{k-2}}(1) - \hat{p}^{4}_{24^{k-2}}(2) - \hat{p}^{4}_{34^{k-2}}(3)]\text{•} \end{cases} \tag{5}$$

(5) 式中 $\Phi^{(4)}_{i,k-2}$ 和 $\Phi^{(4)}_{4,k-2}$ 的计算公式为

$$\begin{cases}\Phi_{i,k-2}^{(4)} = -\Big\{[p_{4^{k-1}}^{i} + u_1 p_{14^{k-2}}^{i} + u_2 p_{24^{k-2}}^{i} + u_3 p_{34^{k-2}}^{i}] + c_{k-2}^{1}[p_4^1 \cdot p_{14^{k-3}}^{i} + p_4^2 \cdot p_{24^{k-3}}^{i} + \\ \quad p_4^3 \cdot p_{34^{k-3}}^{i}] + \cdots + c_{k-2}^{l}[p_{4^l}^1 \cdot p_{14^{k-l-2}}^{i} + p_4^2 \cdot p_{24^{k-l-2}}^{i} + p_4^3 \cdot p_{34^{k-l-2}}^{i}] + \cdots + \\ \quad c_{k-2}^{k-3}[p_{4^{k-3}}^1 \cdot p_{14}^{i} + p_{4^{k-3}}^2 \cdot p_{24}^{i} + p_{4^{k-3}}^3 \cdot p_{34}^{i}] + \dfrac{1}{\rho} \cdot p_{i4^{k-2}}^{4} - \dfrac{\partial^{k-2} F_i}{\partial x_4^{k-2}}\Big\}, \\ \Phi_{4,k-2}^{(4)} = \dfrac{\partial^{k-2}\Phi_4}{\partial x_4^{k-2}} \qquad (k \geqslant 3). \end{cases} \tag{6}$$

现在设 $\tau \in \Omega_3$, 则 τ 由 $J^{k-1}(V, Z)$ 在点 $p(\tau)$ 的如下 3 个切向量生成

$$\begin{cases}\zeta_1 = (1, 0, \delta_1, 0, \hat{u}_i(1), \hat{p}_j^{i\lambda}(1)), \\ \zeta_2 = (0, 1, \delta_2, 0, \hat{u}_i(2), \hat{p}_j^{i\lambda}(2)), \qquad (i = 1, 2, 3, 4, |\lambda| \leqslant k-1). \\ \zeta_3 = (0, 0, \delta_3, 1, \hat{u}_i(3), \hat{p}_j^{i\lambda}(3)), \end{cases}$$

$$p(\tau) = (x_i, u_i, p_j^{i\lambda}) \in J^{k-1}(V, Z) \qquad (i, j = 1, 2, 3, 4, |\lambda| \leqslant k-1),$$

则 $\tau \in \Omega_3 \cap S_{3,k-1}^{0}(D)$ 的充要条件是

$$\varphi_{\delta,k-2} = \delta_1 \varphi_{1,k-2}^{(3)}(\tau) + \delta_2 \varphi_{2,k-2}^{(3)}(\tau) - \varphi_{3,k-2}^{(3)}(\tau) + \nu\delta\varphi_{4,k-2}^{(3)}(\tau) = 0, \tag{7}$$

$$(\delta = 1 + \delta_1^2 + \delta_2^2).$$

式(7)即末方程 $E(S_{3,k-1}^{0}(D))$.

$\varphi_{i,k-2}^{(3)}(\tau)$ $(i = 1, 2, 3, 4)$ 和 $\varphi_{4,k-2}^{(3)}(\tau)$ 的计算公式如下:

$$\begin{cases}\varphi_{i,k-2}^{(3)}(\tau) = -\Phi_{i,k-2}^{(3)}(\tau) + \nu[\delta_1 \hat{p}_{3^{k-1}}^{i}(1) + \delta_2 \hat{p}_{3^{k-1}}^{i}(2) - \hat{p}_{1^{k-1}}^{i}(1) - \\ \qquad \hat{p}_{2^{k-1}}^{i}(2)] \qquad (i = 1, 2, 3), \\ \varphi_{4,k-2}^{(3)}(\tau) = -\hat{p}_{3^{k-1}}^{1}(1) - \hat{p}_{3^{k-1}}^{2}(2), \\ \varphi_{4,k-2}^{(3)}(\tau) = -\ {}_{4,k-2}^{(3)}(\tau) + \dfrac{1}{\rho}[\delta_1 \hat{p}_{3^{k-1}}^{4}(1) + \delta_2 \hat{p}_{3^{k-1}}^{4}(2) - \hat{p}_{1^{k-1}}^{4}(1) - \\ \qquad \hat{p}_{2^{k-1}}^{4}(2)]. \end{cases} \tag{8}$$

(8) 式中 $\Phi_{i,k-2}^{(3)}$ 和 $\Phi_{4,k-2}^{(3)}$ 的计算公式为:

$$\begin{cases}\Phi_{i,k-2}^{(3)} = -\Big\{[p_{3^{k-2}4}^{i} + u_1 p_{13^{k-2}}^{i} + u_2 p_{23^{k-2}}^{i} + u_3 p_{3^{k-1}}^{i}] + c_{k-2}^{1}[p_3^1 \cdot p_{13^{k-3}}^{i} + p_3^2 \cdot p_{23^{k-3}}^{i} + \\ \quad p_3^3 \cdot p_{3^{k-2}}^{i}] + \cdots + c_{k-2}^{k-3}[p_{3^{k-3}}^1 \cdot p_1^{i} + p_{3^{k-3}}^2 \cdot p_{23}^{i} + p_{3^{k-3}}^3 \cdot p_{3^2}^{i}] + \\ \quad \dfrac{1}{\rho} \cdot p_{i3^{k-2}}^{4} - \dfrac{\partial^{k-2} F_i}{\partial x_3^{k-2}}\Big\}, \\ \Phi_{4,k-2}^{(3)} = \dfrac{\partial^{k-2}\Phi_4}{\partial x_3^{k-2}} \qquad (k \geqslant 3). \end{cases} \tag{9}$$

2.2 初值问题形式解的存在条件

设 $x^0 \in V = R^3 \times R$, 并给了 $D(*)$ 一组初始条件 $(\sigma, \gamma) \in Pl(\Delta_3, V) \times Pl(\Delta_3, J^1(V, Z))$, 即满足如下条件的一对 C^∞ 嵌入 (σ, γ)[1]:

$$\alpha_1^1 \circ \gamma = \sigma,$$

$$\gamma^* \omega = 0, \qquad \forall \omega \in I_1(V, Z),$$

$$\mathrm{Im}\,\gamma \subseteq D_1,$$

其中

$$\begin{cases}\sigma(\xi) = (x_1, x_2, x_3, x_4) = (x_1^0 + \xi_1, x_2^0 + \xi_2, x_3^0 + \xi_3, x_4^0 + f(\xi)), \\ \gamma(\xi) = (\sigma(\xi), u_i(\xi), p_j^i(\xi)) \qquad (i, j = 1, 2, 3, 4),\end{cases} \tag{10}$$

这里 $\xi = (\xi_1, \xi_2, \xi_3) \in \Delta_3$ 是重心坐标, 对应 $f: \Delta_3 \to R, f(0) = 0$, 且 $f \in C^\infty$•

则有以下定理:

定理 1　设给出(*)一组初始条件 (σ, γ) 由(10) 定义• 则相对于 (σ, γ) 的初值问题存在形式解的充要条件是存在 C^∞ 嵌入序列 $\{\gamma_{k-1}\}, (k \geqslant 2)$

$$\gamma_{k-1}: \Delta_3 \to J^{k-1}(V, Z),$$

使得

$$\mathrm{Im}\,\gamma_{k-1} \subseteq S^a_{3,k-1}(D) \qquad (a = 0 \text{ 或 } 4), \tag{11}$$

即 γ_{k-1} 是末方程 $E(S^a_{3,k-1}(D))$ 的解• 并满足

$$\alpha^{k-1}_{k-i} \circ \gamma_{k-1} = \gamma_{k-1}, \quad \alpha^{k-1}_{-1} \circ \gamma_{k-1} = \sigma• \tag{12}$$

