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双摆内共振分岔分析*

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摘要: 应用 normal form 理论, 首先分析了复摆自治系统在 1:1 内共振临界点附近的 Hopf 分岔解及其在参数平面上的分岔转迁集的解析表达式, 并与数值解进行了比较; 然后, 应用数值方法, 得到了复摆非自治系统通向混沌的过程

关键词: 内共振; Hopf 分岔; 转迁集; 混沌

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引 言

非线性动力系统在复合临界点附近具有复杂的动力学特性^[1], 可以根据系统在临界点的 Jacobian 矩阵的结构将其分为余维一、余维二等等^[2]. 在一般的两对纯虚根特征值余维二系统, 不仅会产生静态分岔, 当系统的特征指数穿越 1 时^[3,4], 会有 Hopf 分岔发生^[5]. 许多工程及物理系统在复合临界点附近都会存在复杂的动力学行为^[6,7], 而双摆则是其中比较简单的系统之一. 已有许多关于其临界点附近的分岔和稳定性方面的研究报道. Mandadi 和 Huseyin^[8,9]分析了其双零及一对纯虚根的临界情形, 并用摄动方法得到了其近似解, 但他们的工作仅限于静态分岔. Scheidel^[10]等人用平均法研究了其在单零及一对纯虚根复合临界点附近的静态分岔及 Hopf 分岔. Samaranyake 和 Bajaj^[11]给出了其两对纯虚根临界点附近在共振及非共振情形下的动力学特性. Yu 和 Bi 给出了维三的 Norm Form, 进一步指出了导致 2_D 及 3_D 环面的分岔过程^[12]. 这些关于双摆的成果大都局限在动力学特性的理论研究方面, 而数值模拟则不多. 本文应用 Normal Form 理论, 首先分析了复摆自治系统在 1:1 内共振临界点附近的 Hopf 分岔解及其在参数平面上的分岔转迁集的解析表达式, 并与数值解进行了比较; 然后, 应用数值方法, 得到了复摆非自治系统通向混沌的过程.

1 1:1 内共振分岔分析

考虑文[8]中的双摆系统, 其三阶运动方程为

$$\frac{dz_1}{d\tau} = z_2, \quad (1a)$$

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$$\frac{dz_2}{d\tau} = \left\{ -\frac{1}{2}f_1 - f_2 + \frac{1}{2}f_4 + f_5 \right\} z_1 - \left\{ \frac{1}{2}\eta_1 + \eta_2 \right\} z_2 + \left\{ f_2 - \frac{1}{2}f_4 - \frac{1}{2}f_5 \right\} z_3 + \eta_2 z_4 + \left\{ \frac{1}{4}f_1 + \frac{3}{4}f_2 - \frac{1}{3}f_4 - \frac{2}{3}f_5 \right\} z_1^3 - \left\{ \frac{1}{2}f_6 + f_7 \right\} z_2^3 + \left\{ -\frac{3}{4}f_2 - \frac{1}{2}f_3 + \frac{1}{3}f_4 + \frac{7}{12}f_5 \right\} z_3^3 + f_7 z_4^3 + \left\{ \frac{1}{4}\eta_1 + \frac{3}{4}\eta_2 \right\} z_1^2 z_2 - \left\{ \frac{1}{2}f_1 + \frac{9}{4}f_2 - \frac{1}{2}f_3 - f_4 - \frac{3}{2}f_5 \right\} z_1^2 z_3 - \frac{3}{4}\eta_2 z_1^2 z_4 - \frac{1}{2}z_1 z_2^2 + \frac{1}{2}z_2^2 z_3 + 3f_7 z_2 z_4 + \left\{ \frac{1}{4}f_1 + \frac{9}{4}f_2 - f_4 - \frac{3}{2}f_5 \right\} z_1 z_3^2 + \left\{ \frac{1}{4}\eta_1 + \frac{3}{4}\eta_2 \right\} z_2 z_3^2 - \frac{3}{4}\eta_2 z_3 z_4 - \frac{1}{2}z_1 z_4^2 - 3f_7 z_2 z_4^2 + \frac{1}{2}z_3 z_4^2 - \left\{ \frac{1}{2}\eta_1 + \eta_2 \right\} z_1 z_2 z_3 + \frac{3}{2}\eta_2 z_1 z_3 z_4, \quad (1b)$$

