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多重尺度法在圆薄板大挠度弯曲问题中的应用及其渐近性研究(Ⅱ)*

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摘要: 本文是第(Ⅰ)部分的继续, 研究(Ⅰ)中级数解的渐近性质• 求得了级数解的余项按最大模的估计•

关 键 词: 大挠度弯曲; 修正多重尺度法; 渐近性质
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引 言

在(Ⅰ)中已就简支圆薄板和简单支承圆薄板作出了形式上的精确到 $O(\varepsilon^N)$ 的级数解[1], 其中 N 是任意正整数, 本文将证明这些形式级数解是渐近的, 对它们的余项作出按最大模的估计• 所用方法是经典的, 即先导出余项的控制方程和边界条件, 再转化这些边值问题为与它们等价的积分方程, 再应用不动点定理求得余项的估计•

1 形式渐近解和余项

先考察简支板, 它的挠度和应力函数确定于边值问题(采用(Ⅰ)的量纲一的形式):

$$\begin{cases} \varepsilon^2 \mathrm{L}_1 w - \dfrac{1}{r}\dfrac{\mathrm{d}w}{\mathrm{d}r}\phi - \dfrac{qr}{2} = 0, & (1) \\ \mathrm{L}_2\phi + \dfrac{1}{2r}\left(\dfrac{\mathrm{d}w}{\mathrm{d}r}\right)^2 = 0, & (2) \\ w\,|_{r=1} = 0,\ \left(\dfrac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \dfrac{\sigma}{r}\dfrac{\mathrm{d}w}{\mathrm{d}r}\right)\Big|_{r=1} = 0,\ \left(\dfrac{\mathrm{d}\phi}{\mathrm{d}r} - \dfrac{\sigma}{r}\phi\right)\Big|_{r=1} = 0, & (3) \\ \lim\limits_{r\to 0}(\mathrm{d}w/\mathrm{d}r) < \infty,\ \lim\limits_{r\to 0}(\phi/r) < \infty, & (4) \end{cases}$$

其中 $\mathrm{L}_1 = \dfrac{\mathrm{d}^3}{\mathrm{d}r^3} + \dfrac{1}{r}\dfrac{\mathrm{d}^2}{\mathrm{d}r^2} - \dfrac{1}{r^2}\dfrac{\mathrm{d}}{\mathrm{d}r},\ \mathrm{L}_2 = \dfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r} - \dfrac{1}{r^2}$•

在(Ⅰ)中已求得级数解:

$$W_N = \sum_{n=0}^{N}\left[\varepsilon^n w_n(r) + \varepsilon^{(n+2)} v_n(\xi, \eta)\right], \tag{5}$$

$$\Phi_N = \sum_{n=0}^{N}\left[\varepsilon^n \phi_n(r) + \varepsilon^{(n+3)} h_n(\xi, \eta)\right], \tag{6}$$

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到 w_n, ϕ_n($n=0,1,2,\cdots,N$) 所满足的控制方程, 和 $\exp(-\xi)=\exp(-u(r)/\varepsilon)$ 是 ε 的任意阶小量, 知道(7)~(10) 式也成立。

我们称边值问题(1)~(4) 的精确解与形式渐近解之差为形式渐近解的余项:

$$R_N = w - W_N,\quad Z_N = \phi - \Phi_N。\tag{11}$$

将(11) 式代入边值问题(1)~(4), 得到关于余项的边值问题:

$$\begin{cases} \varepsilon^2 L_1 R_N - \dfrac{1}{r}\left(\dfrac{dW_N}{dr}Z_N + \dfrac{dR_N}{dr}\Phi_N + \dfrac{dR_N}{dr}Z_N\right) = -\varepsilon^{N+3} g(r,\varepsilon), & (12)\\ L_2 Z_N + \dfrac{1}{r}\left(\dfrac{dW_N}{dr}\dfrac{dR_N}{dr}\right) + \dfrac{1}{2r}\left(\dfrac{dR_N}{dr}\right)^2 = -\varepsilon^{N+1} h(r,\varepsilon), & (13)\\ R_N\big|_{r=1} = -\varepsilon^{N+1}c_1(\varepsilon),\quad \left(\dfrac{d^2R_N}{dr^2} + \dfrac{\sigma}{r}\dfrac{dR_N}{dr}\right)\Bigg|_{r=1} = -\varepsilon^{N+1}c_2(\varepsilon), & \\ \left(dZ_N/dr - \sigma Z_N/r\right)\big|_{r=1} = -\varepsilon^{N+1}c_3(\varepsilon), & (14)\\ \lim\limits_{r\to 0}(dR_N/dr) < \infty,\quad \lim\limits_{r\to 0}(Z_N/r) < \infty。 & (15) \end{cases}$$

下面再导出与边值问题(12)~(15) 等价的积分方程组。

2　积分方程组

记　$dR_N/dr = \varepsilon^{N+1}R_N$,　$Z_N = \varepsilon^{N+1}Z_N$,

方程(12) 可写成

$$\frac{d^2R_N}{dr^2} + \frac{1}{r}\frac{dR_N}{dr} - \frac{1}{r^2}R_N = \frac{1}{\varepsilon^2}\left[F(R_N, Z_N, \varepsilon) - \varepsilon^2 g(r,\varepsilon)\right],\tag{16}$$

其中

$$F(R_N, Z_N, \varepsilon) = \frac{1}{r}\left(\frac{dW_N}{dr}Z_N + R_N\Phi_N + \varepsilon^{N+1}R_N Z_N\right)。\tag{17}$$

将方程(16) 积分两次, 并考虑到边值条件(14)、(15) 得

$$R_N(r,\varepsilon) = r\int_1^r \frac{1}{t^3}\left\{\int_1^t \frac{s^2}{\varepsilon^2}\left[F(R_N, Z_N, \varepsilon) - \varepsilon^2 g(s,\varepsilon)\right]ds\right\}dt,$$

经分部积分得

$$R_N(r,\varepsilon) = \frac{1}{2\varepsilon^2}\int_1^r\left(r - \frac{t^2}{r}\right)\left[F(R_N, Z_N, \varepsilon) - \varepsilon^2 g(t,\varepsilon)\right]dt。\tag{18}$$

类似地, 将方程(13) 积分两次, 再考虑到边界条件得

$$Z_N(r,\varepsilon) = \frac{1}{2}\int_1^r\left(r - \frac{t^2}{r}\right)\left[G(R_N,\varepsilon) - h(t,\varepsilon)\right]dt,\tag{19}$$

其中

$$G(R_N,\varepsilon) = \frac{-1}{2r}\left(2\frac{dW_N}{dr}R_N + \varepsilon^{N+1}R_N^2\right)。\tag{20}$$

(18) 和(19) 式就是与边值问题(12)~(14) 等价的积分方程组。

下面再应用 Banach 不动点定理证明非线性积分方程组(18)~(19) 解的存在性, 并求出余项的估计式。

3　余项的估计式

为简单起见, 采用矩阵记号

其中 $w_n, \phi_n, v_n, h_n\ (n=0,1,\cdots,N)$ 确定于下面的递推边值问题:

$$
\begin{cases}
\dfrac{1}{r}\dfrac{dw_0}{dr}\phi_0 + \dfrac{qr}{2} = 0,\quad L_2\phi_0 + \dfrac{1}{2r}\left(\dfrac{dw_0}{dr}\right)^2 = 0, \\
M_0 v_0 - \dfrac{\phi_0}{\eta}\delta_r^{(0)} v_0 = 0,\quad N_0 h_0 + \dfrac{1}{2\eta}\left(2\dfrac{dw_0}{dr}\delta_r^{(0)} v_0\right) = 0, \\
w_0|_{r=1} = 0,\quad \left(\dfrac{d\phi_0}{dr} - \sigma\phi_0\right)\Big|_{r=1} = 0,\quad \delta_r^{(0)2} v_0|_{\xi=0,\eta=1} + \left(\dfrac{d^2w_0}{dr^2} + \sigma\dfrac{dw_0}{dr}\right)\Big|_{r=1} = 0, \\
\lim\limits_{r\to 0}(dw_0/dr) < \infty,\quad \lim\limits_{r\to 0}(\phi_0/r) < \infty,\quad \lim\limits_{\xi\to\infty} v_0 = 0,\quad \lim\limits_{\xi\to\infty} h_0 = 0;
\end{cases}
$$

$$
\begin{cases}
\dfrac{dw_0}{dr}\phi_i + \dfrac{dw_i}{dr}\phi_0 = rL_1 w_{i-2} - \sum\limits_{j=1}^{i-1}\dfrac{dw_j}{dr}\phi_{i-j}, \\
L_2\phi_i - \dfrac{1}{r}\dfrac{dw_0}{dr}\dfrac{dw_i}{dr} + \dfrac{1}{2r}\sum\limits_{j=1}^{i-1}\dfrac{dw_j}{dr}\dfrac{dw_{i-j}}{dr} = 0, \\
M_0 v_i - \dfrac{\phi_0}{\eta}\delta_r^{(0)} v_i + \sum\limits_{j=1}^{3} M_j v_{i-j} - \dfrac{1}{\eta}\Big[\phi_i\delta_r^{(0)} v_0 + \sum\limits_{j=0}^{i}\phi_{i-j}\delta_r^{(1)} v_{j-1} + \\
\qquad \sum\limits_{j=0}^{i-2}\left(\phi_{i-j-1}\delta_r^{(0)} v_{j+1} + \dfrac{dw_j}{dr}h_{i-2-j} + \sum\limits_{k=0}^{1}\delta_r^{(k)} v_{j-k} h_{i-3-j}\right)\Big] = 0, \\
N_0 h_i + \sum\limits_{j=1}^{2} N_j h_{i-j} + \dfrac{1}{2\eta}\sum\limits_{j=0}^{i}\sum\limits_{k=0}^{1}\left(2\dfrac{dw_j}{dr}\delta_r^{(k)} v_{i-j-k} + \sum\limits_{m=0}^{1}\delta_r^{(k)} v_{j-k}\delta_r^{(m)} v_{i-j-m-1}\right) = 0, \\
w_i|_{r=1} + v_{i-2}|_{\xi=0,\eta=1} = 0,\quad \left(\dfrac{d\phi_i}{dr} - \sigma\phi_i\right)\Big|_{r=1} + \left(\sum\limits_{k=0}^{1}\delta_r^{(k)} h_{i-k-2} + \sigma h_{i-3}\right)\Big|_{\xi=0,\eta=1} = 0, \\
\left(\sum\limits_{k=0}^{2}\delta_r^{(k)2} v_{i-k} + \dfrac{1}{\eta}\sum\limits_{m=0}^{1}\delta_r^{(m)} v_{i-m-1}\right)\Big|_{\xi=0,\eta=1} + \left(\dfrac{d^2w_i}{dr^2} + \sigma\dfrac{dw_i}{dr}\right)\Big|_{r=1} = 0, \\
\lim\limits_{r\to 0}(dw_i/dr) < \infty,\quad \lim\limits_{r\to 0}(\phi_i/r) < \infty,\quad \lim\limits_{\xi\to\infty} v_i = 0,\quad \lim\limits_{\xi\to\infty} h_i = 0, \\
\qquad\qquad (i = 1, 2, \cdots, N),
\end{cases}
$$

