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集中载荷作用各边上任一点被支承的矩形板弯曲*

边宇虹

(燕山大学 土木工程与力学系, 秦皇岛 066004)

(陈山林推荐)

摘要: 应用功的互等定理求解在集中载荷作用下各边上任一点被支承的矩形板弯曲, 给出了其精确解及算例•

关 键 词: 功的互等定理; 集中载荷; 精确解
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引 言

在工程技术中, 弹性薄板的弯曲问题是具有重要的实际意义的, 并已得到了很多结果• 但由于各边上的点支承问题比较复杂, 因此, 该问题的结果不是很多• 文献[1] 应用叠加法求解了四个角点被支承的矩形板弯曲• 文献[2] 应用 Rayleigh_Ritz 法研究了均布载荷作用下四边中点被支承的矩形板弯曲• 本文应用功的互等定理研究在该板上任意一点作用一集中载荷各边上任一点被支承的矩形板弯曲问题, 给出了该问题的精确解•

1 基本解

为了应用功的互等定理, 取在一流动坐标 (ξ, η) 点处受单位集中载荷作用的四边简支矩形板为基本系统, 如图 1 所示• 该基本系统的解为基本解, 可表示为

$$W_1(x, y, \xi, \eta) = \frac{b^2}{\pi^3 D}\sum_{n=1}^{\infty}\left[1 + \alpha_n \operatorname{cth}\alpha_n - \frac{\alpha_n(a-x)}{a}\operatorname{cth}\frac{\alpha_n(a-x)}{a} - \frac{\alpha_n\xi}{a}\operatorname{cth}\frac{\alpha_n\xi}{a}\right]\times \frac{1}{n^3\operatorname{sh}\alpha_n}\operatorname{sh}\frac{\alpha_n\xi}{a}\operatorname{sh}\frac{\alpha_n(a-x)}{a}\sin\frac{n\pi\eta}{b}\sin\frac{n\pi y}{b} \quad (\xi \leqslant x), \tag{1}$$

$$W_1(x, y, \xi, \eta) = \frac{a^2}{\pi^3 D}\sum_{m=1}^{\infty}\left[1 + \beta_m \operatorname{cth}\beta_m - \frac{\beta_m(b-y)}{b}\operatorname{cth}\frac{\beta_m(b-y)}{b} - \frac{\beta_m\eta}{b}\operatorname{cth}\frac{\beta_m\eta}{b}\right]\times \frac{1}{m^3\operatorname{sh}\beta_m}\operatorname{sh}\frac{\beta_m\eta}{b}\operatorname{sh}\frac{\beta_m(b-y)}{b}\sin\frac{m\pi\xi}{a}\sin\frac{m\pi x}{a} \quad (\eta \leqslant y), \tag{2}$$

式中 $\alpha_n = n\pi a/b$, $\beta_m = m\pi b/a$• 当 $\xi \geqslant x$ 时, 在应用式(1) 时, $a-x$ 必须用 x 代替, ξ 用 $a-\xi$ 代替• 当 $\eta \geqslant y$ 时, 在应用式(2) 时, $b-y$ 必须用 y 代替, η 用 $b-\eta$ 代替• 边界上的等效

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作者简介: 边宇虹(1962~), 女, 副教授, 研究方向: 弹性薄板理论, 发表专业学术论文 30 余篇.

切力，角点力分别表示为 V_{1x0}, V_{1xa}, V_{1y0}, V_{1yb}, R_{100}, R_{1a0}, R_{1ab}, R_{10b} •

2 集中载荷作用矩形板的挠曲面方程

有一集中载荷 P 作用于每一边上任一点被支承的矩形板上的任意一点(x_0, y_0)，如图 2 所示• 它为所求的真实系统，设其挠度为 $W(\xi, \eta)$• 可假设四边挠度分别为

