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薄壳非线性变形理论

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摘要: 对薄壳的非线性变形, 给出了应变与位移之间的精确关系 经过合理的简化, 给出了壳的挠度与厚度同级的大变形基本公式 当薄壳为无限长柱形且作柱形变形时, 精确地求得了壳的挠度与长度同级的大变形基本公式

关 键 词: 薄壳; 非线性变形; 大挠度
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引 言

薄壳的非线性变形问题是一个很复杂而难解的问题, 必须进行简化 在参考文献中, 如对 $= {}^2_{33}/\sqrt{{}^2_{31}+{}^2_{32}+{}^2_{33}}-1$ 简化时, 令 $\sqrt{{}^2_{31}+{}^2_{32}+{}^2_{33}}=1$[1], 则得出 $= {}^2_{33}-1$ 当 $_{33}>1$ 时, 将出现 >0 又如将 $_{11}=\sqrt{(1+e_{11})^2+e_{12}^2+e_{13}^2}-1$ 简化时, 当 e_{11}、e_{12}^2 和 e_{13}^2 均为小量, 令 $\sqrt{1+}=1+/2$, 则得出

$$_{11}=e_{11}+\frac{1}{2}e_{11}^2+\frac{1}{2}e_{12}^2+\frac{1}{2}e_{13}^2$$

如令 $\sqrt{1+}=1+\frac{1}{2}-\frac{1}{8}{}^2+$,

则略去高次小量后, 应得出

$$_{11}=e_{11}+\frac{1}{2}e_{12}^2+\frac{1}{2}e_{13}^2$$

这些不合理的简化, 导致许多不规范的基本公式[2]

弹性薄壳的非线性变形问题有两种 一种是挠度和厚度同级的大变形问题, 此时应变和角转动的平方均为小量, 因而与1 相比时可以略去 另一种是挠度和长度同级的大变形问题, 此时角转动不一定是小量而不能应用公式

$$_{11}=e_{11}+\frac{1}{2}e_{12}^2+\frac{1}{2}e_{13}^2,$$

必须进行精确的推导

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1　薄壳的变形和应变能

根据非线性变形理论[1], 薄壳任一点在坐标线 $_1$、$_2$ 和中面法线方向的位移 u_z、v_z、w_z 可表示为

$$u_z = u + z \; , \; v_z = v + z \; , \; w_z = w + z \; , \tag{1}$$

式中 u、v、w 为壳体中面的位移　由直法线假设可得

$$= \frac{_{31}}{\sqrt{^2_{31} + ^2_{32} + ^2_{33}}}, \quad = \frac{_{32}}{\sqrt{^2_{31} + ^2_{32} + ^2_{33}}}, \quad = \frac{_{33}}{\sqrt{^2_{31} + ^2_{32} + ^2_{33}}} - 1, \tag{2}$$

式中

$$\left.\begin{aligned} &_{31} = - e_{13}(1 + e_{22}) + e_{12}e_{23}, \quad _{32} = - e_{23}(1 + e_{11}) + e_{21}e_{13}, \\ &_{33} = (1 + e_{11})(1 + e_{22}) - e_{12}e_{21}; \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} &e_{11} = \frac{1}{A_1}\frac{u}{_1} + \frac{1}{A_1A_2}\frac{A_1}{_2}v + \frac{w}{R_1}, \quad e_{22} = \frac{1}{A_2}\frac{v}{_2} + \frac{1}{A_1A_2}\frac{A_2}{_1}u + \frac{w}{R_2}, \\ &e_{12} = \frac{1}{A_1}\frac{v}{_1} - \frac{1}{A_1A_2}\frac{A_1}{_2}u, \quad e_{21} = \frac{1}{A_2}\frac{u}{_2} - \frac{1}{A_1A_2}\frac{A_2}{_1}v, \\ &e_{13} = \frac{1}{A_1}\frac{w}{_1} - \frac{u}{R_1}, \quad e_{23} = \frac{1}{A_2}\frac{w}{_2} - \frac{v}{R_2} \end{aligned}\right\} \tag{4}$$

