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速度空间的 D' Alembert 原理*

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(叶庆凯推荐)

摘要: 从质点系的牛顿动力学方程出发, 考虑力是坐标 $\boldsymbol r$、速度 $\dot{\boldsymbol r}$ 和时间 t 的函数的情况, 引入速度空间的“动能”(即加速度能) 的概念, 导出了完整系和非完整系速度空间的 D' Alembert 原理的各种形式·

关 键 词: 牛顿动力学方程; 加速度能; 速度空间; D' Alembert 原理
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引 言

近来, 人们对 D' Alembert 原理各种形式进行了研究[1,2], 而且对速度空间的力学原理给予关注[3~5], 给出了速度空间和相对论力学速度空间的变分原理· 随着对带高阶约束的力学系统的重视, 推导高阶方程的依据 D' Alembert 原理应有所发展, 本文拟推出速度空间的 D' Alembert 原理的各种形式·

1 速度空间 D' Alembert 原理的一般形式

考虑由 N 个质点 $P_i(i=1,2,\cdots,N)$ 构成的力学系统, P_i 的质量为 m_i, 位形为 $\boldsymbol r_i=\boldsymbol r_i(q_s,t)$, $q_s(s=1,2,\cdots,n)$ 为确定体系位形的广义坐标·

对质点 p_i, 牛顿动力学方程为

$$\boldsymbol F_i+\boldsymbol R_i=m_i\ddot{\boldsymbol r}_i, \qquad (1)$$

其中 $\boldsymbol F_i$ 和 $\boldsymbol R_i$ 分别为作用在 p_i 上的主动力和约束反力· 一般情况下, $\boldsymbol F_i$ 和 $\boldsymbol R_i$ 是位矢 $\boldsymbol r$, 速度 $\dot{\boldsymbol r}$ 和时间 t 的函数, 即

$$\boldsymbol F_i=\boldsymbol F_i(\boldsymbol r_i,\dot{\boldsymbol r}_i,t),$$

和

$$\boldsymbol R_i=\boldsymbol R_i(\boldsymbol r_i,\dot{\boldsymbol r}_i,t)·$$

将(1)对 t 微商得:

$$\dot{\boldsymbol F}_i+\dot{\boldsymbol R}_i=m_i\dddot{\boldsymbol r}_i,$$

或

$$\dot{\boldsymbol F}_i+\dot{\boldsymbol R}_i+(-m_i\dddot{\boldsymbol r}_i)=0· \qquad (2)$$

考虑包含 $(-m_i\dddot{\boldsymbol r}_i)$ 在内的力系全部力在速度空间中的任意虚位移 $\delta\dot{\boldsymbol r}$ 的“功”, 即

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$$\sum_{i=1}^{n}[\dot{\boldsymbol F}_i + \dot{\boldsymbol R}_i + (-m_i\dddot{\boldsymbol r}_i)]\cdot\delta\dot{\boldsymbol r}_i = 0, \tag{3}$$

$\dot{\boldsymbol r}_i$ 在速度空间中的变分为

$$\delta\dot{\boldsymbol r}_i = \sum_{s=1}^{n}\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot{\boldsymbol q}_s}\delta\dot q_s. \tag{4}$$

(4) 代入(3) 有

$$\sum_{s=1}^{n}\sum_{i=1}^{N}\left[(\dot{\boldsymbol F}_i + \dot{\boldsymbol R}_i)\cdot\frac{\partial\dot r_i}{\partial\dot q_s} - m_i\dddot{\boldsymbol r}_i\cdot\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s}\right]\delta\dot q_s = 0, \tag{5}$$

令

$$\dot Q_s^F = \sum_{i=1}^{N}\dot{\boldsymbol F}_i\cdot\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s},\quad \dot Q_s^R = \sum_{i=1}^{N}\dot{\boldsymbol R}_i\cdot\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s},$$