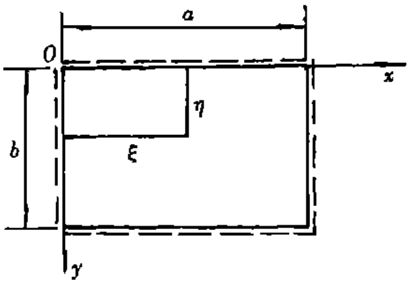

图 1

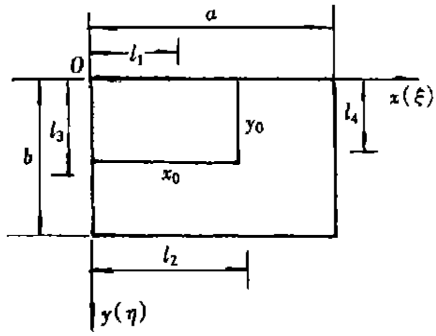

图 2

$$W_{x0} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{b} + K_1 + \frac{K_4 - K_1}{b}y, \tag{3}$$

$$W_{xa} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{b} + K_2 + \frac{K_3 - K_2}{b}y, \tag{4}$$

$$W_{y0} = \sum_{m=1}^{\infty} e_m \sin\frac{m\pi x}{a} + K_1 + \frac{K_2 - K_1}{a}x, \tag{5}$$

$$W_{yb} = \sum_{m=1}^{\infty} d_m \sin\frac{m\pi x}{a} + K_4 + \frac{K_3 - K_4}{a}x, \tag{6}$$

其中 K_1, K_2, K_3, K_4 分别为$(0, 0)$, $(a, 0)$, (a, b), $(0, b)$ 的角点位移; P_1, P_2, P_3, P_4 分别为四个支点的支反力•

在图 1 基本系统与图 2 真实系统之间应用功的互等定理，得

$$W(\xi, \eta) = PW_1(x_0, y_0; \xi, \eta) + \int_a^b V_{1x0}W_{x0}\mathrm{d}y - \int_a^b V_{1xa}W_{xa}\mathrm{d}y + \int_0^a V_{1y0}W_{y0}\mathrm{d}x - \int_0^a V_{1yb}W_{yb}\mathrm{d}x + R_{100}K_1 + R_{1ab}K_3 - R_{1a0}K_2 - R_{10b}K_4 \text{•} \tag{7}$$

将式(1) ~ (6) 代入式(7)，经化简整理，得

$$\begin{aligned} W(\xi, \eta) = {} & W_P + \frac{1}{2}\sum_{n=1}^{\infty}\left\{2 + (1-\mu)\left[\alpha_n \operatorname{cth}\alpha_n - \frac{\alpha_n(a-\xi)}{a}\operatorname{cth}\frac{\alpha_n(a-\xi)}{a}\right]\right\} \times \\ & \frac{1}{\operatorname{sh}\alpha_n}\operatorname{sh}\frac{\alpha_n(a-\xi)}{a}\sin\frac{\alpha_n\eta}{a}(a_n) + \frac{1}{2}\sum_{n=1}^{\infty}\left\{2 + (1-\mu)\left[\alpha_n \operatorname{cth}\alpha_n - \frac{\alpha_n\xi}{a}\operatorname{cth}\frac{\alpha_n\xi}{a}\right]\right\} \times \\ & \frac{1}{\operatorname{sh}\alpha_n}\operatorname{sh}\frac{\alpha_n\xi}{a}\sin\frac{\alpha_n\eta}{a}(b_n) + \frac{1}{2}\sum_{m=1}^{\infty}\left\{2 + (1-\mu)\left[\beta_m \operatorname{cth}\beta_m - \frac{\beta_m(b-\eta)}{b}\operatorname{cth}\frac{\beta_m(b-\eta)}{b}\right]\right\} \times \\ & \frac{1}{\operatorname{sh}\beta_m}\operatorname{sh}\frac{\beta_m(b-\eta)}{b}\sin\frac{\beta_m\xi}{b}(e_m) + \frac{1}{2}\sum_{m=1}^{\infty}\left\{2 + (1-\mu)\left[\beta_m \operatorname{cth}\beta_m - \frac{\beta_m\eta}{b}\operatorname{cth}\frac{\beta_m\eta}{b}\right]\right\} \times \\ & \frac{1}{\operatorname{sh}\beta_m}\operatorname{sh}\frac{\beta_m\eta}{b}\sin\frac{\beta_m\xi}{b}(d_m) + \frac{a-\xi}{a}\frac{b-\eta}{b}K_1 + \frac{\xi}{a}\frac{b-\eta}{b}K_2 + \frac{\xi}{a}\frac{\eta}{b}K_3 + \frac{a-\xi}{a}\frac{\eta}{b}K_4, \end{aligned} \tag{8}$$