证明

充分性　如果满足定理条件的 $\{\gamma_{k-1}\}$ 存在, 设

$$\gamma_{k-1}(\xi) = (\sigma(\xi), u_i(\xi), p_j^{i\lambda}(\xi)) \qquad (i = 1, 2, 3, 4, |\lambda| \leqslant k-1),$$

则由 $\mathrm{Im}\,\gamma_{k-1} \subseteq S^a_{3,k-1}(D)$ $(a = 0$ 或 $4)$ 知, 以 $p_j^{i\lambda}(0)$ 为系数所构成的级数

$$\sum_\lambda \frac{1}{\lambda!} A_j^{i\lambda}(x^0)(x - x^0)^\lambda \tag{13}$$

就是所求的形式解• 这里

$$\left.\begin{aligned} &A_j^{i\lambda}(x^0) = p_j^{i\lambda}(0) \qquad (i = 1, 2, 3, 4), \\ &x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} x_4^{\lambda_4} \in V = R^3 \times R•\end{aligned}\right\} \tag{14}$$

必要性　如果相对于 (σ, γ), (*) 存在形式解

$$\sum_\lambda B_j^{i\lambda}(x - x^0)^\lambda \qquad (i = 1, 2, 3, 4),$$

则由此形式解在点 x^0 及其附近的各阶(形式) 导数, 就可构成序列 $\{\gamma_{k-1}\}, (k \geqslant 2, \gamma_1 = \gamma)$, 满足定理 1 中的全部条件•

定理 1 中的条件等价于以下两组关系式:

$k \geqslant 2$,

$$\begin{cases}\left(\dfrac{\partial f}{\partial \xi_1}\right)^2 + \left(\dfrac{\partial f}{\partial \xi_2}\right)^2 + \left(\dfrac{\partial f}{\partial \xi_3}\right)^2 \neq 0, \\ \left(\dfrac{\partial f}{\partial \xi_1}\right)\varphi^{(4)}_{1,k-2} + \left(\dfrac{\partial f}{\partial \xi_2}\right)\varphi^{(4)}_{2,k-2} + \left(\dfrac{\partial f}{\partial \xi_3}\right)\varphi^{(4)}_{3,k-2} + \\ \qquad \nu\left[\left(\dfrac{\partial f}{\partial \xi_1}\right)^2 + \left(\dfrac{\partial f}{\partial \xi_2}\right)^2 + \left(\dfrac{\partial f}{\partial \xi_3}\right)^2\right]\varphi^{(4)}_{4,k-2} = 0,\end{cases} \tag{15}$$

或

$$\begin{cases}\left(\dfrac{\partial f}{\partial \xi_1}\right)^2 + \left(\dfrac{\partial f}{\partial \xi_2}\right)^2 + \left(\dfrac{\partial f}{\partial \xi_3}\right)^2 = 0, \\ (\varphi^{(4)}_{4,k-2})^2 + \displaystyle\sum_{i=1}^4 (\varphi^{(4)}_{i,k-2})^2 = 0•\end{cases} \tag{16}$$

(15)、(16) 分别对应于 $\mathrm{Im}\,\gamma_{k-1} \subseteq S^0_{3,k-1}(D)$ 和 $\mathrm{Im}\,\gamma_{k-1} \subseteq S^4_{3,k-1}(D)$• 式中 $\varphi^{(4)}_{i,k-2}$ 和 $\varphi^{(4)}_{4,k-2}$, 在

(5) 式中取 $\alpha_i = \partial f/\partial \xi_i$, $(i = 1, 2, 3)$, $\hat{p}_j^{i\lambda}(l) = \partial p_j^{i\lambda}(\xi)/\partial \xi_l$, $(i = 1, 2, 3, |\lambda| \leqslant k-1)$, 即由 $\gamma_{k-1}(\xi)$ 所确定•

如果初始条件(σ, γ) 是:

$$\begin{cases} \sigma(\xi) = (x_1^0 + \xi_1, x_2^0 + \xi_2, x_3^0 + g(\xi), x_4^0 + \xi_4), \\ \gamma(\xi) = (\sigma(\xi), u_i(\xi), p_j^i(\xi)) \qquad (i, j = 1, 2, 3, 4), \end{cases} \tag{17}$$