$$\frac{dz_3}{d\tau} = z_4, \quad (1c)$$

$$\frac{dz_4}{d\tau} = \left\{ \frac{1}{2}f_1 + 2f_2 - f_3 - \frac{1}{2}f_4 - f_5 \right\} z_1 - \left\{ \frac{1}{2}\eta_1 + 2\eta_2 \right\} z_2 + \left\{ -2f_2 - f_3 + \frac{1}{2}f_4 + \frac{3}{2}f_5 \right\} z_3 - 2\eta_2 z_4 + \left\{ \frac{1}{2}f_1 + \frac{5}{4}f_2 - \frac{1}{6}f_3 - \frac{7}{12}f_4 - \frac{7}{6}f_5 \right\} z_1^3 + \left\{ \frac{1}{2}f_6 + 2f_7 \right\} z_2^3 + \left\{ \frac{5}{4}f_2 + \frac{7}{6}f_3 - \frac{7}{12}f_4 - f_5 \right\} z_3^3 - 2f_7 z_4^3 - \left\{ \frac{1}{2}\eta_1 + \frac{5}{4}\eta_2 \right\} z_1^2 z_2 + \left\{ f_1 + \frac{15}{4}f_2 - \frac{1}{2}f_3 - \frac{7}{4}f_4 - \frac{11}{4}f_5 \right\} z_1^2 z_3 + \frac{5}{4}\eta_2 z_1^2 z_4 + \frac{3}{2}z_1 z_2^2 - \frac{3}{2}z_2^2 z_3 - 6f_7 z_2 z_4 - \left\{ \frac{1}{2}f_1 + \frac{15}{4}f_2 - \frac{1}{2}f_3 - \frac{7}{4}f_4 - \frac{5}{2}f_5 \right\} z_1 z_3^2 - \left\{ \frac{1}{2}\eta_1 + \frac{5}{4}\eta_2 \right\} z_2 z_3^2 + \frac{5}{4}\eta_2 z_3 z_4 + \frac{1}{2}z_1 z_4^2 + 6f_7 z_2 z_4^2 - \frac{1}{2}z_3 z_4^2 + \left\{ \eta_1 + \frac{5}{2}\eta_2 \right\} z_1 z_2 z_3 - \frac{5}{2}\eta_2 z_1 z_3 z_4, \quad (1d)$$

其中 η_1, η_2 为量纲为一阻尼, f_4, f_5 为外加作用力. 显然, 当

$$\left. \begin{aligned} f_1 = 10, \quad f_2 = 2, \quad f_3 = 5, \quad f_4 = -6, \\ f_5 = 8, \quad f_6 = -10, \quad f_7 = 30, \quad \eta_1 = \eta_2 = 0, \end{aligned} \right\} \quad (2)$$

系统(1)具有两结纯虚根 $\lambda_{1,2} = \pm i$, 选 η_1 和 η_2 为分岔参数, 令 $\eta_1 = \mu_1, \eta_2 = \mu_2$ 引入线性变换

$$\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \quad (3)$$

有

$$\left. \begin{aligned} \frac{dx_1}{d\tau} &= x_2 + x_3 - \mu_2 x_4 - \frac{8}{3}x_3^2 + 10x_3^3 - 120x_4^3 - \\ &\quad x_1^2 x_3 + 11x_2^2 x_3 - 16x_2 x_3^2 - x_3 x_4^2 + 2x_1 x_3 x_4, \\ \frac{dx_2}{d\tau} &= -x_1 + x_4, \\ \frac{dx_3}{d\tau} &= x_4, \\ \frac{dx_4}{d\tau} &= -x_3 + \frac{1}{2}\mu_1 x_1 - \frac{1}{2}\mu_1 x_4 - 3\mu_2 x_4 - 5x_1^3 - \frac{10}{3}x_2^3 + \frac{62}{3}x_3^3 - 355x_4^3 - \\ &\quad 3x_1^2 x_3 + 15x_1^2 x_4 + 17x_2^2 x_3 - 28x_2 x_3^2 - 15x_1 x_4^2 - 3x_3 x_4^2 + 2x_1 x_3 x_4, \end{aligned} \right\} \quad (4)$$

其在临界点 $\mu_1 = \mu_2 = 0$ 处 Jacobian 矩阵为

$$J = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad (5)$$

引入附录中的非线性变换 $x_i = y_i + g_i(y_j)$ 及极坐标变换

$$y_1 = \rho_1 \cos \theta_1, \quad y_2 = \rho_1 \sin \theta_1, \quad y_3 = \rho_2 \cos \theta_2, \quad y_4 = \rho_2 \sin \theta_2, \quad (6)$$