其中 $W_p = PW_1(x_0, y_0; \xi, \eta)$ 有以下两种形式

$$W_P = \frac{Pb^2}{\pi^3 D}\sum_{n=1}^{\infty}\left[1+\alpha_n \operatorname{cth}\alpha_n - \frac{\alpha_n(a-x_0)}{a}\operatorname{cth}\frac{\alpha_n(a-x_0)}{a} - \frac{\alpha_n\xi}{a}\operatorname{cth}\frac{\alpha_n\xi}{a}\right]\frac{1}{n^3\operatorname{sh}\alpha_n}\times \operatorname{sh}\frac{\alpha_n\xi}{a}\operatorname{sh}\frac{\alpha_n(a-x_0)}{a}\sin\frac{n\pi y_0}{b}\sin\frac{\alpha_n\eta}{a} \quad (\xi \leqslant x_0), \tag{9}$$

$$W_P = \frac{Pa^2}{\pi^3 D}\sum_{m=1}^{\infty}\left[1+\beta_m \operatorname{cth}\beta_m - \frac{\beta_m(b-y_0)}{b}\operatorname{cth}\frac{\beta_m(b-y_0)}{b} - \frac{\beta_m\eta}{b}\operatorname{cth}\frac{\beta_m\eta}{b}\right]\frac{1}{m^3\operatorname{sh}\beta_m}\times \operatorname{sh}\frac{\beta_m\eta}{b}\operatorname{sh}\frac{\beta_m(b-y_0)}{b}\sin\frac{m\pi x_0}{a}\sin\frac{\beta_m\xi}{b} \quad (\eta \leqslant y_0). \tag{10}$$

当 $\xi \geqslant x_0$ 时, 在应用式(9) 时, $a - x_0$ 必须用 x_0 代替, ξ 用 $a-\xi$ 代替. 当 $\eta \geqslant y_0$ 时, 在应用式(10) 时, $b-y_0$ 必须用 y_0 代替, η 用 $b-\eta$ 代替. 式(8) 就是要求的挠曲面方程. 其中 a_n, b_n, e_m, d_m, K_1, K_2, K_3, K_4, P_1, P_2, P_3, P_4 为待定系数, 由边界条件确定.

3 满足边界条件

$$-D\left[\frac{\partial^3 W}{\partial\xi^3}+(2-\mu)\frac{\partial^3 W}{\partial\xi\partial\eta^2}\right]_{\xi=0} = \frac{2P_1}{b}\sum_{n=1}^{\infty}\sin\frac{n\pi l_3}{b}\sin\frac{n\pi\eta}{b}, \tag{11}$$

$$-D\left[\frac{\partial^3 W}{\partial\xi^3}+(2-\mu)\frac{\partial^3 W}{\partial\xi\partial\eta^2}\right]_{\xi=a} = -\frac{2P_2}{b}\sum_{n=1}^{\infty}\sin\frac{n\pi l_4}{b}\sin\frac{n\pi\eta}{b}, \tag{12}$$

$$-D\left[\frac{\partial^3 W}{\partial\eta^3}+(2-\mu)\frac{\partial^3 W}{\partial\eta\partial\xi^2}\right]_{\eta=0} = \frac{2P_3}{a}\sum_{m=1}^{\infty}\sin\frac{m\pi l_1}{a}\sin\frac{m\pi\xi}{a}, \tag{13}$$

$$-D\left[\frac{\partial^3 W}{\partial\eta^3}+(2-\mu)\frac{\partial^3 W}{\partial\eta\partial\xi^2}\right]_{\eta=b} = -\frac{2P_4}{a}\sum_{m=1}^{\infty}\sin\frac{m\pi l_2}{a}\sin\frac{m\pi\xi}{a}, \tag{14}$$

$$\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)_{\xi=0,\eta=0} = 0, \tag{15}$$

$$\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)_{\xi=a,\eta=0} = 0, \tag{16}$$

$$\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)_{\xi=a,\eta=b} = 0, \tag{17}$$

$$\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)_{\xi=0,\eta=b} = 0, \tag{18}$$

