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三角形单元协调与非协调位移的能量正交关系*

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(钟万勰推荐)

摘要: 基于组合稳定化变分原理, 周天孝提出的组合杂交法是绝对收敛和稳定的, 它给出了一种系统化的增强应力/应变方法, 并建立了一簇低阶仿射等价的 n_cube(n = 2, 3) 单元· 本文论证了单元上应力插值为线性, 位移插值为协调线性部分和非协调二次部分之和的三角形组合杂交单元其协调部分与非协调部分的能量正交关系, 进而得到此三角形单元刚度矩阵等同于协调的三角形线性元刚度矩阵, 即非协调部分无应变增强特性·

关 键 词: 组合稳定; 能量正交; 应变增强
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1 组合杂交元格式

将同一泛函乘积空间上相互对偶的两种鞍点型混合/杂交元格式(Primal 和 Dual) 线性组合, 优势互补, 周天孝[1, 2, 3]提出并研究了如下的组合杂交模型

$$\alpha\Pi_R + (1-\alpha)\Pi_N = \text{stationary},$$

其中 α 是组合稳定化系数: $0 < \alpha < 1$, Π_R 和 Π_N 分别为基于区域分解的假设应力杂交法和假设位移杂交法的 Lagrange 泛函:

$$\Pi_R = -\frac{1}{2}a(\tau, \tau) + b_1(\tau, \boldsymbol{v}_C) - \sum_K\left(\int_K \mathrm{div}\,\tau\cdot\boldsymbol{v}_I\mathrm{d}\Omega + \int_K \boldsymbol{f}\cdot\boldsymbol{v}\mathrm{d}\Omega\right),$$

$$\Pi_N = \frac{1}{2}d(\boldsymbol{v}, \boldsymbol{v}) - \sum_K\left(\int_K \boldsymbol{f}\cdot\boldsymbol{v}\mathrm{d}\Omega + \oint_{\partial K}\tau_n\cdot(\boldsymbol{v} - \boldsymbol{v}_C)\mathrm{d}s\right),$$

式中 $K \in \mathscr{T}_h$ 为有限单元, $\mathscr{T}_h$ 为区域 Ω 的正则剖分; 位移 $\boldsymbol{v} = \boldsymbol{v}_C + \boldsymbol{v}_I$, $\boldsymbol{v}_C$ 和 $\boldsymbol{v}_I$ 分别为位移的协调插值和非协调插值部分; 在平面弹性问题中, 记位移矢量 $\boldsymbol{v} = (v_1, v_2)^T$, 应变矢量 $\varepsilon(\boldsymbol{v}) = (\varepsilon_{11}, \varepsilon_{22}, \gamma_{12})^T = \left(\partial v_1/\partial x, \partial v_2/\partial y, \partial v_1/\partial y + \partial v_2/\partial x\right)^T$, 应力矢量 $\tau = (\tau_{11}, \tau_{22}, \tau_{12})^T$, $\boldsymbol{D}$ 为材料弹性矩阵; $\boldsymbol{n}$ 为单元边界外法线单位向量· 于是上两式中能量泛函及有关量可表示为

$$a(\sigma, \tau) = \sum_K\int_K \tau\cdot\boldsymbol{D}^{-1}\sigma\mathrm{d}\Omega,\quad d(\boldsymbol{u}, \boldsymbol{v}) = \sum_K\int_K \varepsilon(\boldsymbol{v})\cdot\boldsymbol{D}\varepsilon(\boldsymbol{u})\mathrm{d}\Omega,$$

$$b_1(\tau, \boldsymbol{v}) = \sum_K\int_K \tau\cdot\varepsilon(\boldsymbol{v})\mathrm{d}\Omega,\quad \tau_{\boldsymbol{n}} = \tau\cdot\boldsymbol{n} = \begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{pmatrix}\begin{pmatrix}n_1\\ n_2\end{pmatrix}·$$

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此混合/杂交元模型不要求应力、位移插值函数选择满足 Babuska_Brezzi 不等式,只需要满足插值逼近条件,无需检验是否通过分片检查,它是绝对收敛和稳定的. 详细理论参阅文献[1,2].