壳体中面任一点沿 $_1$、$_2$ 方向的单位伸长和切应变为

$$\left.\begin{aligned} &_{11} = \sqrt{(1 + e_{11})^2 + e^2_{12} + e^2_{13}} - 1, \quad _{22} = \sqrt{(1 + e_{22})^2 + e^2_{21} + e^2_{23}} - 1, \\ &_{12} = \arcsin\frac{(1 + e_{11})e_{12} + (1 + e_{22})e_{21} + e_{13}e_{23}}{\sqrt{(1 + e_{11})^2 + e^2_{12} + e^2_{13}}\sqrt{(1 + e_{22})^2 + e^2_{21} + e^2_{23}}} \end{aligned}\right\} \tag{5}$$

如在(4)式中的 u、v、w 分别用(1)式中的 u_z、v_z、w_z 代替, 然后将(4)式代入(5)式, 同时在第一式中将 1 改为 $1 + z/R_1$, 并除以 $1 + z/R_1$; 在第二式中将 1 改为 $1 + z/R_2$, 并除以 $1 + z/R_2$; 在第三式中将 $1 + e_{11}$ 改为 $1 + e_{11} + z/R_1$, $1 + e_{22}$ 改为 $1 + e_{22} + z/R_2$, 可得

$$\left.\begin{aligned} &_{11z} = \frac{1}{1 + z/R_1}\left[\sqrt{(1 + e_{11} + zk_{11})^2 + (e_{12} + zk_{12})^2 + (e_{13} + zk_{13})^2} - \left(1 + \frac{z}{R_1}\right)\right], \\ &_{22z} = \frac{1}{1 + z/R_2}\left[\sqrt{(1 + e_{22} + zk_{22})^2 + (e_{21} + zk_{21})^2 + (e_{23} + zk_{23})^2} - \left(1 + \frac{z}{R_2}\right)\right], \\ &_{33z} = \arcsin\Big\{[(1 + e_{11} + zk_{11})(e_{21} + zk_{21}) + (1 + e_{22} + zk_{22})(e_{12} + zk_{12}) + \\ &(e_{13} + zk_{13})(e_{23} + zk_{23})] \Big/ [\sqrt{(1 + e_{11} + zk_{11})^2 + (e_{12} + zk_{12})^2 + (e_{13} + zk_{13})^2} \\ &\sqrt{(1 + e_{22} + zk_{22})^2 + (e_{21} + zk_{21})^2 + (e_{23} + zk_{23})^2}]\Big\}, \end{aligned}\right\} \tag{6}$$

式中

$$\left.\begin{aligned} &k_{11} = \frac{1}{A_1}\frac{}{_1} + \frac{1}{A_1A_2}\frac{A_1}{_2} + \frac{1 +}{R_1}, \quad k_{22} = \frac{1}{A_2}\frac{}{_2} + \frac{1}{A_1A_2}\frac{A_2}{_1} + \frac{1 +}{R_2}, \\ &k_{12} = \frac{1}{A_1}\frac{}{_1} - \frac{1}{A_1A_2}\frac{A_1}{_2}, \quad k_{21} = \frac{1}{A_2}\frac{}{_2} - \frac{1}{A_1A_2}\frac{A_2}{_1}, \\ &k_{13} = \frac{1}{A_1}\frac{}{_1} - \frac{}{R_1}, \quad k_{23} = \frac{1}{A_2}\frac{}{_2} - \frac{}{R_2} \end{aligned}\right\} \tag{7}$$

按照均匀和各向同性体的虎克定律，当忽略壳体中面法应力 $_{33}$ 时，壳体的内力 T_1、T_2、T_{12}、T_{21} 和内矩 M_1、M_2、M_{12}、M_{21} 以及应变能 A 为[2]

$$
\left.
\begin{aligned}
T_1 &= \quad {}_{11z}\left(1+\frac{z}{R_2}\right)\mathrm{d}z, \quad M_1 = \quad {}_{11z}\left(1+\frac{z}{R_2}\right)z\,\mathrm{d}z,\\
T_2 &= \quad {}_{22z}\left(1+\frac{z}{R_1}\right)\mathrm{d}z, \quad M_2 = \quad {}_{22z}\left(1+\frac{z}{R_1}\right)z\,\mathrm{d}z,\\
T_{12} &= \quad {}_{12z}\left(1+\frac{z}{R_2}\right)\mathrm{d}z, \quad M_{12} = \quad {}_{12z}\left(1+\frac{z}{R_2}\right)z\,\mathrm{d}z,\\
T_{21} &= \quad {}_{12z}\left(1+\frac{z}{R_1}\right)\mathrm{d}z, \quad M_{21} = \quad {}_{12z}\left(1+\frac{z}{R_1}\right)z\,\mathrm{d}z,
\end{aligned}
\right\}
\tag{8}
$$