这里 $\xi = (\xi_1, \xi_2, \xi_4) \in \Delta_3$, 是重心坐标, C^∞ 对应 $g: \Delta_3 \to R$, 且 $g(0) = 0$• 则有以下定理:

定理 2 设给出 $D(*)$ 的一组初始条件(σ, γ) 由(17) 定义• 则相对于(σ, γ) 的初值问题存在形式解的充要条件是存在 C^∞ 嵌入序列$\{\gamma_{k-1}\}$, $(k \geqslant 2)$•

$$\gamma_{k-1}: \Delta_3 \to J^{k-1}(V, Z)$$

使得

$$\operatorname{Im} \gamma_{k-1} \subseteq S^0_{3, k-1}(D) \qquad (\gamma_{k-1} \text{ 是末方程 } E(S^0_{3, k-1}(D)) \text{ 的解}), \tag{18}$$

即对任何 $k \geqslant 2$, 成立

$$\left(\frac{\partial g}{\partial \xi_1}\right) \varphi^{(3)}_{1, k-2} + \left(\frac{\partial g}{\partial \xi_2}\right) \varphi^{(3)}_{2, k-2} - \varphi^{(3)}_{3, k-2} + \nu\left[1 + \left(\frac{\partial g}{\partial \xi_1}\right)^2 + \left(\frac{\partial g}{\partial \xi_2}\right)^2\right] \varphi^{(3)}_{4, k-2} = 0 • \tag{19}$$

(19) 式中的 $\varphi^{(3)}_{i, k-2}$ 在(8) 式中取 $\delta_l = \partial g/\partial \xi_l$, $(l = 1, 2, 4)$, $\hat{p}_j^{i\lambda}(l) = \partial p_j^{i\lambda}/\partial \xi_l$, 即由 $\gamma_{k-1}(\xi)$ 所确定:

$$\gamma_{k-1}(\xi) = (\sigma(\xi), u_i(\xi), p_j^{i\lambda}(\xi)) \in J^{k-1}(V, Z) •$$

定理 2 的证明与定理 1 的证明相似•

3 计 算 实 例

为方便起见, 设 $x^0 = 0 \in R^4$• 并设初始条件(σ, γ) 如下:

$$\begin{cases} \sigma(\xi) = (0 + \xi_1, 0 + \xi_2, 0 + \xi_3, 0 + 0), \\ \gamma(\xi) = (\sigma(\xi), u_i(\xi), p_j^i(\xi)) \qquad (i, j = 1, 2, 3, 4) • \end{cases} \tag{20}$$

此即通常在超平面$\{t = 0\} \subseteq R^4$ 上的初值问题• 由于$\partial f/\partial \xi_l = 0$, $(l = 1, 2, 3)$, 因而形式解的存在条件是(16)•

计算过程如下:

1) 根据(σ, γ) 的定义, 必须有 $\operatorname{Im} \gamma \subseteq D_1$, $\gamma^* \omega = 0$, $\forall \omega \in I_1(V, Z)$, 因此初始值 $u_i(\xi)$、$p_j^i(\xi)$ 必须满足

$$\begin{cases} u_1(1) + u_2(2) + u_3(3) = 0, \qquad \left(u_i(l) = \dfrac{\partial u_i}{\partial \xi_l}(\xi)\right), \\ p_l^i(\xi) = u_i(l) \qquad (i = 1, 2, 3, 4; l = 1, 2, 3), \end{cases} \tag{21}$$

p_4^i $(i = 1, 2, 3, 4)$ 待定•

2) $k = 3$, 确定 p_{jk}^i, $(i, j, k = 1, 2, 3, 4; j \leqslant k)$, 即确定 $\gamma_2(\xi)$•