$$(W)_{\xi=0,\eta=l_3} = 0, \tag{19}$$

$$(W)_{\xi=a,\eta=l_4} = 0, \tag{20}$$

$$(W)_{\xi=l_1,\eta=0} = 0, \tag{21}$$

$$(W)_{\xi=l_2,\eta=b} = 0, \tag{22}$$

将式(8) 代入边界条件(11)~(22), 经化简整理, 得到与边界条件相对应的方程为

$$\frac{P}{b}\left[2+(1-\mu)\left(\alpha_n\operatorname{cth}\alpha_n - \frac{\alpha_n(a-x_0)}{a}\operatorname{cth}\frac{\alpha_n(a-x_0)}{a}\right)\right]\frac{1}{\operatorname{sh}\alpha_n}\operatorname{sh}\frac{\alpha_n(a-x_0)}{a}\sin\frac{n\pi y_0}{b} - \frac{D}{2}\left[2(1-\mu^2)\operatorname{ch}\alpha_n+(1-\mu)^2\left(\operatorname{ch}\alpha_n+\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)\right]\left(\frac{n\pi}{b}\right)^3\frac{a_n}{\operatorname{sh}\alpha_n} + \frac{D}{2}[2(1-\mu^2)+(1-\mu)^2(1+\alpha_n\operatorname{cth}\alpha_n)]\left(\frac{n\pi}{b}\right)^3\frac{b_n}{\operatorname{sh}\alpha_n} + 2D(1-\mu)^2\frac{\pi^2}{a^3}\left[\sum_{m=1}^{\infty}\frac{n^3 e_m}{m\left(\frac{b^2}{a^2}+\frac{n^2}{m^2}\right)^2} - \sum_{m=1}^{\infty}\frac{(-1)^n n^3 d_m}{m\left(\frac{b^2}{a^2}+\frac{n^2}{m^2}\right)^2}\right] =$$

$$\frac{2P_1}{b}\sin\frac{n\pi l_3}{b}\qquad (n=1,2,\cdots),\tag{23}$$

$$-\frac{P}{b}\left[2+(1-\mu)\left(\alpha_n\operatorname{cth}\alpha_n-\frac{\alpha_n x_0}{a}\operatorname{cth}\frac{\alpha_n x_0}{a}\right)\right]\frac{1}{\operatorname{sh}\alpha_n}\operatorname{sh}\frac{\alpha_n x_0}{a}\sin\frac{n\pi y_0}{b}-$$
$$\frac{D}{2}[2(1-\mu^2)+(1-\mu)^2(1+\alpha_n\operatorname{cth}\alpha_n)]\left(\frac{n\pi}{b}\right)^3\frac{a_n}{\operatorname{sh}\alpha_n}+$$
$$\frac{D}{2}\left[2(1-\mu^2)\operatorname{ch}\alpha_n+(1-\mu)^2\left(\operatorname{ch}\alpha_n+\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)\right]\left(\frac{n\pi}{b}\right)^3\frac{b_n}{\operatorname{sh}\alpha_n}+$$
$$2D(1-\mu)^2\frac{\pi^2}{a^3}\left[\sum_{m=1}^{\infty}\frac{(-1)^m n^3 e_m}{m\left(\frac{b^2}{a^2}+\frac{n^2}{m^2}\right)^2}-\sum_{m=1}^{\infty}\frac{(-1)^{m+n}n^3 d_m}{m\left(\frac{b^2}{a^2}+\frac{n^2}{m^2}\right)^2}\right]=$$
$$-\frac{2P_2}{b}\sin\frac{n\pi l_4}{b}\qquad (n=1,2,\cdots),\tag{24}$$

$$\frac{P}{a}\left[2+(1-\mu)\left(\beta_m\operatorname{cth}\beta_m-\frac{\beta_m(b-y_0)}{b}\operatorname{cth}\frac{\beta_m(b-y_0)}{b}\right)\right]\frac{1}{\operatorname{sh}\beta_m}\operatorname{sh}\frac{\beta_m(b-y_0)}{a}\sin\frac{m\pi x_0}{a}-$$
$$\frac{D}{2}\left[2(1-\mu^2)\operatorname{ch}\beta_m+(1-\mu)^2\left(\operatorname{ch}\beta_m+\frac{\beta_m}{\operatorname{sh}\beta_m}\right)\right]\left(\frac{m\pi}{a}\right)^3\frac{e_m}{\operatorname{sh}\beta_m}+$$
$$\frac{D}{2}[2(1-\mu^2)+(1-\mu)^2(1+\beta_m\operatorname{cth}\beta_m)]\left(\frac{m\pi}{a}\right)^3\frac{d_m}{\operatorname{sh}\beta_m}+$$
$$2D(1-\mu)^2\frac{\pi^2}{b^3}\left[\sum_{n=1}^{\infty}\frac{m^3 a_n}{n\left(\frac{a^2}{b^2}+\frac{m^2}{n^2}\right)^2}-\sum_{n=1}^{\infty}\frac{(-1)^m m^3 b_n}{n\left(\frac{a^2}{b^2}+\frac{m^2}{n^2}\right)^2}\right]=$$
$$\frac{2P_3}{a}\sin\frac{m\pi l_1}{a}\qquad (m=1,2,\cdots),\tag{25}$$