$$
A = \frac{1}{2} \quad (\quad {}_{11z} \quad {}_{11z} + \quad {}_{22z} \quad {}_{22z} + \quad {}_{12z} \quad {}_{12z})A_1A_2\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\mathrm{d} \quad {}_1\mathrm{d} \quad {}_2\mathrm{d}z,
\tag{9}
$$

式中

$$
\left.
\begin{aligned}
{}_{11z} &= \frac{E}{1- \quad ^2}(\quad {}_{11z} + \quad {}_{22z}),\\
{}_{22z} &= \frac{E}{1- \quad ^2}(\quad {}_{22z} + \quad {}_{11z}),\\
{}_{12z} &= \frac{E}{2(1+ \quad)} \quad {}_{21z},
\end{aligned}
\right\}
\tag{10}
$$

E 为弹性模数， 为泊桑比

2 虚 位 移 原 理

将虚位移原理应用到变形物体，设 u、v、w 为载荷引起的真实位移， u、 v、 w 为由承受载荷的平衡位置算起的虚位移 应变能的增量 A 等于外力在虚位移上所作的功

$$
A = \quad R_1 + \quad R_2,
\tag{11}
$$

式中 R_1 为载荷的功， R_2 为边界力所作的功 作用在壳体上的载荷有体积力和表面力，可近似地认为作用在壳体中面上的分布载荷 通常体积力，如重力和惯性力，其大小和方向是不变的 表面力，如液体或气体的压力，其大小和方向都是变的，且为法线方向 设 p_1、p_2、p_n 为体积力沿 $_1$、 $_2$ 和法线 n 方向的分量，q_n 为法向表面力 载荷的虚功为

$$
\begin{aligned}
R_1 = & \quad (p_1 \quad u + p_2 \quad v + p_n \quad w)A_1A_2\mathrm{d} \quad {}_1\mathrm{d} \quad {}_2 + \\
& \quad q_n(\quad {}_{31} \quad u + \quad {}_{32} \quad v + \quad {}_{33} \quad w)A_1A_2\mathrm{d} \quad {}_1\mathrm{d} \quad {}_2
\end{aligned}
\tag{12}
$$

设 T_1、T_{12}、N_1、M_1 为边界力沿壳体中面轴线 $_2$ 的分量，T_2、T_{21}、N_2、M_2 为边界力沿壳体中面轴线 $_1$ 的分量 P 为角点力 通常这些力是不变的 这些力的虚功是

$$
\begin{aligned}
R_2 = & \quad (T_1 \quad u + T_{12} \quad v + N_1 \quad w + M_1 \quad {}_1)A_2\mathrm{d} \quad {}_2 + \\
& \quad (T_{21} \quad u + T_2 \quad v + N_2 \quad w + M_2 \quad {}_2)A_2\mathrm{d} \quad {}_1 + P \quad w,
\end{aligned}
\tag{13}
$$

式中

$$
{}_1 = \arctan\frac{\quad {}_{31}}{\quad {}_{33}}, \quad {}_2 = \arctan\frac{\quad {}_{32}}{\quad {}_{33}}
\tag{14}
$$

若边界位移是给定的，则边界位移的变分等于零

3　基本理论的简化

对薄壳的挠度与厚度同级的大变形问题, 在(5)式中, 如果应变 e_{11}、e_{12}、e_{21}、e_{22} 和角转动的平方 e_{13}^2、e_{23}^2 均很小, 则 $_{11}$、$_{22}$、$_{12}$ 也必很小　因此, 与 1 相比时可以略去应变和角转动的平方　由(3)、(2)、(7)、(14)、(12) 和(13)6 式简化后可得

