应用数学和力学, 第 21 卷 第 2 期(2000 年 2 月)
Applied Mathematics and Mechanics

应用数学和力学编委会编
重 庆 出 版 社 出 版

文章编号: 1000_0887(2000) 02_0201_08

伴有边界摄动二阶非线性系统的奇摄动*

陈育森

(福建师大 福清分校 数学系, 福建 福清 350300)

(林宗池推荐)

摘要: 讨论了伴有边界摄动的含积分算子的二阶非线性微分方程组边值问题的奇摄动• 在适当的假设条件下, 通过对角化技巧, 证明了解的存在, 并估计了余项•

关 键 词: 奇摄动; 边界摄动; 对角化
中图分类号: O175.1 文献标识码: A

研究含积分算子的二阶非线性系统

$$\varepsilon \boldsymbol{y}'' = \boldsymbol{f}(t, \boldsymbol{y}, \mathrm{T}^* \boldsymbol{y}, \varepsilon)\boldsymbol{y}' + \boldsymbol{g}(t, \boldsymbol{y}, \mathrm{T}^* \boldsymbol{y}, \varepsilon), \tag{1}$$

并伴有边界摄动:

$$\boldsymbol{y}(t, \varepsilon)\,|_{t=\varphi(\varepsilon)} = \alpha(\varepsilon), \quad \boldsymbol{y}(t, \varepsilon)\,|_{t=1+\phi(\varepsilon)} = \beta(\varepsilon) \tag{2}$$

的奇摄动, 其中 $\varepsilon > 0$ 是小参数, $\varphi(\varepsilon)$、$\phi(\varepsilon)$ 是关于 ε 充分光滑的数量函数, 具有性质:

$$\lim_{\varepsilon \to 0^+} \varphi(\varepsilon) = \varphi(0) = 0, \quad \lim_{\varepsilon \to 0^+} \phi(\varepsilon) = \phi(0) = 0,$$

$\boldsymbol{y}, \boldsymbol{g}, \alpha, \beta \in R^n$, $\boldsymbol{f}$ 是 $n \times n$ 阶矩阵函数,

$$\mathrm{T}^* \boldsymbol{y}(t) = \int_t^1 \boldsymbol{k}(t, s, \boldsymbol{y}(s, \varepsilon), \varepsilon)\,\mathrm{d}s, \qquad \boldsymbol{k} \in R^m\,•$$

积分微分方程组的奇摄动已有一些工作[1~4], 本文是研究伴有边界摄动的积分微分方程组的奇摄动• 通过对角化技巧, 利用逐步逼近法证明了解的存在, 并讨论了解的渐近性态•

首先假设

(Ⅰ) 退化问题

$$\left.\begin{aligned} &\boldsymbol{0} = \boldsymbol{f}(t, \boldsymbol{u}, \mathrm{T}^* \boldsymbol{u}, 0)\boldsymbol{u}' + \boldsymbol{g}(t, \boldsymbol{u}, \mathrm{T}^* \boldsymbol{u}, 0), \\ &\boldsymbol{u}(1) = \beta(0) \end{aligned}\right\} \tag{3}$$

有解 $\boldsymbol{u}(t) \in C^{N+2}[-\varepsilon_0, 1+\varepsilon_0]$ ($\varepsilon_0 > 0$ 是小常数) 使得 $\boldsymbol{f}(t, \boldsymbol{u}, \mathrm{T}^* \boldsymbol{u}, 0)$ 的每一个特征值 $\lambda(t)$ 有 $\mathrm{Re}\,\lambda_i(t) \leqslant -8\mu_0 < 0$, $i = 1, 2, 3, \cdots, n$• 同时, 对于满足 $0 < |\theta| \leqslant |\boldsymbol{y}(0, 0) - \boldsymbol{u}(0)|$ 的所有 $\theta + \boldsymbol{u}(0)$ 内积

$$\theta^{\mathrm{T}} \int_0^{\theta} \boldsymbol{f}(0, \boldsymbol{u}(0) + s, \mathrm{T}^* \boldsymbol{u}(0), 0)\,\mathrm{d}s < 0\,•$$

(Ⅱ) $\boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{z}, \varepsilon)$, $\boldsymbol{g}(t, \boldsymbol{y}, \boldsymbol{z}, \varepsilon)$ 在曲线 L 的某个 δ_管中关于 $(t, \boldsymbol{y}, \boldsymbol{z}, \varepsilon)$ 具有直到包括 $(N$

* 收稿日期: 1998_09_01
作者简介: 陈育森(1946~), 男, 福建福清人, 副教授, 系主任, 研究方向为非线性系统奇异摄动理论和应用.

$+2)$ 阶在内的连续偏导数, 其中曲线

$$L=\{(t,\boldsymbol{y},z,\varepsilon):t=0,\boldsymbol{y}=\boldsymbol{u}(0)+\pi_0\boldsymbol{y}(\tau),\tau\geqslant 0,z=\mathrm{T}^*\boldsymbol{u}(0),\varepsilon=0\}\cup\{(t,\boldsymbol{y},z,\varepsilon):-\varepsilon_0\leqslant t\leqslant 1+\varepsilon_0,\boldsymbol{y}=\boldsymbol{u}(t),z=\mathrm{T}^*\boldsymbol{u}(t),\varepsilon=0\}.$$

$\pi_0\boldsymbol{y}(\tau),\tau\geqslant 0$ 将在下文给出.

(Ⅲ) $\boldsymbol{k}(t,s,\boldsymbol{y}(s,\varepsilon),\varepsilon)$ 在曲线

$$L^*=\{(t,s,\boldsymbol{y},\varepsilon):-\varepsilon_0\leqslant t\leqslant s\leqslant 1+\varepsilon_0,\boldsymbol{y}=\boldsymbol{u}(s),\varepsilon=0\}$$

的某个 δ_管中, 对其所有变量具有直到包括 $(N+1)$ 阶在内的连续偏导数.