参见文[1]的(4.10a)~(4.12f),其中 $v_n = \psi(r)v_n,\ h_n = \psi(r)h_n,\ \psi(r)$ 是在 $0\leqslant r\leqslant 1/3$ 取值0, 在 $2/3\leqslant r\leqslant 1$ 取值1的无限次可微的截断函数• 容易验证 W_N 和 Φ_N 是边值问题(1)~(4)的形式渐近解, 即成立

$$
\begin{cases}
\varepsilon^2 L_1 W_N - \dfrac{1}{r}\dfrac{dW_N}{dr}\Phi_N - \dfrac{qr}{2} = \varepsilon^{N+3} g(r,\varepsilon), & (7) \\
L_2\Phi_N + \dfrac{1}{2r}\left(\dfrac{dW_N}{dr}\right)^2 = \varepsilon^{N+1} h(r,\varepsilon), & (8) \\
W_N|_{r=1} = \varepsilon^{N+1} c_1(\varepsilon),\quad \left(\dfrac{d^2W_N}{dr^2} + \dfrac{\sigma}{r}\dfrac{dW_N}{dr}\right)\Big|_{r=1} = \varepsilon^{N+1} c_2(\varepsilon), \\
(d\Phi_N/dr - \sigma\Phi_N/r)|_{r=1} = \varepsilon^{N+1} c_3(\varepsilon), & (9) \\
\lim\limits_{r\to 0}(dW_N/dr) < \infty,\quad \lim\limits_{r\to 0}(\Phi_N/r) < \infty, & (10)
\end{cases}
$$

其中 $g(r,\varepsilon), h(r,\varepsilon)$ $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon)$ 是 r 和 ε 在 $0\leqslant r\leqslant 1,\ 0<\varepsilon\leqslant\varepsilon_0$ 上的一致有界函数• 事实上, 当 $0\leqslant r\leqslant 1/3$ 时, 考虑到 $w_n, \phi_n\ (n=0,1,\cdots,N)$ 所满足的控制方程和边界条件, 立刻知道关系式(7)~(10)成立; 当 $2/3\leqslant r\leqslant 1$ 时, 考虑到 $w_n, \phi_n, v_n, h_n\ (n=0,1,2,\cdots,N)$ 所满足的控制方程和边界条件也立刻知道关系式(7)~(10)成立; 当 $1/3\leqslant r\leqslant 2/3$ 时, 考虑

$$\boldsymbol{R}_N^* = \begin{pmatrix} R_N \\ Z_N \end{pmatrix},\ \boldsymbol{I}^* = \begin{pmatrix} I_0 \\ J_0 \end{pmatrix},\ \boldsymbol{K}^* = \begin{pmatrix} K_1 & K_2 \\ K_3 & 0 \end{pmatrix},\ \boldsymbol{M}^* = \begin{pmatrix} P \\ Q \end{pmatrix},$$

其中

$$I_0(r, \varepsilon) = \frac{1}{2}\int_1^r \left(r - \frac{t^2}{r}\right) g(t, \varepsilon)\mathrm{d}t,\ J_0(r, \varepsilon) = \frac{1}{2}\int_1^r \left(r - \frac{t^2}{r}\right) h(t, \varepsilon)\mathrm{d}t,$$

$$K_1 = \frac{1}{2\varepsilon^2}\left(r - \frac{t^2}{r}\right)\frac{1}{r}\Phi_N,\ K_2 = \frac{1}{2\varepsilon^2}\left(r - \frac{t^2}{r}\right)\frac{1}{r}\frac{\mathrm{d}W_N}{\mathrm{d}r},$$

$$K_3 = \frac{1}{2}\left(r - \frac{t^2}{r}\right)\frac{-1}{r}\frac{\mathrm{d}W_N}{\mathrm{d}r},\ P = \frac{1}{2\varepsilon^2}\int_1^r \left(r - \frac{t^2}{r}\right)\frac{1}{r}R_N Z_N\mathrm{d}t,$$

$$Q = \frac{1}{2}\int_1^r \left(r - \frac{t^2}{r}\right)\frac{-1}{2r}R_N^2\mathrm{d}t.$$