$$_{31} = - e_{13},\quad _{32} = - e_{23},\quad _{33} = 1; \tag{15}$$

$$= - e_{13},\quad = - e_{23},\quad = 0; \tag{16}$$

$$\left.\begin{aligned} k_{11} &= - \frac{1}{A_1}\frac{e_{13}}{_1} - \frac{1}{A_1A_2}\frac{A_1}{_2}e_{23} + \frac{1}{R_1},\quad k_{22} = - \frac{1}{A_2}\frac{e_{23}}{_2} - \frac{1}{A_1A_2}\frac{A_2}{_1}e_{13} + \frac{1}{R_2}, \\ k_{12} &= - \frac{1}{A_1}\frac{e_{23}}{_1} + \frac{1}{A_1A_2}\frac{A_1}{_2}e_{13},\quad k_{21} = - \frac{1}{A_2}\frac{e_{13}}{_2} + \frac{1}{A_1A_2}\frac{A_2}{_1}e_{23}, \\ k_{13} &= e_{13}/R_1,\quad k_{23} = e_{23}/R_2; \end{aligned}\right\} \tag{17}$$

$$_1 = - e_{13},\quad _2 = - e_{23}; \tag{18}$$

$$R_1 = [(p_1 - q_n e_{13}) u + (p_2 - q_n e_{23}) v + (p_n + q_n) w] A_1A_2 \mathrm{d}_1 \mathrm{d}_2; \tag{19}$$

$$\begin{aligned} R_2 = &(T_1 u + T_{12} v + N_1 w - M_1 e_{13}) A_2 \mathrm{d}_2 + \\ &(T_{21} u + T_2 v + N_2 w - M_2 e_{23}) A_1 \mathrm{d}_1 + P w \end{aligned} \tag{20}$$

将(5)式中的平方根按二项式定理展开, 然后进行简化可得

$$_{11} = e_{11} + \frac{1}{2}e_{13}^2,\quad _{22} = e_{22} + \frac{1}{2}e_{23}^2,\quad _{12} = e_{12} + e_{21} + e_{13}e_{23} \tag{21}$$

此外, 由于壳体是薄的, z/R_1、z/R_2 和 1 相比也是一个可以略去的小量　因此(6)、(8) 和(9)3 式简化后, 并应用到(10)式可得

$$\left.\begin{aligned} _{11z} &= e_{11} + e_{13}^2/2 + zk_1, \\ _{22z} &= e_{22} + e_{23}^2/2 + zk_2, \\ _{12z} &= e_{12} + e_{21} + e_{13}e_{23} + z(k_{12} + k_{21}); \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} T_1 &= \frac{Eh}{1 - ^2}\left[e_{11} + \frac{1}{2}e_{13}^2 + \left(e_{22} + \frac{1}{2}e_{23}^2\right)\right],\quad M_1 = \frac{Eh^3}{12(1 - ^2)}(k_1 + k_2), \\ T_2 &= \frac{Eh}{1 - ^2}\left[e_{22} + \frac{1}{2}e_{23}^2 + \left(e_{11} + \frac{1}{2}e_{13}^2\right)\right],\quad M_2 = \frac{Eh^3}{12(1 - ^2)}(k_2 + k_1), \\ T_{12} &= T_{21} = \frac{Eh}{2(1 +)}(e_{12} + e_{21} + e_{13}e_{23}),\quad M_{12} = M_{21} = \frac{Eh^3}{24(1 +)}(k_{12} + k_{21}); \end{aligned}\right\} \tag{23}$$

$$\begin{aligned} A = \frac{1}{2}\Big[&T_1\left(e_{11} + \frac{1}{2}e_{13}^2\right) + T_2\left(e_{22} + \frac{1}{2}e_{23}^2\right) + T_{12}(e_{12} + e_{21} + e_{13}e_{23}) + \\ &M_1k_1 + M_2k_2 + M_{12}(k_{12} + k_{21})\Big] A_1A_2 \mathrm{d}_1 \mathrm{d}_2; \end{aligned} \tag{24}$$