根据 $\varphi^{(4)}_{i, 0} = 0$, $(i = 1, 2, 3, 4)$, $\varphi^{(4)}_{4, 0} = 0$, 得到

$$\begin{cases} p_4^i(\xi) = \nu[p_1^i(1) + p_2^i(2) + p_3^i(3)] - \left[u_1 p_1^i + u_2 p_2^i + u_3 p_3^i + \dfrac{1}{\rho} p_i^4\right] + \\ \qquad F_i(\xi) \qquad (i = 1, 2, 3), \\ p_4^1(1) + p_4^2(2) + p_4^3(3) = 0, \\ \dfrac{1}{\rho}[p_1^4(1) + p_2^4(2) + p_3^4(3)] - \left[\dfrac{\partial F_1}{\partial x_1} + \dfrac{\partial F_2}{\partial x_2} + \dfrac{\partial F_3}{\partial x_3}\right](\xi) + \\ \qquad (p_1^1)^2 + (p_2^2)^2 + (p_3^3)^2 + 2p_2^1 p_1^2 + 2p_3^1 p_1^3 + 2p_3^2 p_2^3 = 0; \end{cases} \tag{22}$$

$$p_j^{i\,l}(\xi) = \frac{\partial p_j^i}{\partial \xi_l}(\xi) = p_j^i(l) \qquad (i, j = 1, 2, 3, 4;\ l = 1, 2, 3); \tag{23}$$

$p_4^{i\,2}(\xi)$ $(i = 1, 2, 3, 4)$ 待定•

(22) 是初始值 $p_4^i(i = 1, 2, 3)$ 的计算公式和约束• 可以看到 p_4^4“缺席”! 因而可任意设值•

这样, 就完成了如何设置初始条件的问题•

由于 $p_4^{i\,2}$ 未知, 因而 γ_2 尚未完全确定•

3) $k = 4$, 确定 $p_j^{i\,\lambda}(i = 1, 2, 3, |\lambda| \leqslant k - 1)$, 即确定 $\gamma_3(\xi)$•

根据 $\varphi_{i,1}^{(4)} = 0(i = 1, 2, 3, 4)$ 和 $\varphi_{4,1}^{(4)} = 0$ 可得 $p_4^{i\,2}$ 的计算公式和约束以及部分 $p_j^{i\,\lambda}(|\lambda| = 3)$ 的计算公式如下:

$$\begin{cases} p_4^{i\,2}(\xi) = \nu[p_{14}^i(1) + p_{24}^i(2) + p_{34}^i(3)] - \left[u_1 p_{14}^i + u_2 p_{24}^i + u_3 p_{34}^i + \right. \\ \qquad \left. \dfrac{1}{\rho} p_{i4}^4 + p_4^1 \cdot p_1^i + p_4^2 \cdot p_2^i + p_4^3 \cdot p_3^i\right] + \dfrac{\partial^2 F_i}{\partial x_4^2}(\xi) \qquad (i = 1, 2, 3), \\ p_4^{1\,2}(1) + p_4^{2\,2}(2) + p_4^{3\,2}(3) = 0, \\ \dfrac{1}{\rho}[p_{14}^4(1) + p_{24}^4(2) + p_{34}^4(3)] - \dfrac{\partial F}{\partial x}(\xi) - (p_1^1)^2 - \\ \qquad (p_2^2)^2 - (p_3^3)^2 - 2p_2^1 p_1^2 - 2p_3^1 p_1^3 - 2p_3^2 p_2^3 = 0, \\ \left(F = \dfrac{\partial F_1}{\partial x_1} + \dfrac{\partial F_2}{\partial x_2} + \dfrac{\partial F_3}{\partial x_3}\right); \end{cases} \tag{24}$$

$$p_{jkl}^i(\xi) = p_{jk}^i(l) \qquad (i, j, k = 1, 2, 3, 4;\ l = 1, 2, 3); \tag{25}$$

$p_4^{i\,2}(\xi)$ $(i = 1, 2, 3, 4)$ 待定•

$p_4^{4\,2}(\xi)$“缺席”! 可任意设值• 至此完整地确定了 γ_2•

4) $k \geqslant 5$, 确定 $p_j^{i\,\lambda}(i = 1, 2, 3, 4, |\lambda| = k)$, 即确定 $\gamma_{k-1}(\xi)$•