可得其规范型

$$\left. \begin{aligned} \frac{d\rho_1}{d\tau} &= \rho_2 \cos(\theta_2 - \theta_1), \\ \frac{d\rho_2}{d\tau} &= -\frac{1}{4} \mu_1 \rho_2 - \frac{3}{2} \mu_2 \rho_2 - \frac{1}{4} \mu_1 \rho_1 \sin(\theta_2 - \theta_1) + 15\rho_1^2 \rho_2 + 30\rho_1 \rho_2^2 \sin(\theta_2 - \theta_1) - 80\rho_1 \rho_2^2 \cos(\theta_2 - \theta_1) + \frac{15}{2} \rho_1^3 \sin(\theta_2 - \theta_1) - 5\rho_1^3 \cos(\theta_2 - \theta_1) + 6\rho_1^2 \rho_2 \sin 2(\theta_2 - \theta_1) - \frac{15}{2} \rho_1^2 \rho_2 \cos 2(\theta_2 - \theta_1) \\ \rho_1 \frac{d\theta_1}{d\tau} &= \rho_1 + \rho_2 \sin(\theta_2 - \theta_1), \\ \rho_2 \frac{d\theta_2}{d\tau} &= \rho_2 - \frac{1}{4} \mu_1 \rho_1 \cos(\theta_2 - \theta_1) - 22\rho_1^2 \rho_2 + 80\rho_1 \rho_2^2 \sin(\theta_2 - \theta_1) + 30\rho_1 \rho_2^2 \cos(\theta_2 - \theta_1) + \frac{15}{2} \rho_1^3 \cos(\theta_2 - \theta_1) + 5\rho_1^3 \sin(\theta_2 - \theta_1) + \frac{15}{2} \rho_1^2 \rho_2 \sin 2(\theta_2 - \theta_1) + 6\rho_1^2 \rho_2 \cos 2(\theta_2 - \theta_1). \end{aligned} \right\} \quad (7)$$

(7) 中四个状态变量中只有三个是独立的。令 $\phi = \theta_2 - \theta_1$, 有

$$\left. \begin{aligned} \frac{d\rho_1}{d\tau} &= \rho_2 \cos \phi, \\ \frac{d\rho_2}{d\tau} &= -\frac{1}{4} \mu_1 \rho_2 - \frac{3}{2} \mu_2 \rho_2 - \frac{1}{4} \mu_1 \rho_1 \sin \phi + 15\rho_1^2 \rho_2 + 30\rho_1 \rho_2^2 \sin \phi - 80\rho_1 \rho_2^2 \cos \phi + \frac{15}{2} \rho_1^3 \sin \phi - 5\rho_1^3 \cos \phi + 6\rho_1^2 \rho_2 \sin 2\phi - \frac{15}{2} \rho_1^2 \rho_2 \cos 2\phi, \\ \frac{d\phi}{d\tau} &= -\frac{1}{4} \mu_1 \frac{\rho_1}{\rho_2} \cos \phi - \frac{\rho_2}{\rho_1} \sin \phi - 22\rho_1^2 + \frac{15}{2} \rho_1^2 \sin 2\phi - 10\rho_1^2 \cos 2\phi + 30\rho_1 \rho_2 \cos \phi + 80\rho_1 \rho_2 \sin \phi + \frac{15}{2} \frac{\rho_1^3}{\rho_2} \cos \phi + 5 \frac{\rho_1^3}{\rho_2} \sin \phi \end{aligned} \right\} \quad (8)$$

注意到, 虽然零解(E. S.) 可由(8) 或(7) 得到, 但其稳定性条件须从(4) 中得到。其零解的特征方程为

$$P(\lambda) = \lambda^4 + \left[\frac{1}{2} \mu_1 + 3\mu_2 \right] \lambda^3 + \left[2 + \frac{1}{2} \mu_1 \mu_2 \right] \lambda^2 + \left[3\mu_2 - \frac{1}{2} \mu_1 \right] \lambda + 1. \quad (9)$$

由 Hurwitz 判据, 注意到 $\mu_1 \geq 0, \mu_2 \geq 0$, 其零解的稳定条件为

$$\mu_2 \mu_1^2 + 8\mu_1 - 36\mu_2^3 < 0 \quad (10)$$

即

$$0 \leq \mu_1 < \frac{4}{\mu_2} \left[-1 + \sqrt{1 + \frac{9}{4} \mu_2^4} \right] \approx \frac{9}{2} \mu_2^3. \quad (11)$$

因此当 $0 \leq \mu_1 < \frac{9}{2} \mu_2^3$ 时, 零解 $x_i = 0$ 是稳定的. 当参数穿越临界边界

$$L: \mu_1 = \frac{9}{2} \mu_2^3 \tag{12}$$

时, 零解失稳, 产生锁相周期解. 分岔图参见图 1.