$$-\frac{P}{a}\left[2+(1-\mu)\left(\beta_m\operatorname{cth}\beta_m-\frac{\beta_m y_0}{b}\operatorname{cth}\frac{\beta_m y_0}{b}\right)\right]\frac{1}{\operatorname{sh}\beta_m}\operatorname{sh}\frac{\beta_m y_0}{b}\sin\frac{m\pi x_0}{a}-$$
$$\frac{D}{2}[2(1-\mu^2)+(1-\mu)^2(1+\beta_m\operatorname{cth}\beta_m)]\left(\frac{m\pi}{a}\right)^3\frac{e_m}{\operatorname{sh}\beta_m}+$$
$$\frac{D}{2}\left[2(1-\mu^2)\operatorname{ch}\beta_m+(1-\mu)^2\left(\operatorname{ch}\beta_m+\frac{\beta_m}{\operatorname{sh}\beta_m}\right)\right]\left(\frac{m\pi}{a}\right)^3\frac{d_m}{\operatorname{sh}\beta_m}+$$
$$2D(1-\mu)^2\frac{\pi^2}{b^3}\left[\sum_{n=1}^{\infty}\frac{(-1)^n m^3 a_n}{n\left(\frac{a^2}{b^2}+\frac{m^2}{n^2}\right)^2}-\sum_{n=1}^{\infty}\frac{(-1)^{m+n}m^3 b_n}{n\left(\frac{a^2}{b^2}+\frac{m^2}{n^2}\right)^2}\right]=$$
$$-\frac{2P_4}{a}\sin\frac{m\pi l_2}{a}\qquad (m=1,2,\cdots),\tag{26}$$

$$-\frac{2Pa}{bD}\sum_{n=1}^{\infty}\left(\operatorname{cth}\alpha_n-\frac{a-x_0}{a}\operatorname{cth}\frac{\alpha_n(a-x_0)}{a}\right)\frac{1}{\operatorname{sh}\alpha_n}\operatorname{sh}\frac{\alpha_n(a-x_0)}{a}\sin\frac{n\pi y_0}{b}+$$
$$\sum_{n=1}^{\infty}\left[\left(\operatorname{ch}\alpha_n+\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)+\mu\left(\operatorname{ch}\alpha_n-\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)\right]\left(\frac{n\pi}{b}\right)^2\frac{a_n}{\operatorname{sh}\alpha_n}-\sum_{n=1}^{\infty}[(1+\alpha_n\operatorname{cth}\alpha_n)+$$
$$\mu(1-\alpha_n\operatorname{cth}\alpha_n)]\left(\frac{n\pi}{b}\right)^2\frac{b_n}{\operatorname{sh}\alpha_n}+\sum_{m=1}^{\infty}\left[\left(\operatorname{ch}\beta_m+\frac{\beta_m}{\operatorname{sh}\beta_m}\right)+\mu\left(\operatorname{ch}\beta_m-\frac{\beta_m}{\operatorname{sh}\beta_m}\right)\right]\left(\frac{m\pi}{a}\right)^2\times$$
$$\frac{e_m}{\operatorname{sh}\beta_m}-\sum_{m=1}^{\infty}[(1+\beta_m\operatorname{cth}\beta_m)+\mu(1-\beta_m\operatorname{cth}\beta_m)]\left(\frac{m\pi}{a}\right)^2\frac{d_m}{\operatorname{sh}\beta_m}-$$
$$\frac{2}{ab}(K_1-K_2+K_3-K_4)=0,\tag{27}$$