(Ⅳ) $\alpha(\varepsilon)=\alpha_0+\sum_{i=1}^{\infty}\alpha_i\varepsilon^i,\ \beta(\varepsilon)=\beta_0+\sum_{i=1}^{\infty}\beta_i\varepsilon^i.$

此外, 对向量函数或矩阵函数 $\boldsymbol{X}(t)=[x_{ij}(t)]\in C[0,1]$, 规定

$$|\boldsymbol{X}(t)|=\Big[\sum_{i,j}x_{ij}^2(t)\Big]^{1/2},\quad \|\boldsymbol{X}(t)\|=\max_{0\leqslant t\leqslant 1}|\boldsymbol{X}(t)|.$$

并将曲线的 δ_管理解为相应空间中所有与曲线的距离在上述模意义下不超过δ的点的集合.

令问题(1)~(2)的外解 $\boldsymbol{y}(t,\varepsilon)$ 有如下形式的展开式:

$$\boldsymbol{y}(t,\varepsilon)\sim\boldsymbol{y}_0(t)+\sum_{i=1}^{\infty}\boldsymbol{y}_i(t)\varepsilon^i.\tag{4}$$

将(4)代入(1), 得到展开式(4)各项系数 $\boldsymbol{y}_i(t)$ 的方程组

$$\boldsymbol{0}=\boldsymbol{f}(t,\boldsymbol{y}_0,\mathrm{T}^*\boldsymbol{y}_0,0)\boldsymbol{y}_0'+\boldsymbol{g}(t,\boldsymbol{y}_0,\mathrm{T}^*\boldsymbol{y}_0,0),\tag{5}$$

$$\boldsymbol{f}(t,\boldsymbol{y}_0,\mathrm{T}^*\boldsymbol{y}_0,0)\boldsymbol{y}_i'+\boldsymbol{F}_{0y}(t)\boldsymbol{y}_i+\boldsymbol{F}_{0z}\mathrm{T}_i^*\boldsymbol{y}_i+\varphi_{i-1}(t)=\boldsymbol{y}_{i-1}''(t),\tag{6}$$

其中

$$\boldsymbol{F}_{0y}(t)=\frac{\partial}{\partial\boldsymbol{y}}[\boldsymbol{f}(t,\boldsymbol{y}_0,\mathrm{T}^*\boldsymbol{y}_0,0)\boldsymbol{y}_0'+\boldsymbol{g}(t,\boldsymbol{y}_0,\mathrm{T}^*\boldsymbol{y}_0,0)],$$

$$\boldsymbol{F}_{0z}(t)=\frac{\partial}{\partial\boldsymbol{z}}[\boldsymbol{f}(t,\boldsymbol{y}_0,\mathrm{T}^*\boldsymbol{y}_0,0)\boldsymbol{y}_0'+\boldsymbol{g}(t,\boldsymbol{y}_0,\mathrm{T}^*\boldsymbol{y}_0,0)],$$

$$\mathrm{T}_0^*\boldsymbol{y}_0(t)=\mathrm{T}^*\boldsymbol{y}_0(t)=\int_t^1\boldsymbol{k}(t,s,\boldsymbol{y}_0(s),0)\,\mathrm{d}s,$$

$$\mathrm{T}_i^*\boldsymbol{y}_i(t)=\int_t^1[\boldsymbol{k}_y(t,s,\boldsymbol{y}_0(s),0)\boldsymbol{y}_i(s)+R_{i-1}(t,s,\boldsymbol{y}_0(s),\boldsymbol{y}_1(s),\cdots,\boldsymbol{y}_{i-1}(s))]\,\mathrm{d}s,$$

$\varphi_{i-1}(t)$ 是 $t,\boldsymbol{y}_j(t)\,(0\leqslant j\leqslant i-1)$ 的已知函数.

为了确定方程(5)、(6)的解, 需要相应的定解条件. 为此, 令 $\boldsymbol{y}[1+\phi(\varepsilon),\varepsilon]=\beta(\varepsilon)$, 并将 $\boldsymbol{y}(1+\phi(\varepsilon),\varepsilon)$ 展开成 ε的幂级数:

$$\boldsymbol{y}[1+\phi(\varepsilon),\varepsilon]\sim\boldsymbol{y}_0(1)+\sum_{i=1}^{\infty}\varepsilon^i[\boldsymbol{y}_i(1)+\boldsymbol{e}_i],\tag{7}$$

其中 $\boldsymbol{e}_i$ 是由 $\boldsymbol{y}_j^{(r)}(1)\,(0\leqslant j\leqslant i-1,\ 1\leqslant j+r\leqslant i)$ 决定, 由(Ⅳ)及(7)得到:

$$\boldsymbol{y}_0(1)=\beta_0(=\beta(0)),\tag{8}$$

$$\boldsymbol{y}_i(1)=\beta_i-\boldsymbol{e}_i,\qquad(i=1,2,\cdots).\tag{9}$$

由(Ⅰ)知问题(5)、(8)的解存在且有 $\boldsymbol{y}_0(t)=\boldsymbol{u}(t),\ -\varepsilon_0\leqslant t\leqslant 1+\varepsilon_0$. 问题(6)、(9)是非奇异线性初值问题, 利用逐步逼近法, 不难验证, 其解存在且唯一. 故由 $\boldsymbol{y}_0(t)$ 可依次确定唯一的 $\boldsymbol{y}_i(t),\ -\varepsilon_0\leqslant t\leqslant 1+\varepsilon_0,\ i=1,2,\cdots$. 展开式(4)由 $\boldsymbol{y}_0(t)$ 就不难产生了.