令 $\frac{d\rho_1}{d\tau} = \frac{d\rho_2}{d\tau} = 0$, 可得锁相周期解满足的代数方程. 由 (8) 中第一式可知 $\cos \phi = 0$, (因为 $\rho_2 \neq 0$,

从而有两个解 $\phi = \pm \frac{1}{2} \pi$. 由 $\phi = \frac{1}{2} \pi$ 可得,

$$\left. \begin{aligned} \mu_1 \rho_1 + (\mu_1 + 6\mu_2) \rho_2 - 30\rho_1^3 - \\ 90\rho_1^2 \rho_2 - 120\rho_1 \rho_2^2 = 0, \\ \rho_2^2 - 5\rho_1^4 + 28\rho_1^3 \rho_2 - 80\rho_1^2 \rho_2^2 = 0, \end{aligned} \right\} \tag{13}$$

消去 ρ_2 并取三阶近似(用 MAPLE 语言) 得

$$\left. \begin{aligned} 16\ 862\ 400\rho_1^8 - 224\ 100\rho_1^6 + 900\rho_1^4 - \\ 60\ \mu_1 \rho_1^2 + \mu_1^2 = 0, \end{aligned} \right\} \tag{14}$$

从而 ρ_2 为

$$\rho_2 = - \frac{\rho_1(1\ 800\rho_1^4 - 30\rho_1^2 + \mu_1)}{10\ 560\rho_1^3 - 90\rho_1^2 + \mu_1 + 6\ \mu_2} \tag{15}$$

对给定参数 μ_1, μ_2 , 从 (15) 和 (14) 中可求出所有的周期锁相解. 对 $\phi = -\frac{1}{2} \pi$ 的情形可同样处理.

2 数值分析

在参数 μ_1, μ_2 空间零解(E.S.) 的稳定域内取一点 $(\mu_1, \mu_2) = (0.03, 0.2)$ (图 1 中的黑点), 图 2 给出了其数值结果. 从图中可知, 从初始值 $(z_1, z_2, z_3, z_4) = (0.06, 0.06, 0.06, 0.06)$ 出发的轨迹收敛到原点 _ 零解(E.S.).

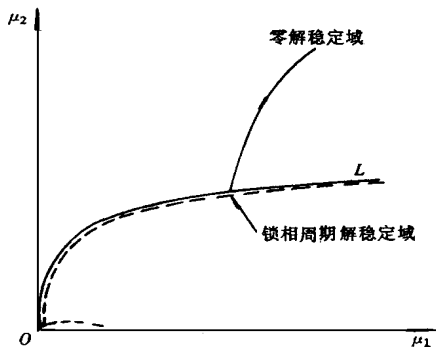


图 1 分岔图

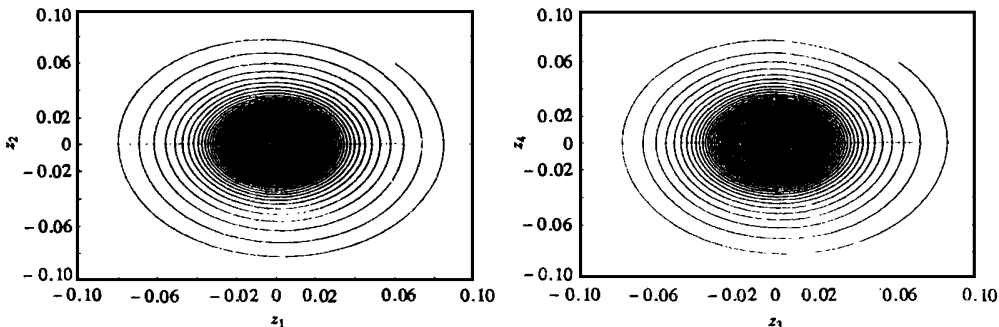


图 2 $(\mu_1, \mu_2) = (0.03, 0.2)$ 的相图

如取 $(\mu_1, \mu_2) = (0.2, 0.2)$, 即在周期锁相解的稳定域中取点, 参见图 1. 从上面的分析可得其近似解为

$$\left. \begin{aligned} \rho_1 = 0.0606, \quad \rho_2 = -0.0057; \quad \rho_1 = 0.1112, \quad \rho_2 = -0.0061 & \left(\text{当 } \phi = +\frac{1}{2}\pi \right), \\ \rho_1 = 0.0606, \quad \rho_2 = 0.0057; \quad \rho_1 = 0.1112, \quad \rho_2 = 0.0061 & \left(\text{当 } \phi = -\frac{1}{2}\pi \right), \end{aligned} \right\} \quad (16)$$