积分方程组(18)、(19)可写成

$$\boldsymbol{R}_N^* = \int_1^r \boldsymbol{K}^*(r, t, \varepsilon)\boldsymbol{R}_N^*(t, \varepsilon)\mathrm{d}t + \boldsymbol{I}_0^*(r, \varepsilon) + \varepsilon^{N+1}\boldsymbol{M}^*(R_N, Z_N, \varepsilon). \tag{21}$$

这是第二种Volterra积分方程. 由于核中所含的函数 $r^{-1}(r - t^2 r^{-1})$ 在 $0 \leqslant t \leqslant r, 0 \leqslant r \leqslant 1$ 连续, 所以其解核 $\Gamma(r, t, \varepsilon)$ 存在. 从(21)式有

$$\boldsymbol{R}_N^* = R_N^{(0)} + \varepsilon^{N+1}\boldsymbol{M}^*(R_N, Z_n, r, \varepsilon) + \int_1^r \Gamma(r, t, \varepsilon)\varepsilon^{N+1}\boldsymbol{M}^*\mathrm{d}t, \tag{22}$$

其中

$$R_N^{(0)} = \boldsymbol{I}_0^*(r, \varepsilon) + \int_1^r \Gamma(r, t, \varepsilon)\boldsymbol{I}_0^*(t, \varepsilon)\mathrm{d}t. \tag{23}$$

(22)式的右端在连续函数空间 $C[0, 1]$ 定义了一个算子 $\mathrm{T}(\Psi)$:

$$\mathrm{T}(\Psi) = R_N^{(0)} + \varepsilon^{N+1}\boldsymbol{M}^*(\xi, \eta, r, \varepsilon) + \int_1^r \Gamma(r, t, \varepsilon)\varepsilon^{N+1}\boldsymbol{M}^*\mathrm{d}t, \tag{24}$$

其中 $\Psi = \begin{pmatrix} \xi(r) \\ \eta(r) \end{pmatrix}$. 在空间 $C[0, 1]$ 定义模为

$$\|\Psi\| = \sup_{0 \leqslant r \leqslant 1, 0 < \varepsilon < \varepsilon_0}\{|\xi(r)|, |\eta(r)|\}. \tag{25}$$

可以证明算子 $\mathrm{T}(\Psi)$ 在 $C[0, 1]$ 中的某个子集 $\mathrm{B}(\rho)$:

$$\mathrm{B}(\rho) = \{\Psi \in C[0, 1] \mid \|\Psi\| \leqslant \rho\},\ \rho = 2\|R_N^{(0)}\|, \tag{26}$$

是一压缩算子, 和映象自身到自身.

因对于 $\mathrm{B}(\rho)$ 中的任两元素 Ψ_1, Ψ_2 有

$$\varepsilon^{N+1}[P(\xi_1, \eta_1) - P(\xi_2, \eta_2)] = \frac{\varepsilon^{N-1}}{2}\int_1^r \left(r - \frac{t^2}{r}\right)\frac{1}{r}(\xi_1\eta_1 - \xi_2\eta_2)\mathrm{d}t =$$

$$\frac{1}{2}\varepsilon^{N-1}\int_1^r \left(r - \frac{t^2}{r}\right)\frac{1}{r}(\xi_1(\eta_1 - \eta_2) + \eta_2(\xi_1 - \xi_2))\mathrm{d}t$$

和

$$\varepsilon^{N+1}[Q(\xi_1, \eta_1) - Q(\xi_2, \eta_2)] = \frac{1}{2}\varepsilon^{N+1}\int_1^r \left(r - \frac{t^2}{r}\right)\frac{-1}{2r}(\xi_1 + \xi_2)(\xi_1 - \xi_2)\mathrm{d}t,$$