考虑到 Green 公式,我们有

$$\alpha\Pi_R + (1-\alpha)\Pi_N = -\frac{\alpha}{2}a(\tau,\tau) + \frac{1-\alpha}{2}d(\boldsymbol{v},\boldsymbol{v}) + \alpha b_1(\tau,\boldsymbol{v}) - \sum_K\int_K \tau\cdot\varepsilon(\boldsymbol{v}_I)\,\mathrm{d}\Omega - \sum_K\int_K \mathrm{div}\,\tau\cdot\boldsymbol{v}_I\,\mathrm{d}\Omega - (\boldsymbol{f},\boldsymbol{v})_\Omega,$$

经变分后,提出如下的离散问题[1]:

求 $(\sigma,\boldsymbol{u}) = (\sigma,\boldsymbol{u}_C+\boldsymbol{u}_I) \in V_h\times U_h$ 使得

$$\alpha a(\sigma,\tau) - \alpha b_1(\tau,\boldsymbol{u}_C) + b_2(\tau,\boldsymbol{u}_I) = 0 \qquad \forall\tau\in V_h, \tag{1}$$

$$\alpha b_1(\sigma,\boldsymbol{v}_C) - b_2(\sigma,\boldsymbol{v}_I) + (1-\alpha)d(\boldsymbol{u}_I+\boldsymbol{u}_C,\boldsymbol{v}_I+\boldsymbol{v}_C) = (\boldsymbol{f},\boldsymbol{v}_I+\boldsymbol{v}_C) \qquad \forall\boldsymbol{v} = \boldsymbol{v}_I+\boldsymbol{v}_C\in U_h \tag{2}$$

成立,式中 V_h 和 U_h 分别为应力和位移的插值函数空间, 并且有

$$b_2(\tau,\boldsymbol{v}_I) = \sum_K\int_K[(1-\alpha)\tau\cdot\varepsilon(\boldsymbol{v}_I) + \mathrm{div}\,\tau\cdot\boldsymbol{v}_I]\,\mathrm{d}\Omega.$$

在下面的各节中依次给出三角形单元上的位移、应力插值函数, 协调位移部分和非协调位移部分能量正交性的证明,以及该单元和基于势能原理的协调三角形单元的等价性.

2 应力和位移的插值函数

设 $P_i(x_i, y_i)$ $i = 1,2,3$ 为三角形单元按逆时针方向排列的三个顶点, 记 λ_1、λ_2、λ_3 为三角形单元上任一点 $P(x,y)$ 的面积坐标, $\triangle$ 为三角形单元面积. 为推证方便列出直角坐标和面积坐标之间的关系:

$$a_i = x_jy_k - x_ky_j,\quad b_i = y_j - y_k,\quad c_i = x_k - x_j,\quad i,j,k \text{ 轮换},$$

$$\lambda_i = (a_i + b_ix + c_iy)/(2\triangle) \qquad (i = 1,2,3).$$

位移 $\boldsymbol{v} = \boldsymbol{v}_C + \boldsymbol{v}_I$ 的插值函数为

$$\boldsymbol{v}_C = \begin{Bmatrix}\lambda_1 & 0 & \lambda_2 & 0 & \lambda_3 & 0\\ 0 & \lambda_1 & 0 & \lambda_2 & 0 & \lambda_3\end{Bmatrix}\boldsymbol{q}^v_C, \tag{3}$$

$$\boldsymbol{v}_I = \begin{Bmatrix}\lambda_1\lambda_2 & 0 & \lambda_2\lambda_3 & 0 & \lambda_3\lambda_1 & 0\\ 0 & \lambda_1\lambda_2 & 0 & \lambda_2\lambda_3 & 0 & \lambda_3\lambda_1\end{Bmatrix}\boldsymbol{q}^v_I, \tag{4}$$

$\boldsymbol{q}^v_C = (v_1(P_1), v_2(P_1), v_1(P_2), v_2(P_2), v_1(P_3), v_2(P_3))^T$ 和 $\boldsymbol{q}^v_I = (v_1(P_4), v_2(P_4), v_1(P_5), v_2(P_5), v_1(P_6), v_2(P_6))^T$ 分别为三角形顶点和内部结点位移参数.