则

$$\dot Q_s = \sum_{i=1}^{N}(\dot{\boldsymbol F}_i + \dot{\boldsymbol R}_i)\cdot\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s} = \dot Q_s^F + \dot Q_s^R, \tag{6}$$

又因

$$\sum_{i=1}^{N}m_i\ddot{\boldsymbol r}_i\cdot\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s} = \sum_{i=1}^{N}m_i\frac{\mathrm d\ddot{\boldsymbol r}_i}{\mathrm dt}\cdot\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s} =$$
$$\sum_{i=1}^{N}\left[m_i\frac{\mathrm d}{\mathrm dt}\left(\ddot{\boldsymbol r}_i\cdot\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s}\right) - m_i\ddot{\boldsymbol r}_i\cdot\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s}\right)\right], \tag{7}$$

而

$$\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s}\right) = \frac{\partial\ddot{\boldsymbol r}_i}{\partial\dot q_s},$$

又

$$\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s} = \frac{\partial\ddot{\boldsymbol r}_i}{\partial\ddot q_s}$$

故(7) 式为

$$\sum_{i=1}^{N}m_i\dddot{\boldsymbol r}_i\cdot\frac{\partial\dot{\boldsymbol r}_i}{\partial\dot q_s} = \frac{\mathrm d}{\mathrm dt}\left(\sum_{i=1}^{N}m_i\ddot{\boldsymbol r}_i\cdot\frac{\partial\ddot{\boldsymbol r}_i}{\partial\ddot q_s}\right) - \sum_{i=1}^{N}m_i\ddot r_i\cdot\frac{\partial\ddot{\boldsymbol r}_i}{\partial\dot q_s} =$$
$$\frac{\mathrm d}{\mathrm dt}\frac{\partial}{\partial\ddot q_s}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\ddot{\boldsymbol r}_i^2\right) - \frac{\partial}{\partial\dot q_s}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\ddot{\boldsymbol r}_i^2\right). \tag{8}$$

令 $S = \sum_{i=1}^{N}\frac{1}{2}m_i\ddot{\boldsymbol r}_i^2$ 为加速度能, 即速度空间的动能, 显然

$$S = S(q_s, \dot q_s, \ddot q_s, t), \tag{9}$$

(9) 代入(8) 有

$$\sum_{i=1}^{N}m_i\dddot{\boldsymbol r}_i\cdot\frac{\partial\dot r_i}{\partial\dot q_s} = \frac{\mathrm d}{\mathrm dt}\frac{\partial S}{\partial\ddot q_s} - \frac{\partial S}{\partial\dot q_s}, \tag{10}$$

将(6)、(10) 代入(5) 可得

$$\sum_{s=1}^{n}\left(\dot Q_s - \frac{\mathrm d}{\mathrm dt}\frac{\partial S}{\partial\ddot q_s} + \frac{\partial S}{\partial\dot q_s}\right)\delta\dot q_s = 0. \tag{11}$$

(11) 即为速度空间的 D' Alembert 原理的一般形式.

2　速度空间中的 D' Alembert 原理的几种其它形式

1) D' Alembert 原理的 Appell 形式

若系统为完整系, 由于 δq_s 为独立变量, $\delta\dot q_s$ 一定是独立的, 由(11) 得

$$\dot Q_s - \frac{\mathrm d}{\mathrm dt}\frac{\partial S}{\partial \ddot q_s} + \frac{\partial S}{\partial \dot q_s} = 0,$$

整理可得

$$\left(\frac{\mathrm d}{\mathrm dt}\frac{\partial}{\partial \ddot q_s} - \frac{\partial}{\partial \dot q_s}\right) S = \dot Q_s. \tag{12}$$

(12) 即为

$$E_s S = \dot Q_s \qquad (s = 1, 2, \cdots, n), \tag{13}$$

其中 E_s 为算子$\frac{\mathrm d}{\mathrm dt}\frac{\partial}{\partial \ddot q_s} - \frac{\partial}{\partial \dot q_s}$. (13) 即为 D' Alembert 原理的 Appell 形式.