由于 ρ_1, ρ_2 均为正数, 故取 $\phi = -\frac{1}{2}\pi$. 将(8)中的两解分别代入其相应的 Jacobian 矩阵中第一个解为稳定的, 而第二个解为不稳定的, 即·

$$\left. \begin{aligned} \text{解 } \rho_1 = 0.0606, \quad \rho_2 = 0.0057, \quad \phi = -\frac{1}{2}\pi & \text{ (稳定),} \\ \text{解 } \rho_1 = 0.1112, \quad \rho_2 = 0.0061, \quad \phi = -\frac{1}{2}\pi & \text{ (不稳定),} \end{aligned} \right\} \quad (17)$$

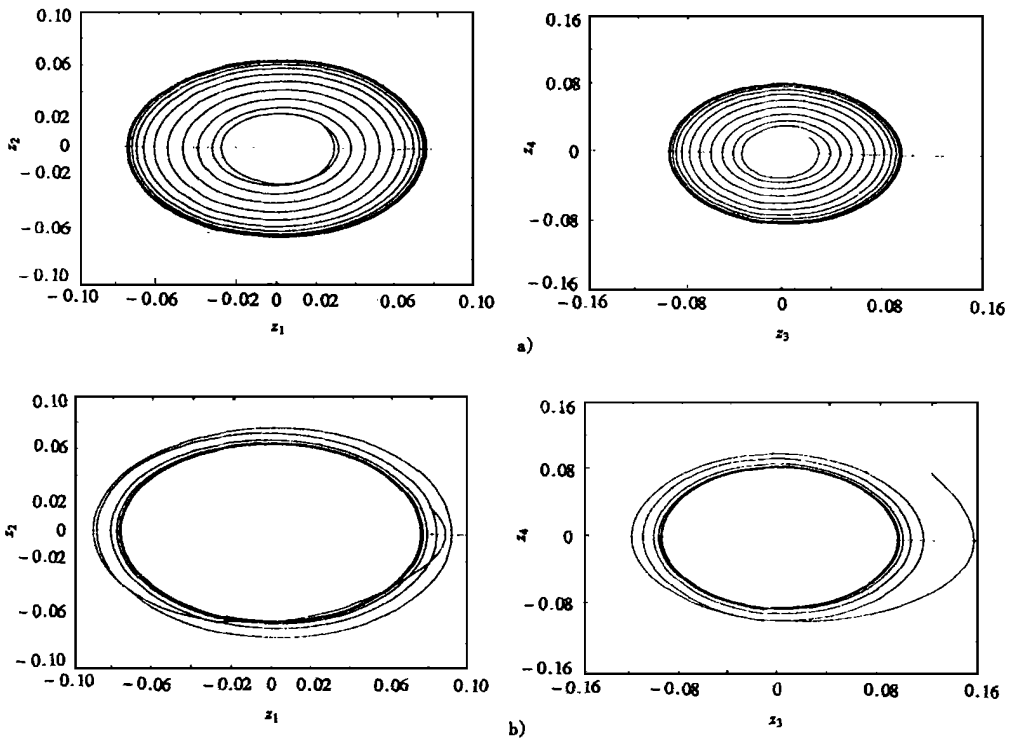


图3 锁相周期解

代回附录中的非线性变换和(4)中的线性变换可得

$$z_1 = 0.0038, \quad z_2 = -0.0573, \quad z_3 = 0.0108, \quad z_4 = -0.0697, \quad (18)$$