应力 τ 的插值函数为分片线性多项式

$$\tau = (\lambda_1I_3, \lambda_2I_3, \lambda_3I_3)\boldsymbol{q}^\tau, \tag{5}$$

$\boldsymbol{q}^\tau = (\tau_x(P_1), \tau_y(P_1), \tau_{xy}(P_1), \tau_x(P_2), \tau_y(P_2), \tau_{xy}(P_2), \tau_x(P_3), \tau_y(P_3), \tau_{xy}(P_3))^T$ 为结点应力参数, $\boldsymbol{I}_3$ 为 3×3 的单位矩阵.

定义矩阵

$$\boldsymbol{L}_i = \begin{bmatrix}b_i & 0 & c_i\\ 0 & c_i & b_i\end{bmatrix}^T \qquad i\in\{1,2,3\}.$$

由(3)和(4)式可得到应变 $\varepsilon(\boldsymbol{v}) = \varepsilon(\boldsymbol{v}_C) + \varepsilon(\boldsymbol{v}_I)$,

$$\varepsilon(\boldsymbol{v}_C) = \left[\frac{\partial v_{1C}}{\partial x}, \frac{\partial v_{2C}}{\partial y}, \frac{\partial v_{1C}}{\partial y} + \frac{\partial v_{2C}}{\partial x}\right]^T\boldsymbol{q}^v_C = \frac{1}{2\triangle}[\boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{L}_3]\boldsymbol{q}^v_C, \tag{6}$$

$$\varepsilon(\boldsymbol{v}_{\mathrm{I}}) = \left[\frac{\partial v_{1\mathrm{I}}}{\partial x}, \frac{\partial v_{2\mathrm{I}}}{\partial y}, \frac{\partial v_{1\mathrm{I}}}{\partial y} + \frac{\partial v_{2\mathrm{I}}}{\partial x}\right]^{\mathrm{T}} \boldsymbol{q}_{\mathrm{I}}^{v} = \frac{1}{2\triangle}[\lambda_1 \boldsymbol{L}_2 + \lambda_2 \boldsymbol{L}_1, \lambda_2 \boldsymbol{L}_3 + \lambda_3 \boldsymbol{L}_2, \lambda_3 \boldsymbol{L}_1 + \lambda_1 \boldsymbol{L}_3]\boldsymbol{q}_{\mathrm{I}}^{v}, \tag{7}$$

和应力的散度

$$\mathrm{div}\,\tau = \begin{bmatrix} \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} \\ \frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} \end{bmatrix} = \frac{1}{2\triangle}[\boldsymbol{L}_1^{\mathrm{T}}, \boldsymbol{L}_2^{\mathrm{T}}, \boldsymbol{L}_3^{\mathrm{T}}]\boldsymbol{q}^{\tau}. \tag{8}$$

3 能量正交性的证明

由于应力 τ 的插值函数为分片线性多项式, 因此在每个单元 K 上可以由(1) 式解出 σ, 即用位移 $\boldsymbol{u}_{\mathrm{C}}$ 和 $\boldsymbol{u}_{\mathrm{I}}$ 表示应力 σ, 其次将 σ 的表达式带入(2) 式, 进而得到单元刚度矩阵.

在单元 K 上由(1) 式可得到

$$a_K(\sigma, \tau) = b_{1K}(\tau, \boldsymbol{u}_{\mathrm{C}}) - \frac{1}{\alpha} b_{2K}(\tau, \boldsymbol{u}_{\mathrm{I}}), \tag{9}$$

将(5) 式代入 $a_K(\sigma, \tau)$, 应用三角形单元上的面积坐标积分公式得

$$a_K(\sigma, \tau) = \int_K \tau \cdot \boldsymbol{D}^{-1} \sigma \,\mathrm{d}\Omega = [\boldsymbol{q}^{\tau}]^{\mathrm{T}} \boldsymbol{M}_{\sigma} \boldsymbol{q}^{\sigma}, \tag{10}$$

其中

$$\boldsymbol{M}_{\sigma} = \frac{\triangle}{12}\begin{bmatrix} 2\boldsymbol{D}^{-1} & \boldsymbol{D}^{-1} & \boldsymbol{D}^{-1} \\ \boldsymbol{D}^{-1} & 2\boldsymbol{D}^{-1} & \boldsymbol{D}^{-1} \\ \boldsymbol{D}^{-1} & \boldsymbol{D}^{-1} & 2\boldsymbol{D}^{-1} \end{bmatrix}, \quad \boldsymbol{M}_{\sigma}^{-1} = \frac{3}{\triangle}\begin{bmatrix} 3\boldsymbol{D} & -\boldsymbol{D} & -\boldsymbol{D} \\ -\boldsymbol{D} & 3\boldsymbol{D} & -\boldsymbol{D} \\ -\boldsymbol{D} & -\boldsymbol{D} & 3\boldsymbol{D} \end{bmatrix}.$$