一般说来, 外解 $\boldsymbol{y}(t,\varepsilon)$ 在 $t=\varphi(\varepsilon)$ 不满足边界条件, 因此我们在 $t=\varphi(\varepsilon)$ 近旁构造具

有边界层性质的函数 $\pi y(\tau,\varepsilon)$，其中 $\tau=(t-\varphi(\varepsilon))/\varepsilon$ 为伸长变量•

令：

$$y=y(t,\varepsilon)+\pi y(\tau,\varepsilon), \tag{10}$$

$$\pi y(\tau,\varepsilon)\sim\pi_0 y(\tau)+\sum_{i=1}^{\infty}\varepsilon^i\pi_i y(\tau)• \tag{11}$$

将(10)代入(1)、(2)并注意到 $y(t,\varepsilon)$ 满足(1)和(2)的第二式，得：

$$\begin{aligned}\frac{\mathrm{d}^2\pi y}{\mathrm{d}\tau^2}=\varepsilon\Big\{&f[t,y(t,\varepsilon)+\pi y(\tau,\varepsilon),T^*(y(t,\varepsilon)+\pi y(\tau,\varepsilon)),\varepsilon]\times\\&\left[y'(t,\varepsilon)+\frac{1}{\varepsilon}\frac{\mathrm{d}\pi y}{\mathrm{d}\tau}\right]+g[t,y(t,\varepsilon)+\pi y(\tau,\varepsilon),T^*(y(t,\varepsilon)+\\&\pi y(\tau,\varepsilon)),\varepsilon]-f[t,y(t,\varepsilon),T^*y(t,\varepsilon),\varepsilon]y'(t,\varepsilon)-\\&g[t,y(t,\varepsilon),T^*y(t,\varepsilon),\varepsilon]\Big\}_{t=\varepsilon\tau+\varphi(\varepsilon)},\end{aligned} \tag{12}$$

$$\pi y(0,\varepsilon)=\alpha(\varepsilon)-y(\varphi(\varepsilon),\varepsilon)• \tag{13}$$

将(11)代入(12)、(13)，并将其右边展成 ε 的幂级数，比较 ε 同次幂的系数并注意到 $\pi y(\tau)$ 仅在 $t=\varphi(\varepsilon)$ $(\tau=0)$ 附近存在，而在 $t=1+\psi(\varepsilon)$ $(\tau\to+\infty)$ 附近迅速趋于零，得：

$$\begin{cases}\dfrac{\mathrm{d}^2\pi_0 y}{\mathrm{d}\tau^2}=f(0,y_0(0)+\pi_0 y(\tau),T^*y_0(0),0)\dfrac{\mathrm{d}\pi_0 y}{\mathrm{d}\tau},\\ \pi_0 y(0)=\alpha_0-y_0(0),\ \pi_0 y(+\infty)=\mathbf{0};\end{cases} \tag{14}$$

$$\begin{cases}\dfrac{\mathrm{d}^2\pi_i y}{\mathrm{d}\tau^2}=f(0,y_0(0)+\pi_0 y(\tau),T^*y_0(0),0)\dfrac{\mathrm{d}\pi_i y}{\mathrm{d}\tau}+\\ \qquad\dfrac{\partial}{\partial y}\left[f(0,y_0(0)+\pi_0 y(\tau),T^*y_0(0),0)\dfrac{\mathrm{d}\pi_0 y}{\mathrm{d}\tau}\right]\pi_i y+E_{i-1}(\tau),\\ \pi_i y(0)=\alpha_i-y_i(0)-d_i,\ \pi_i y(+\infty)=\mathbf{0};\end{cases} \tag{15}$$

其中 $E_{i-1}(\tau)$ 是 $\pi_j y(\tau)$, $\mathrm{d}\pi_j y/\mathrm{d}\tau$, $\pi_j T^*y(\tau)$, $(0\leqslant j\leqslant i-1)$ 的不含常数项的多项式，该多项式的系数是以 τ 的有界函数为系数的 τ 的多项式，d_i 由 $y_j^{(r)}(0)$ $(0\leqslant j\leqslant i-1,\ 0\leqslant j+r\leqslant i)$ 决定，以及

$$\pi_0 T^*y(\tau)=\int_\tau^{+\infty}[k(0,0,y_0(0)+\pi_0 y(\rho),0)-k(0,0,y_0(0),0)]\mathrm{d}\rho,$$

$$\pi_i T^*y(\tau)=\int_\tau^{+\infty}[k_y(0,0,y_0(0)+\pi_0 y(\rho),0)\pi_i y(\rho)+s_{i-1}(\tau,\rho)]\mathrm{d}\rho,$$

s_{i-1} 是 $\pi_j y(\rho)(j=1,2,\cdots,i-1)$ 的不含常数项的多项式，该多项式的系数是以 ρ 的有界函数为系数的 τ、ρ 的多项式•

由(Ⅰ)并参阅文[5]可知问题(14)存在满足 $\pi_0 y(+\infty)=0$ 的指数型衰减解 $\pi_0 y(\tau)$, $\tau\geqslant 0$，从而由问题(15)可依次确定满足 $\pi_i y(+\infty)=0$ 的指数型衰减解 $\pi_i y(\tau)$, $\tau\geqslant 0$, $i=1,2,\cdots$ 于是得到问题(1)～(2)的形式渐近解•

$$\sum_{i=0}^{\infty}\left[y_i(t)+\pi_i y\left(\frac{t-\varphi(\varepsilon)}{\varepsilon}\right)\right]\varepsilon^i• \tag{16}$$

下面将证明形式渐近展开式(16)关于 $\varphi(\varepsilon)\leqslant t\leqslant 1+\psi(\varepsilon)$, $0<\varepsilon\leqslant\varepsilon_0$ 是一致有效的•

令 $y_N(t,\varepsilon)=\sum_{i=0}^{N}[y_i(t)+\pi_i y(\tau)]\varepsilon^i$•

定理　如果(Ⅰ)～(Ⅳ)成立，那么存在 $\varepsilon_0>0$，当 $0<\varepsilon\leqslant\varepsilon_0$ 时，问题(1)～(2)存在解

$\boldsymbol{y}(t,\varepsilon)$ 且满足 $\boldsymbol{y}(t,\varepsilon)=\boldsymbol{y}_N(t,\varepsilon)+O(\varepsilon^{N+1})$, $\varphi(\varepsilon)\leqslant t\leqslant 1+\psi(\varepsilon)$•

证明 令

$$\boldsymbol{U}=\boldsymbol{y}-\boldsymbol{y}_N,\ \boldsymbol{R}=\mathrm{T}^*\boldsymbol{y}-\mathrm{T}^*\boldsymbol{y}_N, \tag{17}$$