所以

$$\|\mathrm{T}(\Psi_1) - \mathrm{T}(\Psi_2)\| \leqslant \varepsilon_0^{N-1}M\|\Psi_1 - \Psi_2\|. \tag{27}$$

当 $N \geqslant 2$ 和 ε_0 充分小时, 总可使常数 $k = \varepsilon_0^{N-1}M < 1$, 所以算子 $\mathrm{T}(\Psi)$ 是压缩的. 又对于任意 $\Psi \in \mathrm{B}(\rho)$ 有

$$\varepsilon^{N+1}P(\xi, \eta) = \frac{1}{2}\varepsilon^{N-1}\int_1^r \left(r - \frac{t^2}{r}\right)\frac{1}{r}\xi\eta\mathrm{d}t,$$

$$\varepsilon^{N+1}Q(\xi,\eta)=\frac{1}{2}\varepsilon_{*}^{N+1}\int_{1}^{r}\left(r-\frac{t^2}{r}\frac{-1}{2r}\xi^2\right)\mathrm{d}t,$$

所以 $\|T(\Psi)\|\leqslant\varepsilon_0^{N-1}M_1\rho^2+\|R_N^{(0)}\|=\left(\varepsilon_0^{N-1}M_1\rho+1/2\right)\rho\cdot$ (28)

当 $N\geqslant2$ 和 ε_0 充分小时总可使常数 $k_1=\varepsilon_0^{N-1}M_1\rho<1/2$, 所以算子 $T(\Psi)$ 是映象自身到自身• 根据 Banach 不动点定理[2] 知在子集 $B(\rho)$ 中存在唯一的元素 $\boldsymbol{R}_N^*$ 使

$$T(\boldsymbol{R}_N^*)=\boldsymbol{R}_N^*\cdot \tag{29}$$

即积分方程组(18)、(19)的解存在并成立估计式

$$\|\boldsymbol{R}_N^*\|\leqslant\rho, \tag{30}$$

即对于余项 R_N, Z_N 成立 $\mathrm{d}R_N/\mathrm{d}r=O(\varepsilon^{N+1})$, $Z_N=O(\varepsilon^{N+1})$, 再根据(14)的边界条件知 $R_N=O(\varepsilon^{N+1})$•

下面再来消除条件 $N\geqslant2$• 即对于任意正整数 N 都成立

$$R_N=O(\varepsilon^{N+1}),\quad Z_N=O(\varepsilon^{N+1})\cdot \tag{31}$$

作函数 $R=W_{N+2}-W_N+R_{N+2}$, $Z=\Phi_{N+2}-\Phi_N+Z_{N+2}$, 因 $R_{N+2}=O(\varepsilon^{N+3})$, $Z_{N+2}=O(\varepsilon^{N+3})$, $W_{N+2}-W_N=O(\varepsilon^{N+1})$, $\Phi_{N+2}-\Phi_N=O(\varepsilon^{N+1})$, 所以 $R=O(\varepsilon^{N+1})$, $Z=O(\varepsilon^{N+1})$• 又函数 $R=w-W_N$, $Z=\phi-\Phi_N$ 满足余项的边值问题(12)~(15), 根据第二种 Volterra 积分方程解的唯一性知 $R=R_N$, $Z=Z_N$, 即对于任意的正整数 N 都成立估计式(31)•

可以类似地求得简单支承板的级数解的余项估计, 只是所对应的第二种 Voltarra 积分方程(21)的形式不同• 这里我们不再重复•

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Application of the Modified Method of Multiple Scales to the Bending Problems for Circular Thin Plate at Very Large Deflection and the Asymptotics of Solutions (Ⅱ)

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Abstract: This paper is a continuation of (Ⅰ), on the asymptotics behaviors of the series solutions investigated in (Ⅰ). The remainder terms of the series solutions are estimated by the maximum norm.

Key words: large deflection; modified method of multiple scales; asymptotic behaviors