$$\frac{2Pa}{bD}\sum_{n=1}^{\infty}\left(\operatorname{cth}\alpha_n-\frac{x_0}{a}\operatorname{cth}\frac{\alpha_n x_0}{a}\right)\frac{1}{\operatorname{sh}\alpha_n}\operatorname{sh}\frac{\alpha_n x_0}{a}\sin\frac{n\pi y_0}{b}+\sum_{n=1}^{\infty}[(1+\alpha_n\operatorname{cth}\alpha_n)+\mu(1-\alpha_n\operatorname{cth}\alpha_n)]\left(\frac{n\pi}{b}\right)^2\frac{a_n}{\operatorname{sh}\alpha_n}-\sum_{n=1}^{\infty}\left[\left(\operatorname{ch}\alpha_n+\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)+\mu\left(\operatorname{ch}\alpha_n-\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)\right]\left(\frac{n\pi}{b}\right)^2\times\frac{b_n}{\operatorname{sh}\alpha_n}+\sum_{m=1}^{\infty}\left[\left(\operatorname{ch}\beta_m+\frac{\beta_m}{\operatorname{sh}\beta_m}\right)+\mu\left(\operatorname{ch}\beta_m-\frac{\beta_m}{\operatorname{sh}\beta_m}\right)\right]\left(\frac{m\pi}{a}\right)^2\frac{(-1)^m e_m}{\operatorname{sh}\beta_m}-\sum_{m=1}^{\infty}[(1+\beta_m\operatorname{cth}\beta_m)+\mu(1-\beta_m\operatorname{cth}\beta_m)]\left(\frac{m\pi}{a}\right)^2\frac{(-1)^m d_m}{\operatorname{sh}\beta_m}-\frac{2}{ab}(K_1-K_2+K_3-K_4)=0, \tag{28}$$

$$\frac{2Pa}{bD}\sum_{n=1}^{\infty}\left(\operatorname{cth}\alpha_n-\frac{x_0}{a}\operatorname{cth}\frac{\alpha_n x_0}{a}\right)\frac{(-1)^n}{\operatorname{sh}\alpha_n}\operatorname{sh}\frac{\alpha_n x_0}{a}\sin\frac{n\pi y_0}{b}+\sum_{n=1}^{\infty}[(1+\alpha_n\operatorname{cth}\alpha_n)+\mu(1-\alpha_n\operatorname{cth}\alpha_n)]\left(\frac{n\pi}{b}\right)^2\frac{(-1)^n a_n}{\operatorname{sh}\alpha_n}-\sum_{n=1}^{\infty}\left[\left(\operatorname{ch}\alpha_n+\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)+\mu\left(\operatorname{ch}\alpha_n-\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)\right]\left(\frac{n\pi}{b}\right)^2\times\frac{(-1)^n b_n}{\operatorname{sh}\alpha_n}+\sum_{m=1}^{\infty}\left[\left(1+\beta_m\operatorname{cth}\beta_m\right)+\mu(1-\beta_m\operatorname{cth}\beta_m)\right]\left(\frac{m\pi}{a}\right)^2\frac{(-1)^m e_m}{\operatorname{sh}\beta_m}-\sum_{m=1}^{\infty}\left[\left(\operatorname{ch}\beta_m+\frac{\beta_m}{\operatorname{sh}\beta_m}\right)+\mu\left(\operatorname{ch}\beta_m-\frac{\beta_m}{\operatorname{sh}\beta_m}\right)\right]\left(\frac{m\pi}{a}\right)^2\frac{(-1)^m d_m}{\operatorname{sh}\beta_m}-\frac{2}{ab}(K_1-K_2+K_3-K_4)=0, \tag{29}$$