在位形空间中可用 Lagrange 方程表示系统的动力学方程; 由(13) 可知, 在速度空间则用 Appell 方程表示, 因为算子 E_s 相同, 前者与广义力有关, 后者与广义力变有关.

2) D' Alembert 原理的 Maggi 形式

若系统还受有 g 个一阶非完整约束

$$f_\beta(q_s, \dot q_s, t) = 0 \qquad (\beta = 1, 2, \cdots, g), \tag{14}$$

的限制, 则在 $\left|\frac{\partial f_\beta}{\partial \dot q_{\varepsilon+\beta}}\right| \neq 0$ 的条件下, 由(14) 可得

$$\dot q_{\varepsilon+\beta} = \dot q_{\varepsilon+\beta}(q_s, \dot q_\sigma, t),$$

$$\delta \dot q_{\varepsilon+\beta} = \sum_{\sigma=1}^{\varepsilon}\frac{\partial \dot q_{\varepsilon+\beta}}{\partial \dot q_\sigma}\delta \dot q_\sigma, \tag{15}$$

其中 $\sigma = 1, 2, \cdots, \varepsilon$, $\varepsilon = n - g$.

(11) 可写成

$$\sum_{s=1}^{n}\left(\dot Q_s - \frac{\mathrm d}{\mathrm dt}\frac{\partial S}{\partial \ddot q_s} + \frac{\partial S}{\partial \dot q_s}\right)\delta \dot q_s = \sum_{\sigma=1}^{\varepsilon}\left(\dot Q_\sigma - \frac{\mathrm d}{\mathrm dt}\frac{\partial S}{\partial \ddot q_\sigma} + \frac{\partial S}{\partial \dot q_\sigma}\right)\cdot \delta \dot q_\sigma$$
$$+ \sum_{\beta=1}^{g}\left(\dot Q_{\varepsilon+\beta} - \frac{\mathrm d}{\mathrm dt}\frac{\partial S}{\partial \ddot q_{\varepsilon+\beta}} + \frac{\partial S}{\partial \dot q_{\varepsilon+\beta}}\right)\cdot \delta \dot q_{\varepsilon+\beta} = 0, \tag{16}$$

由于 δq_σ 为独立变量, 由(16) 可知

$$\dot Q_\sigma - \frac{\mathrm d}{\mathrm dt}\frac{\partial S}{\partial \ddot q_\sigma} + \frac{\partial S}{\partial \dot q_\sigma} + \sum_{\beta=1}^{g}\left(\dot Q_{\varepsilon+\beta} - \frac{\mathrm d}{\mathrm dt}\frac{\partial S}{\partial \ddot q_{\varepsilon+\beta}} + \frac{\partial S}{\partial \dot q_{\varepsilon+\beta}}\right)\frac{\partial \dot q_{\varepsilon+\beta}}{\partial \dot q_\sigma} = 0, \tag{17}$$

(17) 即为受 g 个一阶非完整约束的力学系在速度空间的 Maggi 形式的 D' Alembert 原理.

3) D' Alembert 原理的 ±Á½Í´º¿ 形式

由(15) 可知

$$\ddot q_{\varepsilon+\beta} = \ddot q_{\varepsilon+\beta}(q_s, \dot q_s, \ddot q_\sigma, t),$$

即

$$\ddot q_{\varepsilon+\beta} = \ddot q_{\varepsilon+\beta}[q_s, \dot q_\sigma, \dot q_{\varepsilon+\gamma}(q_s, \dot q_\sigma, t), \ddot q_\sigma, t], \tag{18}$$

其中 $\gamma = 1, 2, \cdots, g$.