将(17)代入(1)、(2)得到:

$$\boldsymbol{R}'=-[\boldsymbol{k}(t,t,\boldsymbol{U}+\boldsymbol{y}_N,\varepsilon)-\boldsymbol{k}(t,t,\boldsymbol{y}_N,\varepsilon)]+\int_t^1[\boldsymbol{k}_t(t,s,\boldsymbol{U}+\boldsymbol{y}_N,\varepsilon)-\boldsymbol{k}_t(t,s,\boldsymbol{y}_N,\varepsilon)]\,\mathrm{d}s, \tag{18}$$

$$\varepsilon\boldsymbol{U}''=\boldsymbol{f}(t,\boldsymbol{U}+\boldsymbol{y}_N,\boldsymbol{R}+\mathrm{T}^*\boldsymbol{y}_N,\varepsilon)(\boldsymbol{U}'+\boldsymbol{y}_N')+\boldsymbol{g}(t,\boldsymbol{U}+\boldsymbol{y}_N,\boldsymbol{R}+\mathrm{T}^*\boldsymbol{y}_N,\varepsilon)-\boldsymbol{f}(t,\boldsymbol{y}_N,\mathrm{T}^*\boldsymbol{y}_N,\varepsilon)\boldsymbol{y}_N'-\boldsymbol{g}(t,\boldsymbol{y}_N,\mathrm{T}^*\boldsymbol{y}_N,\varepsilon)+O(\varepsilon^{N+1}), \tag{19}$$

$$\boldsymbol{U}(\varphi(\varepsilon),\varepsilon)=O(\varepsilon^{N+1}),\quad \boldsymbol{U}(1+\psi(\varepsilon),\varepsilon)=O(\varepsilon^{N+1})• \tag{20}$$

此外,对 $\boldsymbol{M}=(M_1,M_2,\cdots,M_m)\in[-1,1]^m$, 令

$$\boldsymbol{R}(1+\psi(\varepsilon),\varepsilon)=\boldsymbol{M}\varepsilon^{N+1}, \tag{21}$$

将(19)改写为:

$$\varepsilon\boldsymbol{U}''-\boldsymbol{F}_{y'}(t,\varepsilon)\boldsymbol{U}'-\boldsymbol{F}_{y}(t,\varepsilon)\boldsymbol{U}=\boldsymbol{h}(t,\boldsymbol{U},\boldsymbol{U}',\boldsymbol{R},\varepsilon), \tag{22}$$

其中, $\boldsymbol{F}(t,\varepsilon)=\boldsymbol{F}(t,\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}',\varepsilon)\equiv\boldsymbol{f}(t,\boldsymbol{y},\boldsymbol{z},\varepsilon)\boldsymbol{y}'+\boldsymbol{g}(t,\boldsymbol{y},\boldsymbol{z},\varepsilon)$, Jacobi 矩阵 $\boldsymbol{F}_{y'}(t,\varepsilon)$, $\boldsymbol{F}_{y}(t,\varepsilon)$ 及下文中的 $\boldsymbol{F}_{z}(t,\varepsilon)$ 在点$(t,\boldsymbol{y}_N,\mathrm{T}^*\boldsymbol{y}_N,\boldsymbol{y}_N',\varepsilon)$ 取值,

$$\boldsymbol{h}(t,\boldsymbol{U},\boldsymbol{U}',\boldsymbol{R},\varepsilon)=\boldsymbol{f}(t,\boldsymbol{U}+\boldsymbol{y}_N,\boldsymbol{R}+\mathrm{T}^*\boldsymbol{y}_N,\varepsilon)(\boldsymbol{U}'+\boldsymbol{y}_N')+\boldsymbol{g}(t,\boldsymbol{U}+\boldsymbol{y}_N,\boldsymbol{R}+\mathrm{T}^*\boldsymbol{y}_N,\varepsilon)-\boldsymbol{f}(t,\boldsymbol{y}_N,\mathrm{T}^*\boldsymbol{y}_N,\varepsilon)\boldsymbol{y}_N'-\boldsymbol{g}(t,\boldsymbol{y}_N,\mathrm{T}^*\boldsymbol{y}_N,\varepsilon)+O(\varepsilon^{N+1})-\boldsymbol{F}_{y'}(t,\varepsilon)\boldsymbol{U}'-\boldsymbol{F}_{y}(t,\varepsilon)\boldsymbol{U}•$$

引理 1[6,7] 对于上述 $\boldsymbol{F}_{y'}(t,\varepsilon)$, $\boldsymbol{F}_{y}(t,\varepsilon)$ 存在 $\varepsilon_1>0$, 当 $0<\varepsilon\leqslant\varepsilon_1$ 时矩阵微分方程:

$$\varepsilon\boldsymbol{P}'=\boldsymbol{F}_{y'}(t,\varepsilon)\boldsymbol{P}-\varepsilon\boldsymbol{P}^2+\boldsymbol{F}_{y}(t,\varepsilon),\quad \boldsymbol{P}(\varphi(\varepsilon),\varepsilon)=\boldsymbol{0}, \tag{23}$$

$$\varepsilon\boldsymbol{Q}'=\varepsilon\boldsymbol{P}(t,\varepsilon)\boldsymbol{Q}+\boldsymbol{Q}[-\boldsymbol{F}_{y'}(t,\varepsilon)+\varepsilon\boldsymbol{P}(t,\varepsilon)]-\boldsymbol{I},\quad \boldsymbol{Q}(1+\psi(\varepsilon),\varepsilon)=\boldsymbol{0} \tag{24}$$

分别有解 $\boldsymbol{P}_{n\times n}=\boldsymbol{P}(t,\varepsilon)$, $\boldsymbol{Q}_{n\times n}=\boldsymbol{Q}(t,\varepsilon)$ 关于 $\varphi(\varepsilon)\leqslant t\leqslant 1+\psi(\varepsilon)$, $0<\varepsilon\leqslant\varepsilon_1$ 一致有界: 存在正常数 p_0, q_0 使得

$$\|\boldsymbol{P}(t,\varepsilon)\|\leqslant p_0,\quad \|\boldsymbol{Q}(t,\varepsilon)\|\leqslant q_0 \tag{25}$$