可得其解析近似解 $\sqrt{z_1^2 + z_2^2} = 0.0574$ 和 $\sqrt{z_3^2 + z_4^2} = 0.0705$ ·

图3中给出了其相应的数值解, 从而可得其数值解的振幅为

$$\sqrt{z_1^2 + z_2^2} = 0.065, \quad \sqrt{z_3^2 + z_4^2} = 0.080, \quad (19)$$

与解析近似解接近·

进一步的数值模拟显示, 在临界边界 L 和 μ_1 轴所围成的区域内, 两解中的一个总是不稳定的, 而另一个解除了在图一中虚线所围成的区域外, 都是稳定的·

3 混 沌

本节我们应用数值模拟, 研究系统(1)的混沌特性. 改 f_5 从常数为周期激励

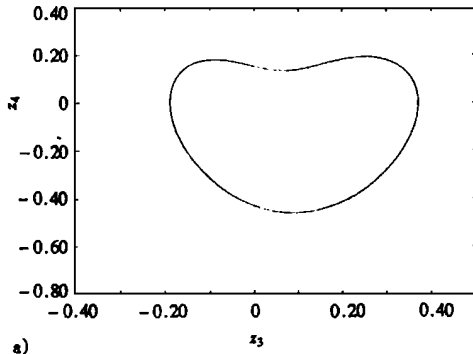
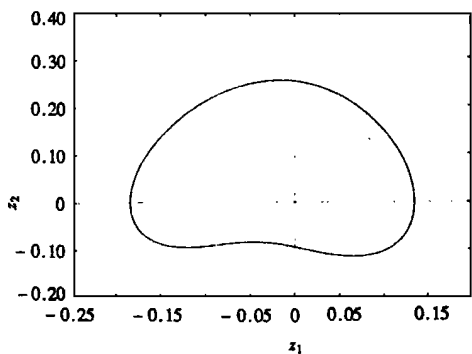
$$f_5 = 2 + \alpha \cos(\Omega \tau), \tag{20}$$

1) 现为一非自治系统. 取参数为

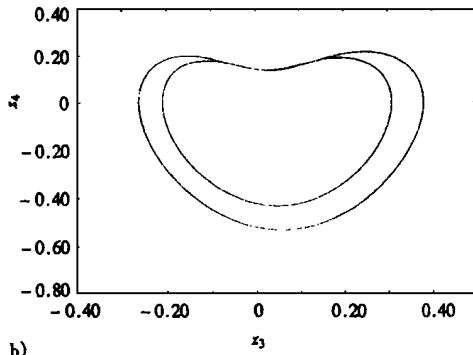
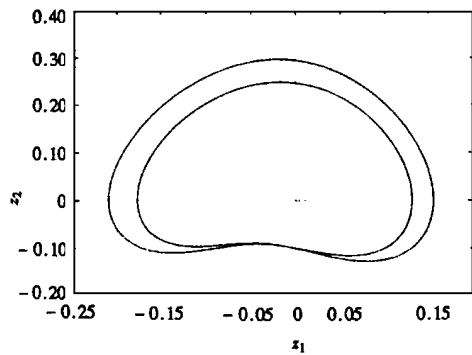
$$f_1 = 4, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 0, \quad f_6 = f_7 = 0, \quad \Gamma_1 = 0, \quad \Gamma_2 = 0, \tag{21}$$

系统在没有周期激励项时, 显然, 其 Jacobian 矩阵在临界点 $\mu_1 = \mu_2 = 0$ 处有两对重纯虚根 $\lambda_{1,2} = \lambda_{3,4} = \pm i$.

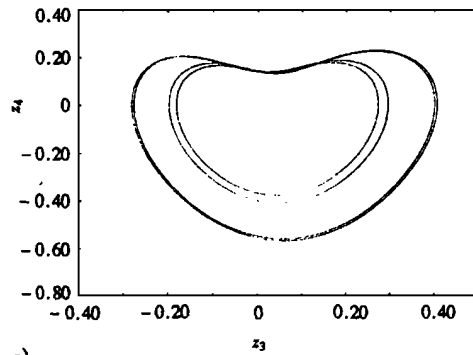
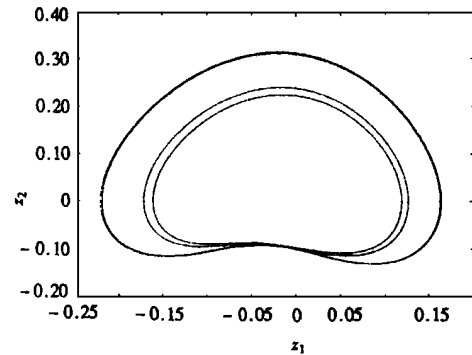
对(1)进行数值积分, 发现其会通过周期倍化分岔——一种典型混沌道路通向混沌. 我们取 Ω 接近 1(文中取 1.05), 从 1 开始改变激励幅值 α , 有周期倍化分岔发生. 当 $\alpha < 1.08$ 时, 有



a)



b)



c)