$$-\frac{2Pa}{bD}\sum_{n=1}^{\infty}\left(\operatorname{cth}\alpha_n-\frac{a-x_0}{a}\operatorname{cth}\frac{\alpha_n(a-x_0)}{a}\right)\frac{(-1)^n}{\operatorname{sh}\alpha_n}\operatorname{sh}\frac{\alpha_n(a-x_0)}{a}\sin\frac{n\pi y_0}{b}+\sum_{n=1}^{\infty}\left[\left(\operatorname{ch}\alpha_n+\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)+\mu\left(\operatorname{ch}\alpha_n-\frac{\alpha_n}{\operatorname{sh}\alpha_n}\right)\right]\left(\frac{n\pi}{b}\right)^2\frac{(-1)^n a_n}{\operatorname{sh}\alpha_n}-\sum_{n=1}^{\infty}[(1+\alpha_n\operatorname{cth}\alpha_n)+\mu(1-\alpha_n\operatorname{cth}\alpha_n)]\left(\frac{n\pi}{b}\right)^2\frac{(-1)^n b_n}{\operatorname{sh}\alpha_n}+\sum_{m=1}^{\infty}\left[\left(1+\beta_m\operatorname{cth}\beta_m\right)+\mu(1-\beta_m\operatorname{cth}\beta_m)\right]\left(\frac{m\pi}{a}\right)^2\times\frac{e_m}{\operatorname{sh}\beta_m}-\sum_{m=1}^{\infty}\left[\left(\operatorname{ch}\beta_m+\frac{\beta_m}{\operatorname{sh}\beta_m}\right)+\mu\left(\operatorname{ch}\beta_m-\frac{\beta_m}{\operatorname{sh}\beta_m}\right)\right]\left(\frac{m\pi}{a}\right)^2\frac{d_m}{\operatorname{sh}\beta_m}-\frac{2}{ab}(K_1-K_2+K_3-K_4)=0, \tag{30}$$

$$\sum_{n=1}^{\infty}a_n\sin\frac{n\pi l_3}{b}+K_1+\frac{K_4-K_1}{b}l_3=0, \tag{31}$$

$$\sum_{n=1}^{\infty}b_n\sin\frac{n\pi l_4}{b}+K_2+\frac{K_3-K_2}{b}l_4=0, \tag{32}$$

$$\sum_{m=1}^{\infty}e_m\sin\frac{m\pi l_1}{a}+K_1+\frac{K_2-K_1}{a}l_1=0, \tag{33}$$

$$\sum_{m=1}^{\infty}d_m\sin\frac{m\pi l_2}{a}+K_4+\frac{K_3-K_4}{a}l_2=0\cdot \tag{34}$$

至此, 我们得到四组无穷联立方程(23)~(26)和8个单独方程(27)~(34)• 利用它们可解出未知量 a_n、b_n、e_m、d_m、K_1、K_2、K_3、K_4、P_1、P_2、P_3 和 P_4, 进而可计算板的挠度, 内矩分量, 内力分量和内应力分量•

4 算 例

当 $l_1=0$, $l_2=a$, $l_3=b$, $l_4=0$ 时, 所求问题为四个角点被支承的情况• 这里取 $\mu=0.3$,

$a/b = 1, x_0 = y_0 = 0.5a$, 我们给出自由边的挠度幅值如图 3 表 1•

表 1 自由边 $y = 0$ 挠度 (Pa^2/D) 和弯矩 $M_x(P)$

x/a \ M(W)	0.1	0.2	0.3	0.4	0.5
M_x	0.074 385	0.124 728	0.165 935	0.193 939	0.203 751
W_{y0}	0.007 194	0.013 579	0.018 598	0.021 810	0.022 958

下表给出本文解与文献[1]对应值的比较, 从中可看出两种方法所得结果相差不是很大•

表 2 自由边 $y = 0$ 中点挠度 (Pa^2/D) 和弯矩 $M_x(P)$

文献 \ 数值	[1]	本文
$M_x\left(\frac{a}{2}, 0\right)$	0.205 5	0.203 8
$W\left(\frac{a}{2}, 0\right)$	0.023 16	0.022 96

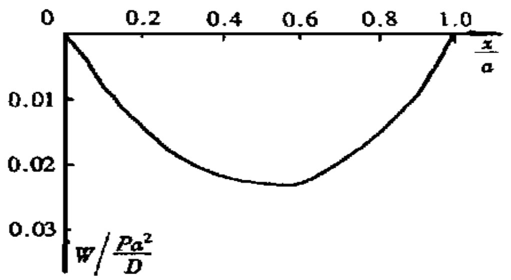

图 3 自由边 $y = 0$ 挠度曲线

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Bending of Rectangular Plate With Each Edges Arbitrary a Point Supported Under a Concentrated Load

Bian Yuhong
(Department of Civil Engineering and Mechanics, Yanshan University,
Qinhuangdao, Hebei 066004, P R China)

Abstract: The reciprocal theorem was applied to solve the bending of the rectangular plates with each edges arbitrary a point supported under a concentrated load, the exact solutions and computation example are given.

Key words: the reciprocal theorem; concentrated load; exact solution