把(3) 式和(5) 式代入 $b_{1K}(\tau, \boldsymbol{u}_{\mathrm{C}})$ 得

$$b_{1K}(\tau, \boldsymbol{u}_{\mathrm{C}}) = \int_K \tau \cdot \varepsilon(\boldsymbol{u}_{\mathrm{C}}) \,\mathrm{d}\Omega = [\boldsymbol{q}^{\tau}]^{\mathrm{T}} \boldsymbol{M}_{\mathrm{C}} \boldsymbol{q}_{\mathrm{C}}^{u}, \tag{11}$$

式中

$$\boldsymbol{M}_{\mathrm{C}} = \frac{1}{6}\begin{bmatrix} \boldsymbol{L}_1 & \boldsymbol{L}_2 & \boldsymbol{L}_3 \\ \boldsymbol{L}_1 & \boldsymbol{L}_2 & \boldsymbol{L}_3 \\ \boldsymbol{L}_1 & \boldsymbol{L}_2 & \boldsymbol{L}_3 \end{bmatrix}.$$

类似地, 我们有

$$b_{2K}(\tau, \boldsymbol{u}_{\mathrm{I}}) = \int_K [(1-\alpha)\tau \cdot \varepsilon(\boldsymbol{u}_{\mathrm{I}}) + \mathrm{div}(\tau) \cdot \boldsymbol{u}_{\mathrm{I}}] \,\mathrm{d}\Omega = [\boldsymbol{q}^{\tau}]^{\mathrm{T}}[\boldsymbol{M}_{\mathrm{I}1} + \boldsymbol{M}_{\mathrm{I}2}]\boldsymbol{q}_{\mathrm{I}}^{u}, \tag{12}$$

式中

$$\boldsymbol{M}_{\mathrm{I}1} = \frac{1-\alpha}{24}\begin{bmatrix} \boldsymbol{L}_1 + 2\boldsymbol{L}_2 & \boldsymbol{L}_2 + \boldsymbol{L}_3 & 2\boldsymbol{L}_3 + \boldsymbol{L}_1 \\ 2\boldsymbol{L}_1 + \boldsymbol{L}_2 & \boldsymbol{L}_2 + 2\boldsymbol{L}_3 & \boldsymbol{L}_3 + \boldsymbol{L}_1 \\ \boldsymbol{L}_1 + \boldsymbol{L}_2 & 2\boldsymbol{L}_2 + \boldsymbol{L}_3 & \boldsymbol{L}_3 + 2\boldsymbol{L}_1 \end{bmatrix},$$

$$\boldsymbol{M}_{\mathrm{I}2} = \frac{1}{24}\begin{bmatrix} \boldsymbol{L}_1 & \boldsymbol{L}_1 & \boldsymbol{L}_1 \\ \boldsymbol{L}_2 & \boldsymbol{L}_2 & \boldsymbol{L}_2 \\ \boldsymbol{L}_3 & \boldsymbol{L}_3 & \boldsymbol{L}_3 \end{bmatrix}.$$

把(10)~(12) 式代入(9) 式, 由 τ 的任意性知

$$\boldsymbol{q}^{\sigma} = \boldsymbol{M}_{\sigma}^{-1}\boldsymbol{M}_{\mathrm{C}}\boldsymbol{q}_{\mathrm{C}}^{u} - \frac{1}{\alpha}\boldsymbol{M}_{\sigma}^{-1}(\boldsymbol{M}_{\mathrm{I}1} + \boldsymbol{M}_{\mathrm{I}2})\boldsymbol{q}_{\mathrm{I}}^{u}, \tag{13}$$

