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Banach 空间中二阶常微分方程初值问题解的存在唯一性*

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(林宗池推荐)

摘要: 用单调迭代方法给出了二阶常微分方程初值问题的解的存在唯一性的结果•

关键词: 序 Banach 空间; 非紧性测度; 混合单调算子
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引 言

设$(E, \|\cdot\|)$ 是有序 Banach 空间,其中偏序由正规锥 K 导入• 本文考虑以下初值问题(IVP):

$$\begin{cases} x'' = f(t, x, x') & (t \in J \quad \text{a.e.}), \\ x(0) = a, \quad x'(0) = b, \end{cases} \tag{Ⅰ}$$

这里 $f: J \times E^2 \to E$, $J = [0, T]$, $T > 0$• x 是 IVP(Ⅰ) 的解意味着 $x \in C^1(J, E)$, x' 在 J 上绝对连续,对几乎所有的 $t \in J$, $x(t)$ 满足(Ⅰ)•

用单调迭代方法研究常微分方程的初(边)值问题近年来相当广泛(见[1~8])• 用此方法,首先要解决两个问题:一是构造迭代序列;一是判断所得序列的收敛性• 前者往往依赖于微分不等式,这导致对高阶问题研究的困难;后者在有限维空间不难解决,但在无穷维空间则取决于所给条件了(见[5,6])• 本文所构造的迭代序列避免应用微分不等式,且易推广到高阶问题上去• 在 Banach 空间中,用非紧性测度证明了序列的收敛性•

1 基本假设

本文始终采用以下记号和假设:

$h: J \to R_+$, $h(t) \neq 0 (t \in J)$ 为 Lebesgue 可积函数•

定义 $H(t) = \int_0^t h(s)\,ds \quad (t \in J)$;

$T_i: E \to E \ (i = 1, 2)$ 为非负线性有界算子,且满足

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$$w''(t) = h(t)[\mathscr{T}_1 w(t) + \mathscr{T}_2 w'(t)], \quad w(0) = w'(0) = 0.$$

仅有零解 $w(t) \equiv 0 \quad (t \in J)$;

当 $u \in C^1(J, E)$, 规定 $u = (u, u')$;

$W = \{u \in C^1(J, E): u'$ 在 J 上绝对连续$\}$; 若 $D \subset E$, 令 $D(t) = \{x(t): x \in D\}$; $D(J) = \bigcup_{t \in J} D(t)$;

规定 $E \times E$ 中范数为 $\|x\| = |x_1| + |x_2|$, 其中 $x = (x_1, x_2)$. $|\cdot|_0$, $\|\cdot\|_0$ 依次表 $C(J, E)$, $C(J, E \times E)$ 中上确界范数;

假设 f 满足以下条件:

(S_1) 存在 $\alpha, \beta \in W$, 满足 $\alpha \leqslant \beta$, $\alpha' \leqslant \beta'$, $\alpha(0) \leqslant a \leqslant \beta(0)$, $\alpha'(0) \leqslant b \leqslant \beta'(0)$, 使得

$$\alpha'' \leqslant f(t, \alpha, \alpha') + h(t)[\mathscr{T}_1(\alpha(t) - \beta(t)) + \mathscr{T}_2(\alpha'(t) - \beta'(t))],$$
$$\beta'' \geqslant f(t, \beta, \beta') + h(t)[\mathscr{T}_1(\beta(t) - \alpha(t)) + \mathscr{T}_2(\beta'(t) - \alpha'(t))],$$

(S_2) 任给 $x = (x_1, x_2)$, $y \in (y_1, y_2) \in C^1(J, E) \times C^1(J, E)$, $\alpha \leqslant x_1 \leqslant \beta$, $\alpha' \leqslant x_2 \leqslant \beta'$, $\alpha \leqslant y_1 \leqslant \beta$, $\alpha' \leqslant y_2 \leqslant \beta'$, $y \leqslant x$, 则

$$-h(t)[\mathscr{T}_1(x_1(t) - y_1(t)) + \mathscr{T}_2(x_2(t) - y_2(t))] \leqslant f(t, x_1, x_2) - f(t, y_1, y_2) \leqslant$$
$$h(t) \cdot [\mathscr{T}_1(x_1(t) - y_1(t)) + \mathscr{T}_2(x_2(t) - y_2(t))].$$

(S_3) f 满足 Caratheodory 条件, 即 $f(\cdot, x_1, x_2)$ 对每个 $(x_1, x_2) \in E \times E$ 强可测; 对几乎所有的 $t \in J$, $f(t, x_1, x_2)$ 关于 $x_1, x_2 \in E$ 连续;

(S_4) 设 $D_1, D_2 \subset E$ 有界, 那么

$$\gamma(f(t, D_1, D_2)) \leqslant h(t)(\gamma(D_1) + \gamma(D_2)) \qquad (t \in J).$$

其中, γ 表 Kuratowski 非紧性测度, 关于其定义及基本性质依据[8, 9].