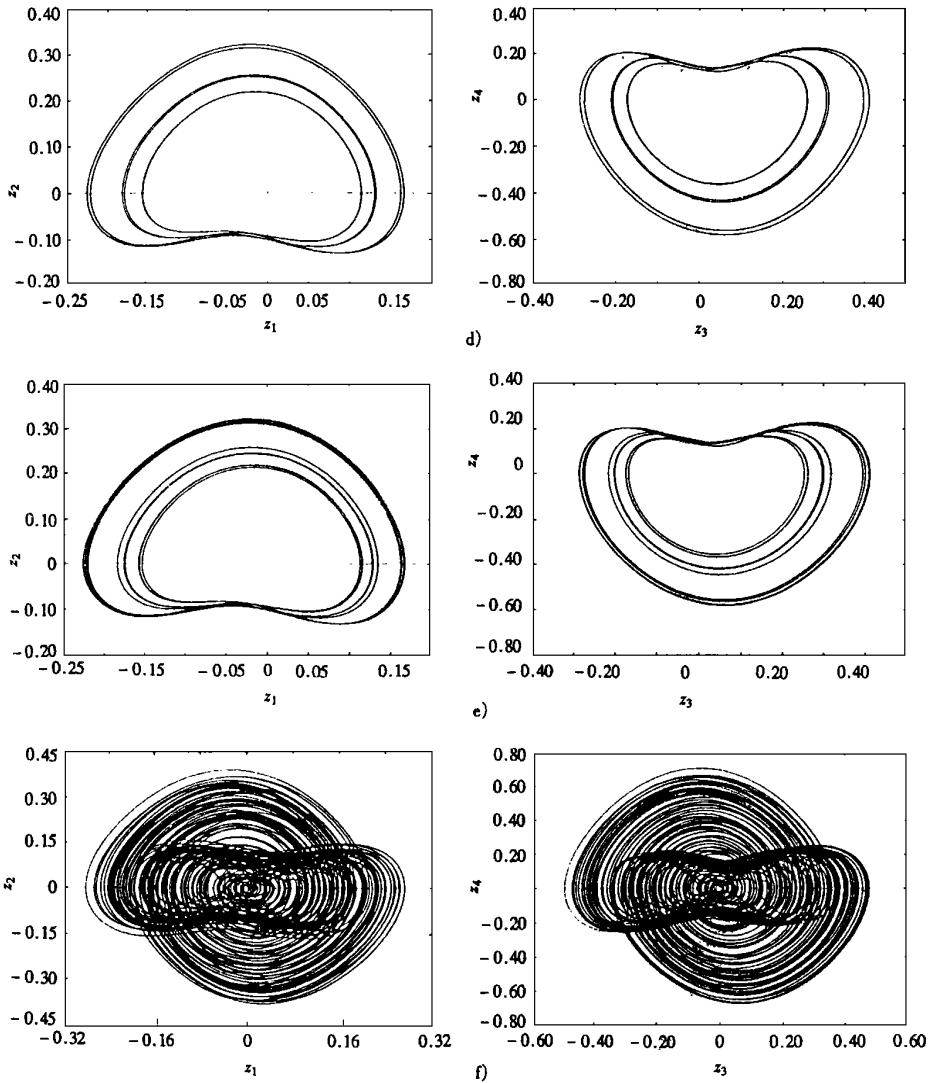


图4 周期倍化分岔

周期一解(见图4(a), $\alpha = 1.05$)。当 $\alpha = 1.08$ 时, 周期一解失稳, 分岔为周期二解(见图4(b), $\alpha = 1.08$), 该周期二解稳定直到 $\alpha = 1.095$, 产生周期四解(见图4(c))。当 $\alpha = 1.0989$ 时为周期八解(见图4(d)), $\alpha = 1.0999$ 时为周期十六解(见图4(e)), 当 $\alpha = 1.109$ 时, 解呈混沌状(见图4(f))。

应该注意到, 图4中的轨迹是四维空间分别在 (z_1, z_2) 和 (z_3, z_4) 子空间中的投影。同时为描述清晰, 暂态过程均已略去。另外由于(1)存在对称性(即以 $-z_i$ 代替 z_i , 方程形式不变), 从对称的初始条件出发, 可得关于原点对称的周期倍化分岔过程。从图4(e)中可以看出, 其混沌解也是关于原点对称的并局限在一定的范围内。周期解先在各自对称的轨道上运动, 并周期倍化, 到一定分岔程度后, 混沌解将轮流在两对称的轨道上运动, 即将其连接一起。

4 结 论

本文研究了双摆在系统在 1:1 内共振时的分岔特性。当没有周期激励时,参数的改变会影响其解的稳定性,在参数平面上存在一临界转迁集,将解的按区域分类。当有周期激励时,存在周期倍化通向混沌的过程。周期解先在各自对称的轨道上运动,并周期倍化,当分岔到一定程度后,混沌解将轮流在两对称的轨道上运动,即将其连接一起。