考虑(9) 式, 令

$$S = S\{q_s, \dot q_\sigma, \dot q_{\varepsilon+\beta}(q_s, \dot q_\sigma, t), \ddot q_\sigma, \ddot q_{\varepsilon+\beta}[q_s, \dot q_\sigma, \dot q_{\varepsilon+\gamma}(q_s, \dot q_\sigma, t), \ddot q_\sigma, t], t\} =$$
$$S(q_s, \dot q_\sigma, \ddot q_\sigma, t) \tag{19}$$

则

$$\frac{\partial S}{\partial \dot q_\sigma} = \frac{\partial S}{\partial \dot q_\sigma} + \sum_{\beta=1}^{g}\frac{\partial S}{\partial \dot q_{\varepsilon+\beta}}\cdot\frac{\partial \dot q_{\varepsilon+\beta}}{\partial \dot q_\sigma} + \sum_{\beta=1}^{g}\sum_{\gamma=1}^{g}\frac{\partial S}{\partial \ddot q_{\varepsilon+\beta}}\cdot\frac{\partial \ddot q_{\varepsilon+\beta}}{\partial \dot q_{\varepsilon+\gamma}}\cdot\frac{\partial \dot q_{\varepsilon+\gamma}}{\partial \dot q_\sigma}, \tag{20}$$

以及

$$\frac{\partial S}{\partial \ddot{q}_\sigma} = \frac{\partial S}{\partial \ddot{q}_\sigma} + \sum_{\beta=1}^{g} \frac{\partial S}{\partial \ddot{q}_{\varepsilon+\beta}} \cdot \frac{\partial \ddot{q}_{\varepsilon+\beta}}{\partial \ddot{q}_\sigma}, \tag{21}$$

从而

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\sigma}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\sigma}\right) + \sum_{\beta=1}^{g} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial S}{\partial \ddot{q}_{\varepsilon+\beta}} \cdot \frac{\partial \ddot{q}_{\varepsilon+\beta}}{\partial \ddot{q}_\sigma} + \frac{\partial S}{\partial \ddot{q}_{\varepsilon+\beta}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ddot{q}_{\varepsilon+\beta}}{\partial \ddot{q}_\sigma}, \tag{22}$$

又

$$\ddot{q}_{\varepsilon+\beta} = \frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_{\varepsilon+\beta}(q_s, \dot{q}_\sigma, t) =$$

$$\sum_{s=1}^{n} \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_s}\dot{q}_s + \sum_{s=1}^{n} \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial t}$$

所以

$$\frac{\partial \ddot{q}_{\varepsilon+\beta}}{\partial \ddot{q}_\sigma} = \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}, \tag{23}$$

以及

$$\frac{\partial \ddot{q}_{\varepsilon+\beta}}{\partial \dot{q}_s} = \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_s} = \begin{cases} \dfrac{\partial \ddot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma} = \dfrac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_\sigma} & (s = \sigma), \\ \dfrac{\partial \ddot{q}_{\varepsilon+\beta}}{\partial \dot{q}_{\varepsilon+\gamma}} = \dfrac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_{\varepsilon+\gamma}} & (s = \varepsilon+\gamma) \cdot \end{cases} \tag{24}$$

将(23)、(24)代入(20)、(22)得

$$\frac{\partial S}{\partial \dot{q}_\sigma} = \frac{\partial S}{\partial \dot{q}_\sigma} + \sum_{\beta=1}^{g} \frac{\partial S}{\partial \dot{q}_{\varepsilon+\beta}} \cdot \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma} + \sum_{\beta,\gamma=1}^{g} \frac{\partial S}{\partial \ddot{q}_{\varepsilon+\beta}} \cdot \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_{\varepsilon+\gamma}} \cdot \frac{\partial \dot{q}_{\varepsilon+\gamma}}{\partial \dot{q}_\sigma}, \tag{25}$$