和

$$\lim_{\varepsilon\to 0^+}\boldsymbol{Q}(t,\varepsilon)=[\boldsymbol{F}_{y'}(t,0)]^{-1}• \tag{26}$$

注 对参数 $\varepsilon>0$ 充分小的要求,在下文中还将多次出现,我们约定今后用同一个 ε_1 来表示满足对 ε 充分小的所有要求: $0<\varepsilon\leqslant\varepsilon_1$•

利用变量替换

$$\boldsymbol{V}=\boldsymbol{U}'-\boldsymbol{P}\boldsymbol{U},\quad \boldsymbol{W}=\boldsymbol{U}+\varepsilon\boldsymbol{Q}\boldsymbol{V} \tag{27}$$

将(20)、(22)变换为

$$\boldsymbol{W}'=\boldsymbol{P}(t,\varepsilon)\boldsymbol{W}+\boldsymbol{Q}(t,\varepsilon)\boldsymbol{h}^*(t,\boldsymbol{W},\boldsymbol{V},\boldsymbol{R},\varepsilon); \tag{28}$$

$$\varepsilon\boldsymbol{V}'=[\boldsymbol{F}_{y'}(t,\varepsilon)-\varepsilon\boldsymbol{P}(t,\varepsilon)]\boldsymbol{V}+\boldsymbol{h}^*(t,\boldsymbol{W},\boldsymbol{V},\boldsymbol{R},\varepsilon); \tag{29}$$

$$\left.\begin{aligned}&\boldsymbol{W}(t,\varepsilon)-\varepsilon\boldsymbol{Q}(t,\varepsilon)\boldsymbol{V}(t,\varepsilon)\big|_{t=\varphi(\varepsilon)}=O(\varepsilon^{N+1}),\\&\boldsymbol{W}(1+\psi(\varepsilon),\varepsilon)=O(\varepsilon^{N+1})•\end{aligned}\right\} \tag{30}$$

令 $\boldsymbol{J}=\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{W}\end{bmatrix}$ 并将(18),(21),(28) ~ (30) 重新写成

$$J' = A(t,\varepsilon)J + B(t,\varepsilon)V + G_1(t,\varepsilon); \tag{31}$$

$$\varepsilon V' = C(t,\varepsilon)J + D(t,\varepsilon)V + G_2(t,\varepsilon); \tag{32}$$

$$\left.\begin{aligned} &J(1+\psi(\varepsilon),\varepsilon) = O(\varepsilon^{N+1}) = \theta_1(\varepsilon),\\ &(\mathbf{0}\ I_n)J(t,\varepsilon) - \varepsilon Q(t,\varepsilon)V(t,\varepsilon)\mid_{t=\varphi(\varepsilon)} = O(\varepsilon^{N+1}) = \theta_2(\varepsilon);\end{aligned}\right\} \tag{33}$$

其中

$$A(t,\varepsilon) = \begin{bmatrix} 0 & A_{12}(t,\varepsilon) \\ A_{21}(t,\varepsilon) & P(t,\varepsilon)\end{bmatrix},$$

$$A_{12}(t,\varepsilon) = -k_y(t,t,y_N(t,\varepsilon),\varepsilon) + \int_t^1 k_{ty}(t,s,y_N(t,\varepsilon),\varepsilon)\mathrm{d}s,$$

$$A_{21}(t,\varepsilon) = Q(t,\varepsilon)F_z(t,\varepsilon);$$

$$B(t,\varepsilon) = \begin{bmatrix} B_{11}(t,\varepsilon) \\ 0 \end{bmatrix},\quad B_{11}(t,\varepsilon) = -\varepsilon A_{12}(t,\varepsilon)Q(t,\varepsilon);$$

$$C(t,\varepsilon) = [F_z(t,\varepsilon)\quad 0];\quad D(t,\varepsilon) = F_{y'}(t,\varepsilon) - \varepsilon P(t,\varepsilon);$$

$$G_1(t,\varepsilon) = \begin{bmatrix} G_{11}(t,\varepsilon) \\ G_{12}(t,\varepsilon)\end{bmatrix},$$

$$\begin{aligned} G_{11}(t,\varepsilon) = {}& -[k(t,t,U+y_N,\varepsilon) - k(t,t,y_N,\varepsilon)] + \\ & \int_t^1 [k_t(t,s,U+y_N,\varepsilon) - k_t(t,s,y_N,\varepsilon)]\mathrm{d}s - \\ & A_{12}(t,\varepsilon)W - B_{11}(t,\varepsilon)V,\end{aligned}$$

$$G_{12}(t,\varepsilon) = Q(t,\varepsilon)(h^*(t,\varepsilon) - F_z(t,\varepsilon)R),$$

$$G_2(t,\varepsilon) = h^*(t,\varepsilon) - F_z(t,\varepsilon)R;$$

$$h^*(t,\varepsilon) \equiv h^*(t,W,V,R,\varepsilon) \equiv h(t,U,U',R,\varepsilon).$$

引理 2[6,7] 对于上述 A、B、C、D 存在 $\varepsilon_1 > 0$, 当 $0 < \varepsilon \leqslant \varepsilon_1$ 时矩阵微分方程

$$\varepsilon T' = D(t,\varepsilon)T - \varepsilon TA(t,\varepsilon) + \varepsilon TB(t,\varepsilon)T - C(t,\varepsilon),\qquad T[\varphi(\varepsilon),\varepsilon] = \mathbf{0}; \tag{34}$$

$$\left.\begin{aligned} &\varepsilon S' = \varepsilon[A(t,\varepsilon) - B(t,\varepsilon)T(t,\varepsilon)]S - S[D(t,\varepsilon) + \varepsilon T(t,\varepsilon)B(t,\varepsilon)] - B(t,\varepsilon),\\ &S(1+\psi(\varepsilon),\varepsilon) = \mathbf{0}\end{aligned}\right\} \tag{35}$$