定义 $Z = \{\mu = (\mu_1, \mu_2) \in C^1(J, E) \times C^1(J, E): \alpha \leqslant \eta_1 \leqslant \beta, \alpha' \leqslant \eta_2 \leqslant \beta'\}$.

任给 $u = (u_1, u_2)$, $v = (v_1, v_2) \in Z$,

定义 $G: J \times E^2 \to E$:

$$G(t, u, v) = \frac{1}{2}[f(t, u_1, u_2) + f(t, v_1, v_2)] + \frac{1}{2}h(t)[\mathscr{T}_1(u_1(t) - v_1(t)) + \mathscr{T}_2(u_2(t) - v_2(t))].$$

2 辅助结果

引理 1 设(S_1), (S_2)成立, 则算子 G 有性质:

i) $\alpha'' \leqslant G(t, \alpha, \beta)$, $G(t, \beta, \alpha) \leqslant \beta''$;

ii) $G(t, u, u) = f(t, u, u')$; (1)

对任意 $X_i = (x_1^i, x_2^i)$, $Y_i = (y_1^i, y_2^i) \in Z$, $(i = 1, 2)$, $X_1 \leqslant X_2$, $Y_1 \leqslant Y_2$, 有

$$-h(t)[\mathscr{T}_1(y_1^2(t) - y_1^1(t)) + \mathscr{T}_2(y_2^2(t) - y_2^1(t))] \leqslant G(t, X_2, Y_2) - G(t, X_1, Y_1) \leqslant h(t)[\mathscr{T}_1(x_1^2(t) - x_1^1(t)) + \mathscr{T}_2(x_2^2(t) - x_2^1(t))]; \tag{2}$$

特别

$$G(t, X_1, Y_1) - G(t, Y_1, X_1) = h(t)[\mathscr{T}_1(x_1^1(t) - y_1^1(t)) + \mathscr{T}_2(x_2^1(t) - y_2^1(t))]; \tag{3}$$

iii) X_i, Y_i 如 ii), 则有正实数 N, 使得

$$|G(t,X_2,Y_2)-G(t,X_1,Y_1)|\leqslant Nh(t)\max\{\|X_1(t)-X_2(t)\|,\|Y_1(t)-Y_2(t)\|\}.$$

证 i) 由(S_1)(S_2)得

$$G(t,\alpha,\beta)=\frac{1}{2}[f(t,\alpha,\alpha')+f(t,\beta,\beta')+h(t)[\mathscr{T}_1(\alpha(t)-\beta(t))+\mathscr{T}_2(\alpha'(t)-\beta'(t))]]\geqslant$$

$$\frac{1}{2}[f(t,\alpha,\alpha')+f(t,\alpha,\alpha')-h(t)[\mathscr{T}_1(\beta(t)-\alpha(t))+\mathscr{T}_2(\beta'(t)-\alpha'(t))]+h(t)[\mathscr{T}_1(\alpha(t)-\beta(t))+\mathscr{T}_2(\alpha'(t)-\beta'(t))]]=$$

$$f(t,\alpha,\alpha')+h(t)[\mathscr{T}_1(\alpha(t)-\beta(t))+\mathscr{T}_2(\alpha'(t)-\beta'(t))]\geqslant\alpha''(t).$$

同理可证: $G(t,\beta,\alpha)\leqslant\beta''(t)$.