及

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\sigma}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial S}{\partial \ddot{q}_\sigma} + \sum_{\beta=1}^{g}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial S}{\partial \ddot{q}_{\varepsilon+\beta}} \cdot \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma} + \frac{\partial S}{\partial \ddot{q}_{\varepsilon+\beta}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right)\right] \cdot \tag{26}$$

将(25)(26)代入Maggi型方程(17)得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\sigma}\right) - \frac{\partial S}{\partial \dot{q}_\sigma} + \sum_{\beta=1}^{g} \frac{\partial S}{\partial \ddot{q}_{\varepsilon+\beta}}\left[\sum_{r=1}^{g} \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_{\varepsilon+\gamma}} \cdot \frac{\partial \dot{q}_{\varepsilon+\gamma}}{\partial \dot{q}_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right)\right] = \tilde{Q}_\sigma, \tag{27}$$

(27)式即为 ±Á½ Í´º¿ 形式的 D' Alembert 原理·

其中

$$Q_\sigma = \tilde{Q}_\sigma + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta} \cdot \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_\sigma},$$

并且

$$\left.\begin{aligned} \tilde{Q}_\sigma^F &= Q_\sigma^F + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^F \cdot \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}, \\ \tilde{Q}_\sigma^R &= Q_\sigma^R + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^R \cdot \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma} \end{aligned}\right\} \tag{28}$$

以及

$$\tilde{Q}_\sigma = \tilde{Q}_\sigma^F + \tilde{Q}_\sigma^R \cdot \tag{29}$$

3　非完整系的速度空间的 D' Alembert 原理

若系统为非完整系, 由于一阶非线性、非完整系统的Appell方程为

$$\frac{\partial S}{\partial \ddot{q}_\sigma} = Q_\sigma^F, \tag{30}$$

其中 Q_σ^F 为主动力所产生的广义力• 把(30) 两边对 t 微商得

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial S}{\partial \ddot{q}_\sigma} = \dot{Q}_\sigma^F• \tag{31}$$

由于

$$Q_\sigma^F = Q_\sigma^F + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^F • \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma},$$

则

$$\dot{Q}_\sigma^F = \dot{Q}_\sigma^F + \sum_{\beta=1}^{g} \dot{Q}_{\varepsilon+\beta}^F • \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma} + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^F \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right)• \tag{32}$$

考虑到(28) 式, (32) 变为

$$\dot{Q}_\sigma^F = \tilde{\dot{Q}}_\sigma^F + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^F \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right),$$

则

$$\tilde{\dot{Q}}_\sigma^F = \dot{Q}_\sigma^F - \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^F \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right)• \tag{33}$$

由理想约束条件可知

$$\sum_{i=1}^{N} \boldsymbol{R}_i • \delta \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{R}_i • \sum_{s=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_s}\delta q_s = \sum_{s=1}^{n}\left(\sum_{i=1}^{N} \boldsymbol{R}_i • \frac{\partial \boldsymbol{r}_i}{\partial q_a}\right)\delta q_s = 0• \tag{34}$$

若系统为完整系, 由于 δq_s 为独立变量, 由(34) 可知理想约束的广义力为

$$Q_s^R = \sum_{i=1}^{N} \boldsymbol{R}_i • \frac{\partial \boldsymbol{r}_i}{\partial q_s} = 0, \tag{35}$$

又因为不管约束是否完整, 下式总成立

$$\frac{\partial \boldsymbol{r}_i}{\partial q_s} = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_s},$$

则(35) 为

$$Q_s^R = \sum_{i=1}^{N} \boldsymbol{R}_i • \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_s} = 0• \tag{36}$$

由(35) 和(36) 知, 理想约束力在位形和速度空间均为零, 我们不妨用同一符号 Q_s^R 表示, 则有

$$\sum_{i=1}^{N} \boldsymbol{R}_i • \delta \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{R}_i • \delta \dot{\boldsymbol{r}}_i = 0, \tag{37}$$