分别有解 $T_{n\times(m+n)} = T(t,\varepsilon)$, $S_{(n+m)\times n} = S(t,\varepsilon)$ 关于 $\varphi(\varepsilon) \leqslant t \leqslant 1+\psi(\varepsilon)$, $0 < \varepsilon \leqslant \varepsilon_1$ 一致有界: 存在正常数 T_0, S_0 使得

$$\|T(t,\varepsilon)\| \leqslant T_0,\quad \|S(t,\varepsilon)\| \leqslant S_0. \tag{36}$$

利用变量替换

$$\xi = J + \varepsilon S\eta,\quad \eta = V + TJ, \tag{37}$$

将(31)~(33)变换为

$$\xi' = [A(t,\varepsilon) - B(t,\varepsilon)T(t,\varepsilon)]\xi + G_1(\xi,\eta); \tag{38}$$

$$\varepsilon\eta' = [D(t,\varepsilon) + \varepsilon T(t,\varepsilon)B(t,\varepsilon)]\eta + G_2(\xi,\eta); \tag{39}$$

$$\left.\begin{aligned} &\xi(1+\psi(\varepsilon),\varepsilon) = \theta_1(\varepsilon),\\ &(\mathbf{0}_{n\times m}\quad I_n)\xi(t,\varepsilon) - \varepsilon[(\mathbf{0}_{n\times m}\quad I_n)S(t,\varepsilon)\times Q(t,\varepsilon)]\eta(t,\varepsilon)\mid_{t=\varphi(\varepsilon)} = \theta_2(\varepsilon);\end{aligned}\right\} \tag{40}$$

其中:

$$G_1(\xi,\eta) = (I_{m\times n} + \varepsilon ST)G_1 + SG_2,$$

$$G_2(\xi, \eta) = \varepsilon T G_1 + G_2.$$

由(26)知，当 $\varepsilon > 0$ 充分小时，$(\mathbf{0}\quad I)S(\varphi(\varepsilon), \varepsilon) + Q(\varphi(\varepsilon), \varepsilon)$ 的逆存在且有界，故问题(38)～(40)等价于积分方程

$$\xi(t, \varepsilon) = Z(t)\theta_1(\varepsilon) + \int_{1+\psi(\varepsilon)}^{t} Z(t)Z^{-1}(s)G_1(s, \xi(s), \eta(s), \varepsilon)\mathrm{d}s, \tag{41}$$

$$\begin{aligned}\varepsilon\eta(t, \varepsilon) = {}& H(t)[(\mathbf{0}_{n\times m}\quad I_n)S(\varphi(\varepsilon), \varepsilon) + Q(\varphi(\varepsilon), \varepsilon)]^{-1} \times \\ & [(\mathbf{0}_{n\times m}\quad I_n)\xi(\varphi(\varepsilon), \varepsilon) - \theta_2(\varepsilon)] + \\ & \int_{\varphi(\varepsilon)}^{t} H(t)H^{-1}(s)G_2(s, \xi(s), \eta(s), \varepsilon)\mathrm{d}s.\end{aligned} \tag{42}$$

由条件(Ⅰ)知矩阵 $D(t, \varepsilon) - \varepsilon T(t, \varepsilon)B(t, \varepsilon)$ 的每一个特征值实部 $\leqslant -\mu_0 < 0$，所以齐线性系统

$$\varepsilon\eta' = [D(t, \varepsilon) + \varepsilon T(t, \varepsilon)B(t, \varepsilon)]\eta \tag{43}$$

有一个指数二分法：存在 $l > 0$，使得

$$|H(t)H^{-1}(s)| \leqslant l e^{-\mu_0(t-s)/2\varepsilon}, \qquad \varphi(\varepsilon) \leqslant s \leqslant t \leqslant 1 + \psi(\varepsilon). \tag{44}$$

由 $A(t, \varepsilon) - B(t, \varepsilon)T(t, \varepsilon)$ 的有界性知对足够大的 $l > 1$，有

$$|Z(t)Z^{-1}(s)| \leqslant 1, \qquad \varphi(\varepsilon) \leqslant s, t \leqslant 1 + \psi(\varepsilon), \tag{45}$$

这里 $Z(t) = Z(t, \varepsilon)$，$H(t) = H(t, \varepsilon)$ 分别是(38)、(39)对应的齐线性系统的基本解矩阵，满足 $Z(1+\psi(\varepsilon), \varepsilon) = I_{m+n}$，$H(\varphi(\varepsilon), \varepsilon) = I_n$.

令 $(\xi_0, \eta_0) = (\mathbf{0}, \mathbf{0})$ 并作如下迭代

$$\xi_{k+1}(t, \varepsilon) = Z(t)\theta_1(\varepsilon) + \int_{1+\psi(\varepsilon)}^{t} Z(t)Z^{-1}(s)G_1(s, \xi_k(s), \eta_k(s), \varepsilon)\mathrm{d}s, \tag{46}$$

$$\begin{aligned}\varepsilon\eta_{k+1}(t, \varepsilon) = {}& H(t)[(\mathbf{0}_{n\times m}\quad I_n)S(\varphi(\varepsilon), \varepsilon) + Q(\varphi(\varepsilon), \varepsilon)]^{-1} \times \\ & [(\mathbf{0}_{n\times m}\quad I_n)\xi_{k+1}(\varphi(\varepsilon), \varepsilon) - \theta_2(\varepsilon)] + \\ & \int_{\varphi(\varepsilon)}^{t} H(t)H^{-1}(s)G_2(s, \xi_k(s), \eta_k(s), \varepsilon)\mathrm{d}s,\end{aligned} \tag{47}$$

易知，当 $\varphi(\varepsilon) \leqslant t \leqslant 1 + \psi(\varepsilon)$，$0 < \varepsilon \leqslant \varepsilon_1$，$G_i(t, \xi, \eta, \varepsilon)$，$(i = 1, 2)$ 具有：