式中 h 为壳的厚度　将(23)式代入(24)式可得

$$\begin{aligned} A = \frac{Eh}{2(1 - ^2)}\Bigg\{&\left(e_{11} + \frac{1}{2}e_{13}^2\right)^2 + \left(e_{22} + \frac{1}{2}e_{23}^2\right)^2 + 2\left(e_{11} + \frac{1}{2}e_{13}^2\right)\left(e_{22} + \frac{1}{2}e_{23}^2\right) + \\ &\frac{1 - }{2}(e_{12} + e_{21} + e_{13}e_{23})^2 + \frac{h^2}{12}\Big[k_1^2 + k_2^2 + 2 k_1k_2 + \end{aligned}$$

$$\frac{1-}{2}(k_{12}+k_{21})^2\Big]\Big\}A_1A_2\mathrm{d}\ _1\mathrm{d}\ _2, \qquad (25)$$

式中 $k_1=k_{11}-1/R_1,\ k_2=k_{22}-1/R_2$

对上式进行变分, 并应用到(23) 式可得

$$A=\Big[T_1\Big(e_{11}+\frac{1}{2}e_{13}^2\Big)+T_2\Big(e_{22}+\frac{1}{2}e_{23}^2\Big)+T_{12}(e_{12}+e_{21}+e_{13}e_{23})+$$
$$M_1\ k_1+M_2\ k_2+M_{12}(k_{12}+k_{21})\Big]A_1A_2\mathrm{d}\ _1\mathrm{d}\ _2 \qquad (26)$$

将(4) 式代入(17) 、(19) 、(20) 和(26) 4 式, 然后代入(11) 式, 积分后可得

$$(A-R_1-R_2)=-\Big\{\Big[\frac{A_2T_1}{1}+\frac{A_1T_{12}}{2}+\frac{A_1}{2}T_{12}-\frac{A_2}{1}T_2+\frac{1}{R_1}\Big(\frac{A_2M_1}{1}+$$
$$\frac{A_1M_{12}}{2}+\frac{A_1}{2}M_{12}-\frac{A_2}{1}M_2\Big)+\frac{A_1A_2}{R_1}(T_1e_{13}+T_{12}e_{23})+A_1A_2(p_1-q_ne_{13})\Big]\ u+$$
$$\Big[\frac{A_1T_2}{2}+\frac{A_2T_{12}}{1}+\frac{A_2}{1}T_{12}-\frac{A_1}{2}T_1+\frac{1}{R_2}\Big(\frac{A_1M_2}{2}+\frac{A_2M_{12}}{1}+\frac{A_2}{1}M_{12}-$$
$$\frac{A_1}{5\ A_2}M_1\Big)+\frac{A_1A_2}{R_2}(T_2e_{23}+T_{12}e_{13})+A_1A_2(p_2-q_ne_{23})\Big]Dv+\Big\{\frac{5}{5A_1}\Big[\frac{1}{A_1}\Big(\frac{5A_2M_1}{5\ A_1}+$$
$$\frac{5A_1M_{12}}{5\ A2}+\frac{5A_1}{5\ A2}M_{12}-\frac{5A_2}{5\ A1}M_2\Big)\Big]+\frac{5}{5A_2}\Big[\frac{1}{A_2}\Big(\frac{5A_1M_2}{5\ A2}+\frac{5A_2M_{12}}{5\ A1}+\frac{5A_2}{5\ A1}M_{12}-$$
$$\frac{5A_1}{5\ A2}M_1\Big)\Big]-A_1A_2\Big(\frac{T_1}{R_1}+\frac{T_2}{R_2}\Big)+\frac{5A_2T_1e_{13}}{5\ A1}+\frac{5A_1T_{12}e_{13}}{5\ A2}+\frac{5A_2T_{12}e_{23}}{5\ A1}+\frac{5A_1T_2e_{23}}{5\ A2}+$$
$$A_1A_2(p_n+q_n)\Big\}Dw\Big\}\mathrm{d}\,\mathrm{A}_1\mathrm{d}\,\mathrm{A}_2+\oint\Big\{(T_1-T_1)Du+\Big(T_{12}+\frac{M_{12}}{R_2}-T_{12}\Big)\mathrm{D}v+\Big[\frac{1}{A_1A_2}@$$
$$\Big(\frac{5A_2M_1}{5\ A_1}+\frac{5A_1M_{12}}{5\ A_2}+\frac{5A_1}{5\ A_2}M_{12}-\frac{5A_2}{5\ A_1}M_2\Big)+\frac{1}{A_2}\frac{5M_{12}}{5\ A_2}+T_1e_{13}+T_{12}e_{23}-N_1\Big]\mathrm{D}w-$$
$$(M_1-M_1)De_{13}\Big\}A_2\mathrm{d}\mathrm{A}_2+\oint\Big\{\Big(T_{12}+\frac{M_{12}}{R_1}-T_{21}\Big)\mathrm{D}u+(T_2-T_2)Dv+\Big[\frac{1}{A_1A_2}@$$
$$\Big(\frac{5A_1M_2}{5\ A_2}+\frac{5A_2M_{12}}{5\ A_1}+\frac{5A_2}{5\ A_1}M_{12}-\frac{5A_1}{5\ A_2}M_1\Big)+\frac{1}{A_1}\frac{5M_{12}}{5\ A_1}+T_2e_{23}+T_{12}e_{13}-N_2\Big]\mathrm{D}w-$$
$$(M_2-M_2)De_{23}\Big\}A_1\mathrm{d}\mathrm{A}_1+(2M_{12}-P)Dw=0\# \qquad (27)$$