在一阶非完整约束下, 由(37) 可得

$$\begin{aligned}
&\sum_{s=1}^{n}\left(\sum_{i=1}^{N} \boldsymbol{R}_i • \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_s}\right)\delta \dot{q}_s = \sum_{s=1}^{n} Q_s^R • \delta \dot{q}_s = \\
&\sum_{\sigma=1}^{\varepsilon} Q_\sigma^R • \delta \dot{q}_\sigma + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^R \delta \dot{q}_{\varepsilon+\beta} = \\
&\sum_{\sigma=1}^{\varepsilon}\left(Q_\sigma^R • \delta \dot{q}_\sigma + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^R • \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\delta \dot{q}_\sigma\right) = \\
&\sum_{\sigma=1}^{\varepsilon} Q_\sigma^R • \delta \dot{q}_s = 0,
\end{aligned} \tag{38}$$

其中 $Q_\sigma^R = Q_\sigma^R + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^R \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}$•

由于 $\delta\dot{q}_s$ 为独立变量, 可知理想约束广义力在一阶非完整约束系统中用广义坐标表示的形式下为

$$Q_\sigma^R = 0, \tag{39}$$

可得

$$\dot{Q}_\sigma^R = 0, \tag{40}$$

即

$$\dot{Q}_\sigma^R = \dot{Q}_\sigma^R + \sum_{\beta=1}^{g} \dot{Q}_{\varepsilon+\beta}^R \cdot \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma} + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^R \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right) = 0\cdot \tag{41}$$

考虑到(28)、(41) 为

$$\dot{Q}_\sigma^R = \tilde{\dot{Q}}_\sigma^R + \sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^R \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right) = 0,$$

即

$$\dot{Q}_\sigma^R = -\sum_{\beta=1}^{g} Q_{\varepsilon+\beta}^R \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right)\cdot \tag{42}$$

考虑到(33) 和(42) 有

$$\begin{aligned} \tilde{\dot{Q}}_\sigma &= \tilde{\dot{Q}}_\sigma^F + \tilde{\dot{Q}}_\sigma^R = \\ &\dot{Q}_\sigma^F - \sum_{\beta=1}^{g} (Q_{\varepsilon+\beta}^F + Q_{\varepsilon+\beta}^R) \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right) = \\ &\dot{Q}_\sigma^F - \sum_{\beta=1}^{g} Q_{\varepsilon+\beta} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right), \end{aligned} \tag{43}$$

其中 $Q_{\varepsilon+\beta} = Q_{\varepsilon+\beta}^F + Q_{\varepsilon+\beta}^R$ 为对应的约束分量为 $\delta q_{\varepsilon+\beta}$ 之广义力· 把(31) 和(43) 代入(27), 两边消去 $\dot{Q}_\sigma^F$ 且两边同剩(－ 1) 得

$$\frac{\partial S}{\partial \dot{q}_\sigma} - \sum_{\beta=1}^{g} \frac{\partial S}{\partial \ddot{q}_{\varepsilon+\beta}}\left[\sum_{\gamma=1}^{g} \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_{\varepsilon+\gamma}} \cdot \frac{\partial \dot{q}_{\varepsilon+\gamma}}{\partial \dot{q}_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right)\right] = \sum_{\beta=1}^{g} Q_{\varepsilon+\beta} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\right)\cdot \tag{44}$$

(44) 即为一阶非线性非完整系速度空间的 D' Alembert 原理·

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D' Alembert Principle in the Velocity Space

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Abstract: According to Newton' s dynamical equation of the system of particles, the force is considered to be the functional relationship of the coordinate $\boldsymbol{r}$, velocity $\dot{\boldsymbol{r}}$ and time t, and the various formulae for D' Alembert principle of the velocity space in both the holonomic and nonholonomic systems are deduced by introducing the concept of kinetic energy in the velocity space (i. e. the accelerated energy).

Key words: Newton dynamical equation; accelerated energy; velocity space; D' Alembert principle