根据 $\varphi_{i,k-2}^{(4)} = 0(i = 1, 2, 3, 4)$, $\varphi_{4,k-2}^{(4)} = 0$, 可得 $p_4^{i\,k-1}$ 和部分 $p_j^{i\,\lambda}(|\lambda| = k)$ 的计算公式及约束条件如下:

$$\begin{cases} p_{4}^{i_{k-1}}(\xi) = \nu[p_{14}^{i_{k-2}}(1) + p_{24}^{i_{k-2}}(2) + p_{34}^{i_{k-2}}(3)] - \Big\{[u_1 p_{14}^{i_{k-2}} + u_2 p_{24}^{i_{k-2}} + \\ \quad u_3 p_{34}^{i_{k-2}}] + c_{k-2}^{1}[p_{4}^{1} p_{14}^{i_{k-3}} + p_{4}^{2} \cdot p_{24}^{i_{k-3}} + p_{4}^{3} \cdot p_{34}^{i_{k-3}}] + \cdots + \\ \quad c_{k-2}^{l}[p_{4}^{1_l} \cdot p_{14}^{i_{k-l-2}} + p_{4}^{2_l} \cdot p_{24}^{i_{k-l-2}} + p_{4}^{3_l} \cdot p_{34}^{i_{k-l-2}}] + \cdots + \\ \quad c_{k-2}^{k-3}[p_{4}^{1_{k-3}} \cdot p_{14}^{i} + p_{4}^{2_{k-3}} \cdot p_{24}^{i} + p_{4}^{3_{k-3}} \cdot p_{34}^{i}] + [p_{4}^{1_{k-2}} \cdot p_{1}^{i} + \\ \quad p_{4}^{2_{k-2}} \cdot p_{2}^{i} + p_{4}^{3_{k-2}} \cdot p_{3}^{i}] + \dfrac{1}{\rho} p_{i4}^{4_{k-2}}\Big\} + \\ \quad \dfrac{\partial^{k-2} F_i}{\partial x_4^{k-2}}(\xi) \qquad (i = 1, 2, 3), \\ p_{4}^{1_{k-1}}(1) + p_{4}^{2_{k-1}}(2) + p_{4}^{3_{k-1}}(3) = 0, \\ \dfrac{1}{\rho}[p_{14}^{4_{k-2}}(1) + p_{24}^{4_{k-2}}(2) + p_{34}^{4_{k-2}}(3)] - \dfrac{\partial^{k-3} F}{\partial x_4^{k-3}}(\xi) = 0; \end{cases} \tag{26}$$

$$p_{j}^{i_{\lambda_l}}(\xi) = p_j^{i_\lambda}(l) \qquad (l = 1, 2, 3, |\lambda| = k - 1); \tag{27}$$

$p_4^{i_k}(\xi)\,(i = 1, 2, 3, 4)$ 待定(即由 $|\lambda| = k + 1$ 时确定)•

求出了所有的 $p_j^{i_\lambda}(\xi)$ 之后, 根据定理1, 就可得到形式解为

$$u_i(x) \sim \sum_{|\lambda|} \frac{1}{\lambda!} A_j^{i_\lambda}(0) x^\lambda \qquad (i = 1, 2, 3, 4),$$

式中,

$$A_j^{i_\lambda}(0) = p_j^{i_\lambda}(0)\text{•}$$

[参 考 文 献]

[1] SHIH Wei_hui. Solutions Analytiques de Quelques Equations aux Derivees Partielles en Mecanique des Fluides[M]. Paris: Hermann, 1992.

[2] Landau J, Lifchitz E. Mecanique des Fluides[M]. Moscou: Editions Mir, 1971.

[3] 施惟慧, Navier_Stokes 方程稳定性研究(III)[J]. 应用数学和力学, 1994, **15**(12): 1067—1073.

On the Formal Solution of Initial Value Problem of Navier_Stokes Equation

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Abstract: A necessary and sufficient conditions of the existence of formal solution to the initial value problem of Navier_Stokes equation on $R^3 \times R$ are presented. A computation case is also given.

Key words: Navier_Stokes equation; formal solution; stratification; equation secondaire