由于变分的任意性, 上式双重积分号下各项等于零组成壳体非线性变形的平衡方程# 线积分号下各项等于零组成外力的边界条件和角点条件#

4 挠度与长度同级时的大变形问题

对于壳的挠度与长度同级的大变形问题, 虽然 E_{11}、E_{22}、E_{12} 必须是小的, 然而 e_{11}、e_{12}、e_{21}、e_{22}、e_{13}、e_{23} 与 1 相比可能不小# 由于这一原因, 我们不能展开(5) 式中的平方根# 当薄壳为无限长柱形, 并作柱状变形时, 能够形成很大的变形且为弹性# 此时, 取无限长方向为 A_2, 则应有

$$v=\frac{1}{R_2}=\frac{5}{5\ A_2}=0,\ A_2=\text{常数}\#$$

将上式代入(4) 式可得

$$e_{12}=e_{21}=e_{22}=e_{23}=0,\ e_{11}=\frac{1}{A_1}\frac{du}{dA_1}+\frac{w}{R_1},\ e_{13}=\frac{1}{A_1}\frac{dw}{dA_1}-\frac{u}{R_1}\# \tag{28}$$

将上式代入(3)、(14)和(5)3式可得

$$A_{31}=-e_{13},\ A_{32}=0,\ A_{33}=1+e_{11}, \tag{29}$$

$$U_1=\arctan\frac{-e_{13}}{1+e_{11}},\ U_2=0; \tag{30}$$

$$E_{11}=\sqrt{(1+e_{11})^2+e_{13}^2}-1,\ E_{22}=E_{12}=0\# \tag{31}$$

将(29)式代入(2)式,并注意到(30)式可得[3]

$$;=-\frac{e_{13}}{\sqrt{(1+e_{11})^2+e_{13}^2}}=\sin U_1,\ W=0,\ V=\frac{1+e_{11}}{\sqrt{(1+e_{11})^2+e_{13}^2}}-1=\cos U_1-1\# \tag{32}$$

注意到以上各式,(7)、(12)和(3)3式成为

$$k_{12}=k_{21}=k_{22}=k_{23}=0,\ k_{11}=\cos U_1\left(\frac{1}{A_1}\frac{dU_1}{dA_1}+\frac{1}{R_1}\right),\ k_{13}=-\sin U_1\left(\frac{1}{A_1}\frac{dU_1}{dA_1}+\frac{1}{R_1}\right); \tag{33}$$

$$DR_1=Q\{(p_1-q_ne_{13})Du+[p_n+q_n(1+e_{11})]Dw\}A_1dA_1; \tag{34}$$

$$DR_2=T_1Du+N_1Dw+M_1DU_1\# \tag{35}$$

将(33)式代入(6)式可得

$$E_{11z}=\frac{1}{1+z/R_1}(E_{11}+zk_1),\ E_{22z}=0,\ E_{12z}=0, \tag{36}$$

式中

$$k_1=\frac{1}{A_1}\frac{dU_1}{dA_1}\# \tag{37}$$

将(30)式代入上式可得

$$k_1=-\frac{(1+e_{11})\frac{1}{A_1}\frac{de_{13}}{dA_1}-e_{13}\frac{1}{A_1}\frac{de_{11}}{dA_1}}{(1+e_{11})^2+e_{13}^2}\# \tag{38}$$