到此我们得到了用 $\boldsymbol{q}_{\mathrm{C}}^{u}$ 和 $\boldsymbol{q}_{\mathrm{I}}^{u}$ 表示 $\boldsymbol{q}^{\sigma}$ 的具体形式• 为推导方便将上式改写为矩阵形式

$$\boldsymbol{q}^{\sigma}=\boldsymbol{M}_{\sigma}^{-1}\left[\boldsymbol{M}_{\mathrm{C}},-\frac{1}{\alpha}(\boldsymbol{M}_{\mathrm{I}1}+\boldsymbol{M}_{\mathrm{I}2})\right]\begin{bmatrix}\boldsymbol{q}_{\mathrm{C}}^{u}\\ \boldsymbol{q}_{\mathrm{I}}^{u}\end{bmatrix}\text{•} \tag{13'}$$

下面我们推导出单元刚度矩阵• 类似于(11)式和(12)式的推证过程，有

$$\alpha b_{1K}(\sigma,\boldsymbol{v}_{\mathrm{C}})-b_{2K}(\sigma,\boldsymbol{v}_{\mathrm{I}})=\alpha\begin{bmatrix}\boldsymbol{q}_{\mathrm{C}}^{v}\\ \boldsymbol{q}_{\mathrm{I}}^{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{M}_{\mathrm{C}}^{\mathrm{T}}\\ -\dfrac{1}{\alpha}(\boldsymbol{M}_{\mathrm{I}1}^{\mathrm{T}}+\boldsymbol{M}_{\mathrm{I}2}^{\mathrm{T}})\end{bmatrix}\boldsymbol{q}^{\sigma}\text{•} \tag{14}$$

把(13)′式代入上式，得

$$\begin{aligned}&\alpha b_{1K}(\sigma,\boldsymbol{v}_{\mathrm{C}})-b_{2K}(\sigma,\boldsymbol{v}_{\mathrm{I}})=\\&\begin{bmatrix}\boldsymbol{q}_{\mathrm{C}}^{v}\\ \boldsymbol{q}_{\mathrm{I}}^{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\alpha\boldsymbol{M}_{\mathrm{C}}^{T}\boldsymbol{M}_{\sigma}^{-1}\boldsymbol{M}_{\mathrm{C}} & -\boldsymbol{M}_{\mathrm{C}}^{T}\boldsymbol{M}_{\sigma}^{-1}(\boldsymbol{M}_{\mathrm{I}1}+\boldsymbol{M}_{\mathrm{I}2})\\ -(\boldsymbol{M}_{\mathrm{I}1}^{\mathrm{T}}+\boldsymbol{M}_{\mathrm{I}2}^{\mathrm{T}})\boldsymbol{M}_{\sigma}^{-1}\boldsymbol{M}_{\mathrm{C}} & \dfrac{1}{\alpha}(\boldsymbol{M}_{\mathrm{I}1}^{\mathrm{T}}+\boldsymbol{M}_{\mathrm{I}2}^{\mathrm{T}})\boldsymbol{M}_{\sigma}^{-1}(\boldsymbol{M}_{\mathrm{I}1}+\boldsymbol{M}_{\mathrm{I}2})\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_{\mathrm{C}}^{u}\\ \boldsymbol{q}_{\mathrm{I}}^{u}\end{bmatrix}\triangleq\\&\begin{bmatrix}\boldsymbol{q}_{\mathrm{C}}^{v}\\ \boldsymbol{q}_{\mathrm{I}}^{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{D}_{11}^{(1)} & \boldsymbol{D}_{12}^{(1)}\\ \boldsymbol{D}_{21}^{(1)} & \boldsymbol{D}_{22}^{(1)}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_{\mathrm{C}}^{u}\\ \boldsymbol{q}_{\mathrm{I}}^{u}\end{bmatrix},\end{aligned} \tag{15}$$

经简单的矩阵运算得到

$$\boldsymbol{D}_{11}^{(1)}=\alpha\boldsymbol{M}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{M}_{\sigma}^{-1}\boldsymbol{M}_{\mathrm{C}}=\frac{\alpha}{4\triangle}\begin{bmatrix}\boldsymbol{L}_{11} & \boldsymbol{L}_{12} & \boldsymbol{L}_{13}\\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} & \boldsymbol{L}_{23}\\ \boldsymbol{L}_{31} & \boldsymbol{L}_{32} & \boldsymbol{L}_{33}\end{bmatrix}, \tag{16}$$