附录 非线性函数表达式

$$x_1 = y_1 - \frac{1}{16}\mu_1 y_2 + \frac{5}{8}\mu_2 y_3 + \frac{255}{16}y_1^3 + \frac{205}{32}y_2^3 + \frac{205}{32}y_3^3 - \frac{7693}{384}y_4^3 + \frac{105}{32}y_1^2 y_2 - \frac{16785}{64}y_1^2 y_3 - \frac{1177}{32}y_1^2 y_4 + \frac{243}{16}y_1 y_2^2 + \frac{16785}{64}y_2^2 y_3 + \frac{345}{32}y_2^2 y_4 - \frac{1623}{128}y_1 y_2^3 + \frac{45}{256}y_2 y_2^3 + \frac{1613}{128}y_3^2 y_4 + \frac{4503}{128}y_1 y_4^2 - \frac{1005}{256}y_2 y_4^2 + \frac{12495}{128}y_3 y_4^2 + \frac{615}{16}y_1 y_2 y_3 - \frac{16815}{32}y_1 y_2 y_4 + \frac{45}{128}y_1 y_3 y_4 - \frac{1321}{64}y_2 y_3 y_4,$$

$$x_2 = y_2 - \frac{1}{16}\mu_1 y_1 - \frac{5}{8}\mu_2 y_4 - \frac{45}{32}y_1^3 + \frac{159}{16}y_2^3 - \frac{1681}{128}y_3^3 - \frac{27045}{128}y_4^3 + \frac{115}{16}y_1^2 y_2 - \frac{985}{32}y_1^2 y_3 + \frac{17265}{64}y_1^2 y_4 - \frac{105}{32}y_1 y_2^2 + \frac{153}{32}y_2^2 y_3 - \frac{17265}{64}y_2^2 y_4 - \frac{915}{256}y_1 y_2^3 - \frac{3433}{128}y_2 y_2^3 - \frac{32625}{128}y_3^2 y_4 - \frac{45}{256}y_1 y_4^2 + \frac{553}{128}y_2 y_4^2 - \frac{1037}{128}y_3 y_4^2 - \frac{17295}{32}y_1 y_2 y_3 - \frac{391}{16}y_1 y_2 y_4 + \frac{2391}{64}y_1 y_3 y_4 - \frac{45}{128}y_2 y_3 y_4,$$

$$x_3 = y_3 - \frac{1}{8}\mu_1 y_1 - \left(\frac{1}{16}\mu_1 + \frac{5}{4}\mu_2 \right) y_4 + \frac{15}{8}y_1^3 - \frac{21}{4}y_2^3 - \frac{1691}{128}y_3^3 - \frac{80085}{256}y_4^3 - \frac{41}{4}y_1^2 y_2 + \frac{299}{16}y_1^2 y_3 + \frac{315}{32}y_1^2 y_4 + \frac{75}{8}y_1 y_2^2 - \frac{123}{16}y_2^2 y_3 + \frac{405}{32}y_2^2 y_4 - \frac{33765}{63}y_1 y_2^3 + \frac{117}{32}y_2 y_2^3 - \frac{217665}{256}y_3^2 y_4 - \frac{33915}{64}y_1 y_4^2 + \frac{363}{32}y_2 y_4^2 - \frac{10095}{128}y_3 y_4^2 - \frac{165}{16}y_1 y_2 y_3 + \frac{501}{8}y_1 y_2 y_4 + \frac{1237}{16}y_1 y_3 y_4 - \frac{315}{32}y_2 y_3 y_4,$$

$$x_4 = y_4 + \frac{1}{8}\mu_1 y_2 - \left(\frac{1}{16}\mu_1 + \frac{1}{4}\mu_2 \right) y_3 + \frac{23}{12}y_1^3 - \frac{25}{8}y_2^3 - \frac{10795}{256}y_3^3 - \frac{2897}{384}y_4^3 - \frac{45}{8}y_1^2 y_2 - \frac{405}{32}y_1^2 y_3 + \frac{101}{16}y_1^2 y_4 + \frac{1}{4}y_1 y_2^2 + \frac{165}{32}y_2^2 y_3 + \frac{683}{16}y_2^2 y_4 - \frac{1227}{32}y_1 y_2^3 - \frac{34395}{64}y_2 y_2^3 + \frac{3217}{128}y_3^2 y_4 - \frac{917}{32}y_1 y_4^2 - \frac{34245}{64}y_2 y_4^2 - \frac{56895}{256}y_3 y_4^2 + \frac{181}{8}y_1 y_2 y_3 - \frac{315}{16}y_1 y_2 y_4 + \frac{165}{32}y_1 y_3 y_4 + \frac{363}{16}y_2 y_3 y_4.$$

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Bifurcation Analysis of a Double Pendulum With Internal Resonance

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Abstract: By employing the normal form theory, the Hopf bifurcation and the transition boundary of an autonomous double pendulum with 1:1 internal resonance at the critical point is studied. The results are compared with numerical solutions. Further, by numerical methods, the road to chaos of a non-autonomous system is presented in the end.

Key words: internal resonance; Hopf bifurcation; transition boundary; chaos