将(36)式代入(10)、(8)和(9)3式,略去和1相比的z/R_1可得

$$R_{11z}=\frac{E}{1-L^2}(E_{11}+zk_1),\ R_{22z}=\frac{LE}{1-L^2}(E_{11}+zk_1),\ R_{12z}=0; \tag{39}$$

$$\left.\begin{aligned}&T_1=\frac{Eh}{1-L^2}E_{11},\ T_2=\frac{LEh}{1-L^2}E_{11},\ T_{12}=T_{21}=0,\\&M_1=\frac{Eh^3}{12(1-L^2)}k_1,\ M_2=\frac{LEh^3}{12(1-L^2)}k_1,\ M_{12}=M_{21}=0;\end{aligned}\right\} \tag{40}$$

$$A=\frac{1}{2}Q(T_1E_{11}+M_1k_1)A_1dA_1=\frac{Eh}{2(1-L^2)}Q\left(E_{11}^2+\frac{h^2}{12}k_1^2\right)A_1dA_1, \tag{41}$$

$$DA=Q(T_1DE_{11}+M_1Dk_1)A_1dA_1\# \tag{42}$$

将(28)式代入(30)、(31)和(37)3式,然后代入(42)、(34)和(35)3式,最后代入(11)式,积分后可得

$$D(A-R_1-R_2)=(T_1\cos U_1+N_1\sin U_1-T_1)Du-(T_1\sin U_1-N_1\cos U_1+N_1)Dw+$$
$$(M_1-M_1)DU_1-Q\left\{\left[\frac{1}{A_1}\frac{d}{dA_1}(T_1\cos U_1)-\frac{T_1}{R_1}\sin U_1+\frac{1}{A_1}\frac{d}{dA_1}(N_1\sin U_1)+\right.\right.$$

$$\frac{N_1}{R}\cos U_1 + p_1 - q_n e_{13}\Big] D u - \Big[\frac{1}{A_1}\frac{d}{dA_1}(T_1\sin U_1) + \frac{T_1}{R_1}\cos U_1 - \frac{1}{A_1}\frac{d}{dA_1}(N_1\cos U_1) + \frac{N_1}{R_1}\sin U_1 - p_n - q_n(1 + e_{11})\Big] Dw\Big\} A_1 dA_1 = 0, \qquad (43)$$

式中

$$N_1 = \frac{1}{1+E_{11}}\frac{1}{A_1}\frac{dM_1}{dA_1}\# \qquad (44)$$

当变形很大而单位伸长仍然很小, 可令 $E_{11} = 0$# 由(31) 式和(38) 式可得

$$e_{11} = ? \sqrt{1 - e_{13}^2} - 1, \quad k_1 = \frac{\frac{1}{A_1}\frac{de_{13}}{dA_1}}{\sqrt{1 - e_{13}^2}}\# \qquad (45)$$

例如当圆柱壳的两边承受力矩 M_1 作用时, 应有 $A_1 = R_1 =$ 常数# 由(37) 式和(40) 式可得

$$\frac{dU_1}{dA_1} = \frac{M_1 R_1}{D}, \quad D = \frac{Eh^3}{12(1 - L^2)},$$

即变形后中面的曲率将变为

$$\frac{1}{R} = \frac{M_1}{D} + \frac{1}{R_1},$$

当 $1/R = 0$, 即 $M_1 = - D/R_1$ 时, 中面变成一平面#

对于平板, 可令 $1/R_1 = 0$, $A_1 dA_1 = dx$# 当板的两边承受力矩 M 作用时[3], 应有 $dU_1/dx = M/D$, 变形后成为曲率为 $1/R = M/D$ 的圆柱面# 当 $M = - M_1$ 时, $R = R_1$#

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Nonlinear Deformation Theory of Thin Shell

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Abstract: The exact relation between strain and displacement is given for nonlinear deformation of thin shell. The fundamental formula of large deformation when the deflection is on the same class with the thickness of the shell is derived after simplified rationally. The fundamental formula of large deformation when the deflection is on the same class with the length of the shell is derived exactly for cylinder shell deformed cylindrical shaped.

Key words: thin shell; nonlinear deformation; large deflection