$$\boldsymbol{D}_{12}^{(1)}=-\boldsymbol{M}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{M}_{\sigma}^{-1}\boldsymbol{M}_{\mathrm{I}1}-\boldsymbol{M}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{M}_{\sigma}^{-1}\boldsymbol{M}_{\mathrm{I}2}=\frac{1-\alpha}{12\triangle}\begin{bmatrix}\boldsymbol{L}_{13} & \boldsymbol{L}_{11} & \boldsymbol{L}_{12}\\ \boldsymbol{L}_{23} & \boldsymbol{L}_{21} & \boldsymbol{L}_{22}\\ \boldsymbol{L}_{33} & \boldsymbol{L}_{31} & \boldsymbol{L}_{32}\end{bmatrix}+[0]_{6\times6}\text{•} \tag{17}$$

在上两式中 $\boldsymbol{L}_{ij}\triangleq\boldsymbol{L}_i^{\mathrm{T}}\boldsymbol{D}\boldsymbol{L}_j$ 是 2×2 的矩阵，i 和 j 属于集合 $\{1,2,3\}$• 在(17)式的推导过程中用到了关系式 $\boldsymbol{L}_1+\boldsymbol{L}_2+\boldsymbol{L}_3=[0]_{3\times2}$，这是因为 $b_1+b_2+b_3=0$ 和 $c_1+c_2+c_3=0$ 成立•

(17)式中的第二部分为零矩阵，即

$$\boldsymbol{M}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{M}_{\sigma}^{-1}\boldsymbol{M}_{\mathrm{I}2}=[0]\text{•} \tag{18}$$

上式即说明了位移插值函数的协调插值部分和非协调插值部分关于分片线性应力插值函数是能量正交的• 此式更为深刻的力学含义仍然保持开放，对它的剖析将影响到对增强精度格式概念的更为深刻的理解•

为了得到此单元和协调的三角形线性单元的等价性，我们还需计算出 $d(\boldsymbol{u}_{\mathrm{I}}+\boldsymbol{u}_{\mathrm{C}},\boldsymbol{v}_{\mathrm{I}}+\boldsymbol{v}_{\mathrm{C}})$•

$$\begin{aligned}d_K(\boldsymbol{u},\boldsymbol{v})=&\int_K\varepsilon(\boldsymbol{v})\cdot\boldsymbol{D}\varepsilon(\boldsymbol{u})\,\mathrm{d}\Omega=\\&\int_K[\varepsilon(\boldsymbol{v}_{\mathrm{I}})+\varepsilon(\boldsymbol{v}_{\mathrm{C}})]\cdot\boldsymbol{D}[\varepsilon(\boldsymbol{u}_{\mathrm{I}})+\varepsilon(\boldsymbol{u}_{\mathrm{C}})]\,\mathrm{d}\Omega\triangleq\\&\begin{bmatrix}\boldsymbol{q}_{\mathrm{C}}^{v}\\ \boldsymbol{q}_{\mathrm{I}}^{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{D}_{11}^{(2)} & \boldsymbol{D}_{12}^{(2)}\\ \boldsymbol{D}_{21}^{(2)} & \boldsymbol{D}_{22}^{(2)}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_{\mathrm{C}}^{u}\\ \boldsymbol{q}_{\mathrm{I}}^{u}\end{bmatrix},\end{aligned} \tag{19}$$

其中

$$\begin{aligned}[\boldsymbol{q}_{\mathrm{C}}^{v}]^{\mathrm{T}}\boldsymbol{D}_{11}^{(2)}\boldsymbol{q}_{\mathrm{C}}^{u}=&\int_K\varepsilon(\boldsymbol{v}_{\mathrm{C}})\cdot\boldsymbol{D}\varepsilon(\boldsymbol{u}_{\mathrm{C}})\,\mathrm{d}\Omega=\\&[\boldsymbol{q}_{\mathrm{C}}^{v}]^{\mathrm{T}}\frac{1}{4\triangle}\begin{bmatrix}\boldsymbol{L}_{11} & \boldsymbol{L}_{12} & \boldsymbol{L}_{13}\\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} & \boldsymbol{L}_{23}\\ \boldsymbol{L}_{31} & \boldsymbol{L}_{32} & \boldsymbol{L}_{33}\end{bmatrix}\boldsymbol{q}_{\mathrm{C}}^{u},\end{aligned} \tag{20}$$