① $\|G_i(t, 0, 0, \varepsilon)\| \leqslant M_0\varepsilon^{N+1}$; (48)

② 存在常数 $K > 0$，使得

$$\|G_i(t, \xi, \eta, \varepsilon) - G_i(t, \xi^*, \eta^*, \varepsilon)\| \leqslant K\|\chi(\xi, \xi^*, \eta, \eta^*)\|, \tag{49}$$

其中 $\chi(\xi, \xi^*, \eta, \eta^*)$ 是下面三个值中得最大者：

$$|\xi - \xi^*| \cdot \max(|\xi|, |\xi^*|, |\eta|, |\eta^*|),$$

$$\varepsilon|\eta - \eta^*| \cdot \max(|\xi|, |\xi^*|, |\eta|, |\eta^*|),$$

$$|\eta - \eta^*| \cdot \max(|\xi|, |\xi^*|, \varepsilon|\eta|, \varepsilon|\eta^*|).$$

对于 $R_0 = l[h_0(l\theta_0 + 2lM_0) + h_0\theta_0 + 4M_0\mu_0^{-1} + \theta_0 + 2M_0]$，取 $\varepsilon > 0$ 如此之小，使得 $r = 2lKR_0(2lh_0 + 4\mu_0^{-1} + 2)\varepsilon^N < 1/2$，其中正常数 θ_0、h_0 分别使下列不等式成立：

$$\|\theta_1(\varepsilon)\|, \|\theta_2(\varepsilon)\| \leqslant \theta_0\varepsilon^{N+1},$$

$$\|[(\mathbf{0}_{n\times m}\quad I_n)S(\varphi(\varepsilon), \varepsilon) + Q(\varphi(\varepsilon), \varepsilon)]^{-1}\| \leqslant h_0.$$

利用数学归纳法可推得当 $\varphi(\varepsilon) \leqslant t \leqslant 1 + \psi(\varepsilon)$，$0 < \varepsilon \leqslant \varepsilon_1$ 时

$$\|\xi_K(t, \varepsilon)\|, \varepsilon\|\eta_K(t, \varepsilon)\| \leqslant 2R_0\varepsilon^{N+1},$$

$$\|\xi_K(t, \varepsilon) - \xi_{K-1}(t, \varepsilon)\|, \varepsilon\|\eta_K(t, \varepsilon) - \eta_{K-1}(t, \varepsilon)\| \leqslant R_0 r^{K-1}\varepsilon^{N+1}.$$

由此得出序列

$$\left\{ \xi_K(t, \varepsilon) \right\}_1^\infty, \left\{ \eta_K(t, \varepsilon) \right\}_1^\infty$$

关于 $\varphi(\varepsilon) \leqslant t \leqslant 1 + \phi(\varepsilon)$, $0 < \varepsilon \leqslant \varepsilon_0$ 一致地收敛于问题(38) ~ (40) 的解$(\xi(t, \varepsilon), \eta(t, \varepsilon))$, 从而问题(18) ~ (21) 存在解$(\boldsymbol{R}(t, \varepsilon), \boldsymbol{U}(t, \varepsilon))$ 且满足

$$\left.\begin{array}{l} \boldsymbol{R}(t, \varepsilon), \boldsymbol{U}(t, \varepsilon) = O(\varepsilon^{N+1}), \quad \boldsymbol{U}'(t, \varepsilon) = O(\varepsilon^N), \\ \varphi(\varepsilon) \leqslant t \leqslant 1 + \phi(\varepsilon), \qquad 0 < \varepsilon \leqslant \varepsilon_1 \cdot \end{array}\right\} \tag{50}$$

由(50) 可知当 $\varepsilon > 0$ 充分小时成立:

$$\varepsilon^{-(N+1)} \left\| \int_{1+\phi(\varepsilon)}^1 [\boldsymbol{k}(t, s, \boldsymbol{y}_N + \boldsymbol{U}, \varepsilon) - \boldsymbol{k}(t, s, \boldsymbol{y}_N, \varepsilon)] \mathrm{d}s \right\| \leqslant 1 \cdot$$

定义 $[-1, 1]^m \rightarrow [-1, 1]^m$ 上的连续映射 $J^*(\boldsymbol{M})$:

$$J^*(\boldsymbol{M}) = \varepsilon^{-(N+1)} \int_{1+\phi(\varepsilon)}^1 [\boldsymbol{k}(t, s, \boldsymbol{y}_N + \boldsymbol{U}, \boldsymbol{S}) - \boldsymbol{k}(t, s, \boldsymbol{y}_N, \varepsilon)] \mathrm{d}s,$$

可证明 $J^*(\boldsymbol{M})$ 为压缩映射· 事实上任取 $\overset{(i)}{\boldsymbol{M}} = (\overset{(i)}{M}_1 \cdots \overset{(i)}{M}_m) \in [-1, 1]^m (i = 1, 2,)$, 则边值问题(18) ~ (21) 分别有解$(\boldsymbol{R}_1(t, \varepsilon), \boldsymbol{U}_1(t, \varepsilon))$ 和$(\boldsymbol{R}_2(t, \varepsilon), \boldsymbol{U}_2(t, \varepsilon))$ ·