$$[\boldsymbol{q}_C^v]^T \boldsymbol{D}_{12}^{(2)} \boldsymbol{q}_I^u = \int_K \varepsilon(\boldsymbol{v}_C) \cdot \boldsymbol{D} \varepsilon(\boldsymbol{u}_I) \, d\Omega = [\boldsymbol{q}_C^v]^T \frac{-1}{12\triangle} \begin{bmatrix} \boldsymbol{L}_{13} & \boldsymbol{L}_{11} & \boldsymbol{L}_{12} \\ \boldsymbol{L}_{23} & \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \\ \boldsymbol{L}_{33} & \boldsymbol{L}_{31} & \boldsymbol{L}_{32} \end{bmatrix} \boldsymbol{q}_I^u . \tag{21}$$

在(21)式的推理中用到了关系式 $\boldsymbol{L}_{i1} + \boldsymbol{L}_{i2} + \boldsymbol{L}_{i3} = [0]_{2\times 2}$.

现在将(15)~(21)式代入(2)式左端

$$\text{left} = \alpha b_{1K}(\sigma, \boldsymbol{v}_C) - b_{2K}(\sigma, \boldsymbol{v}_I) + (1-\alpha) d(\boldsymbol{u}_I + \boldsymbol{u}_C, \boldsymbol{v}_I + \boldsymbol{v}_C) \triangleq \begin{bmatrix} \boldsymbol{q}_C^v \\ \boldsymbol{q}_I^v \end{bmatrix}^T \begin{bmatrix} \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_C^u \\ \boldsymbol{q}_I^u \end{bmatrix}, \tag{22}$$

式中

$$\boldsymbol{D}_{ij} = \boldsymbol{D}_{ij}^{(1)} + \boldsymbol{D}_{ij}^{(2)}, \qquad i, j \in \{1, 2\},$$

$$\boldsymbol{D}_{11} = \frac{1}{4\triangle} \begin{bmatrix} \boldsymbol{L}_{11} & \boldsymbol{L}_{12} & \boldsymbol{L}_{13} \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} & \boldsymbol{L}_{23} \\ \boldsymbol{L}_{31} & \boldsymbol{L}_{32} & \boldsymbol{L}_{33} \end{bmatrix}, \tag{23}$$

$$\boldsymbol{D}_{12} = [0]_{6\times 6} . \tag{24}$$

考虑单元刚阵的对称性, 有

$$\text{left} = \begin{bmatrix} \boldsymbol{q}_C^v \\ \boldsymbol{q}_I^v \end{bmatrix}^T \begin{bmatrix} \boldsymbol{D}_{11} & 0 \\ 0 & \boldsymbol{D}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_C^u \\ \boldsymbol{q}_I^u \end{bmatrix} . \tag{25}$$

(25)式表明内部自由度和三角形单元顶点自由度无耦合, 静力凝聚内部自由度后, $\boldsymbol{D}_{11}$ 保持不变, 仍为协调的线性单元的刚度矩阵. 至此, 我们证明了应力插值为分片线性多项式, 位移插值函数为协调的线性部分和非协调的二次部分的三角形组合杂交元等价于基于最小势能原理的协调线性三角形单元, 非协调位移插值部分无应变增强特性. 上述结果产生的关键在于(18)式和关系式 $\boldsymbol{D}_{12}^{(1)} + (1-\alpha)\boldsymbol{D}_{12}^{(2)} = [0]_{6\times 6}$ 成立, 要构造三角形单元增强精度格式需着眼于突破这两个恒等关系.

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The Energy Orthogonal Relation Between Conforming and Non_Conforming Displacements of Triangular Element

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Abstract: Based on the variational principle of combinative stability, combined hybrid methods posed by Zhou Tianxiao are absolutely convergent and stabilized. Zhou has advocated a systematic approach to enhanced stress/strain schemes and has designed a family of lower_order elements which are affine_equivalent to n_cube($n = 2, 3$). The energy orthogonal relation between the conforming part and the non_conforming part of displacements interpolation functions in triangular element is given, in which the stress is interpolated by linear polynomials on each element, but the displacements are interpolated by the sum of conforming linear and non_conforming quadratic polynomials. Furthermore, this element is equivalent to the conforming triangular linear element, that is, the non_conforming parts have no contribution to enhanced strains.

Key words: combinative stability; energy orthogonality; enhanced strain