令 $\boldsymbol{R} = \boldsymbol{R}_1 - \boldsymbol{R}_2$, $\boldsymbol{U} = \boldsymbol{U}_1 - \boldsymbol{U}_2$, 那么 $\boldsymbol{R}, \boldsymbol{U}$ 可满足

$$\begin{aligned} \boldsymbol{R}' = & -[\boldsymbol{k}(t, t, \boldsymbol{U} + \boldsymbol{U}_2 + \boldsymbol{y}_N, \varepsilon) - \boldsymbol{k}(t, t, \boldsymbol{U}_2 + \boldsymbol{y}_N, \varepsilon)] + \\ & \int_t^1 [\boldsymbol{k}_t(t, s, \boldsymbol{U} + \boldsymbol{U}_2 + \boldsymbol{y}_N, \varepsilon) - \boldsymbol{k}_t(t, s, \boldsymbol{U}_2 + \boldsymbol{y}_N, \varepsilon)] \mathrm{d}s, \end{aligned} \tag{51}$$

$$\begin{aligned} \varepsilon \boldsymbol{U}'' = & F(t, \boldsymbol{U} + \boldsymbol{U}_2 + \boldsymbol{y}_N, \boldsymbol{R} + \boldsymbol{R}_2 + \mathrm{T}^* \boldsymbol{y}_N, \boldsymbol{U}' + \boldsymbol{U}_2' + \boldsymbol{y}_N', \varepsilon) - \\ & F(t, \boldsymbol{U}_2 + \boldsymbol{y}_N, \boldsymbol{R}_2 + \mathrm{T}^* \boldsymbol{y}_N, \boldsymbol{U}_2' + \boldsymbol{y}_N', \varepsilon), \end{aligned} \tag{52}$$

$$\boldsymbol{R}(1 + \phi(\varepsilon), \varepsilon) = (\overset{(1)}{\boldsymbol{M}} - \overset{(2)}{\boldsymbol{M}}) \varepsilon^{N+1}, \boldsymbol{U}(\varphi(\varepsilon), \varepsilon) = \boldsymbol{U}(1 + \varphi(\varepsilon), \varepsilon) = 0 \cdot \tag{53}$$

关于边值问题(51) ~ (53) 再次应用求解(18) ~ (21) 的步骤, 可得

$$\boldsymbol{R}(t, \varepsilon), \boldsymbol{U}(t, \varepsilon) = O(\varepsilon^{N+1}), \quad \boldsymbol{U}'(t, \varepsilon) = O(\varepsilon^N),$$

从而有

$$\lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-(N+1)} \int_{1+\phi(\varepsilon)}^1 [\boldsymbol{k}(t, \varepsilon, \boldsymbol{U} + \boldsymbol{U}_2 + \boldsymbol{y}_N, \varepsilon) - \boldsymbol{k}(t, s, \boldsymbol{U}_2 + \boldsymbol{y}_N, \varepsilon)] \mathrm{d}s \right\| = 0 \cdot$$

对 $0 < \mu \ll 1$ 存在 $\varepsilon_2 > 0$ 使得 $0 < \varepsilon \leqslant \varepsilon_2$ 时有

$$\| J^*(\overset{(1)}{\boldsymbol{M}}) - J^*(\overset{(2)}{\boldsymbol{M}}) \| \leqslant \mu \| (\overset{(1)}{\boldsymbol{M}}) - (\overset{(2)}{\boldsymbol{M}}) \| \cdot$$

根据不动点原理知 J^* 在$[-1, 1]^m$ 中有唯一的不动点$\boldsymbol{M}^*$: $J^*(\boldsymbol{M}^*) = \boldsymbol{M}^*$· 于是对于$\boldsymbol{M}^* = (M_1^*, \ldots, M_m^*) \in [-1, 1]^m$, 问题(18) ~ (21) 的解 $\boldsymbol{R}(t, \varepsilon)$、$\boldsymbol{U}(t, \varepsilon)$ 存在且有

$$\boldsymbol{R}(t, \varepsilon), \boldsymbol{U}(t, \varepsilon) = O(\varepsilon^{N+1}), \quad \boldsymbol{U}'(t, \varepsilon) = O(\varepsilon^N)$$

以及

$$\boldsymbol{R}(1 + \phi(\varepsilon), \varepsilon) = \boldsymbol{M}^* \varepsilon^{N+1} \cdot$$

返回到原来的变量知存在 $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$, 当 $0 < \varepsilon \leqslant \varepsilon_0$ 时, 问题(1)、(2) 存在解 $\boldsymbol{y}(t, \varepsilon)$, $\varphi(\varepsilon) \leqslant t \leqslant 1 + \varphi(\varepsilon)$, 且成立

$$\begin{aligned} \boldsymbol{y}(t, \varepsilon) &= \boldsymbol{y}_N(t, \varepsilon) + O(\varepsilon^{N+1}), \\ \boldsymbol{y}'(t, \varepsilon) &= \boldsymbol{y}_N'(t, \varepsilon) + O(\varepsilon^N) \cdot \end{aligned}$$

定理证毕·

[参 考 文 献]

[1] 莫嘉琪, 许玉兴. 一类奇摄动非线性反应扩散积分微分方程组[J]. 应用数学学报, 1994, **17**(2): 278 ~ 286.

[2] 黄蔚章, 莫嘉琪. 四阶半线性椭圆型积分微分方程的奇摄动[J]. 应用数学学报, 1997, **20**(1): 70~ 76.

[3] 莫嘉琪, 陈育森. 一类具有非局部边值条件的反应扩散方程奇摄动问题[J]. 数学物理学报, 1997, **17**(1): 25~ 30.

[4] Huang Weizhang (黄蔚章). A class of nonlocal singularly perturbed nonlinear boundary value problems[J]. Ann Diff Eqs, 1997, **13**(2): 140~ 145.

[5] Howes F A, Ó Malley R E. Singular perturbation of semilinear second order systems[A]. In: On O D E and P D E at Dundee[C]. 1978, 131~ 150. Lecture Notes in Math [M]. Berlan: Springer, 1980, 827.

[6] 林宗池. 非线性系统边值问题的奇摄动[J]. 福建师范大学学报(自然科学版), 1989, **5**(4): 1~ 8.

[7] Chen Yusen (陈育森). Singular perturbation of two_point boundary value problems for nonlinear system[J]. Ann Diff Eqs, 1995, **11**(1): 25~ 36.

Singular Perturbation of Second Order Nonlinear System With Boundary Perturbation

Chen Yusen

(Department of Mathematics, Fuqing Branch, Fujian Normal University, Fuqing, Fujian 350300, P R China)

Abstract: The singular perturbation of boundary value problem of second order nonlinear system of differential equations with integral operators and boundary perturbation is discussed. Under the suitable assumed conditions, by the technique of diagonalization, the existence of the solutions is proved and its remainder term is estimated.

Key words: singular perturbation; boundary perturbation; diagonalization