ii) 只须证(2)式, 由(S_1)得

$$G(t,X_2,Y_2)-G(t,X_1,Y_1)=\frac{1}{2}[f(t,x_1^2,x_2^2)+f(t,y_1^2,y_2^2)+h(t)[\mathscr{T}_1(x_1^2(t)-y_1^2(t))+\mathscr{T}_2(x_2^2(t)-y_2^2(t))]]-\frac{1}{2}[f(t,x_1^1,x_2^1)+f(t,y_1^1,y_2^1)+h(t)[\mathscr{T}_1(x_1^1(t)-y_1^1(t))+\mathscr{T}_2(x_2^1(t)-y_2^1(t))]]=$$

$$\frac{1}{2}[f(t,x_1^2,x_2^2)-f(t,x_1^1,x_2^1)]+\frac{1}{2}[f(t,y_1^2,y_2^2)-f(t,y_1^1,y_2^1)]+\frac{1}{2}h(t)[\mathscr{T}_1(x_1^2(t)-x_1^1(t))+\mathscr{T}_2(x_2^2(t)-x_2^1(t))]-\frac{1}{2}h(t)[\mathscr{T}_1(y_1^2(t)-y_1^1(t))+\mathscr{T}_2(y_2^2(t)-y_2^1(t))]\geqslant$$

$$-\frac{1}{2}h(t)[\mathscr{T}_1(x_1^2(t)-x_1^1(t))+\mathscr{T}_2(x_2^2(t)-x_2^1(t))]-\frac{1}{2}h(t)[\mathscr{T}_1(y_1^2(t)-y_1^1(t))+\mathscr{T}_2(y_2^2(t)-y_2^1(t))]+\frac{1}{2}h(t)[\mathscr{T}_1(x_1^2(t)-x_1^1(t))+\mathscr{T}_2(x_2^2(t)-x_2^1(t))]-\frac{1}{2}h(t)[\mathscr{T}_1(y_1^2(t)-y_1^1(t))+\mathscr{T}_2(y_2^2(t)-y_2^1(t))]=$$

$$-h(t)[\mathscr{T}_1(y_1^2(t)-y_1^1(t))+\mathscr{T}_2(y_2^2(t)-y_2^1(t))].$$

同理得 $g(t,X_2,Y_2)-G(t,X_1,Y_1)\leqslant h(t)[\mathscr{T}_1(x_1^2(t)-x_1^1(t))+\mathscr{T}_2(x_2^2(t)-x_2^1(t))]$.

iii) 由 ii) 之(2)易得

$$0\leqslant G(t,X_2,Y_2)-G(t,X_1,Y_2)\leqslant h(t)[\mathscr{T}_1(x_1^2(t)-x_1^1(t))+\mathscr{T}_2(x_2^2(t)-x_2^1(t))]$$

$$0\leqslant G(t,X_1,Y_1)-G(t,X_1,Y_2)\leqslant h(t)[\mathscr{T}_1(y_1^2(t)-y_1^1(t))+\mathscr{T}_2(y_2^2(t)-y_2^1(t))]$$

由于 K 为正规锥, 从而存在 $0<N_1\in R$, 使

$$|G(t,X_2,Y_2)-G(t,X_1,Y_2)|\leqslant N_1h(t)|\mathscr{T}_1(x_1^2(t)-x_1^1(t))+\mathscr{T}_2(x_2^2(t)-x_2^1(t))|\leqslant MN_1h(t)\max\{\|X_1(t)-X_2(t)\|,\|Y_1(t)-Y_2(t)\|\},\quad)$$

$$|G(t,X_1,Y_1)-G(t,X_1,Y_2)|\leqslant MN_1h(t)\max\{\|X_1(t)-X_2(t)\|,\|Y_1(t)-Y_2(t)\|\}.$$

其中 $M=\max\{|\mathscr{T}_1|,|\mathscr{T}_2|\}$, 令 $N=2N_1M$, 得

$$|G(t,X_2,Y_2)-G(t,X_1,Y_1)|\leqslant|G(t,X_2,Y_2)-G(t,X_1,Y_2)|+|G(t,X_1,Y_2)-G(t,X_1,Y_1)|\leqslant Nh(t)\max\{\|X_1(t)-X_2(t)\|,\|Y_1(t)-Y_2(t)\|\}.$$

由(1)式得 IVP(Ⅰ)等价于

$$u''(t) = G(t, u(t), u(t)), \quad u(0) = (a, b) \tag{II}$$

定义　若 $v, w \in W$ 满足以下方程, 则称 v, w 为 IVP(Ⅱ) 的偶合解:

$$v''(t) = G(t, v(t), w(t)), \quad v(0) = (a, b) \tag{4}$$

$$w''(t) = G(t, w(t), v(t)), \quad w(0) = (a, b) \tag{5}$$

引理 2　v, w 是 IVP(Ⅱ) 的偶合解的充要条件是 $x(t) = (x_1(t), x_2(t)) = (v(t), v'(t))$, $y(t) = (y_1(t), y_2(t)) = (w(t), w'(t))$ 是以下算子方程的解:

$$A(x, y) = x, \quad A(y, x) = y, \tag{6}$$

其中 $A(x, y)(t) = (A_1(x, y)(t), A_2(x, y)(t))$ $(t \in J)$ 满足

$$A_1(x, y)(t) = e^{-t}\left[a + \int_0^t e^s(x_1(s) + x_2(s))ds\right] \tag{7}$$

$$A_2(x, y)(t) = e^{-H(t)}\left[b + \int_0^t e^{H(s)}(h(s)x_2(s) + G(s, x(s), y(s)))ds\right] \tag{8}$$

证　若 v, w 是 IVP(Ⅱ) 的偶合解, 则 v, $w \in W$, 从而 v'', w'' 可积, 再由(4)(5) 式知 $A_1(x, y), A_2(x, y)$ 有意义. 下面验证(6) 式:

$$e^tA_1(x, y)(t) = a + \int_0^t[e^sv(s) + e^sv'(s)]ds = a + e^tv(t) - v(0) = e^tx_1(t) \quad (t \in J).$$

$$e^{H(t)}A_2(x, y)(t) = b + \int_0^t e^{H(s)}[h(s)v'(s) + e^{H(s)}G(s, v(s), w(s))]ds = b + \int_0^t[e^{H(s)}h(s)v'(s) + e^{H(s)}v''(s)]ds = b + e^{H(t)}v'(t) - v'(0) = e^{H(t)}x_2(t) \quad (t \in J).$$

所以 $A(x, y)(t) = x(t)$. 同理可得 $A(y, x)(t) = y(t)$.

反过来, 设 x, y 满足(6) 式, 那么

$$e^tx_1(t) = e^tA_1(x, y)(t) = a + \int_0^t e^s[x_1(s) + x_2(s)]ds. \tag{9}$$

两边关于 t 微分得

$$x_1'(t) = x_2(t) \quad (t \in J), \tag{10}$$

$$e^{H(t)}x_2(t) = e^{H(t)}A_2(x, y)(t) = b + \int_0^t e^{H(s)}[h(s)x_2(s) + G(s, x(s), y(s))]ds. \tag{11}$$

两边关于 t 微分得

$$x_2'(t) = G(t, x(t), y(t)), \tag{12}$$

令 $x_1(t) = v(t)$, $y_1(t) = w(t)$ $(t \in J)$. 由(10) 得 $x_2(t) = v'(t)$. 由(12) 得 $v''(t) = G(t, x(t), y(t))$. 类似地可得 $y_2(t) = w'(t), w''(t) = G(t, x(t), y(t))$. 这样 v, w 满足(4)(5) 式, 且由(9) 式得 $v(0) = x_1(0) = a$, 由(11) 式得 $v'(0) = x_2(0) = b$. 同理 $w(0) = a, w'(0) = b$. 再由(11) 式得 $v(t) \in W$, 同理有 $w(t) \in W$. 故 v, w 是 IVP(Ⅱ) 的偶合解.

引理 3　假设(S_1)~(S_3) 成立, 那么由(7)(8) 式定义的算子 A 有下列性质:

(a) $A(\alpha, \beta) \geqslant \alpha, A(\beta, \alpha) \leqslant \beta$;

(b) $A(\cdot, z)$ 单调增而 $A(z, \cdot)$ 单调减($\forall z \in Z$);

(c) $A: Z \times Z \to Z$.

证　显然由(S_3) 及引理 1 之 iii) 得 $A_i(x, y)$ $(i = 1, 2)$ 有意义. 下证(a). 对每个 $t \in J$,

由 (S_1) 及引理 1 之 i) 得

$$e^t A_1(\alpha,\beta)(t) = a + \int_0^t e^s[\alpha(s) + \alpha'(s)]ds = e^t\alpha(t) - \alpha(0) + a \geqslant e^t\alpha(t).$$

$$e^{H(t)}A_2(\alpha,\beta)(t) = b + \int_0^t e^{H(s)}[h(s)\alpha'(s) + G(s,\alpha,\beta)]ds \geqslant b + \int_0^t e^{H(s)}[h(s)\alpha'(s) + \alpha''(s)]ds = b + e^{H(t)}\alpha'(t) - \alpha'(0) \geqslant e^{H(t)}\alpha'(t).$$

故 $A(\alpha,\beta)(t) \geqslant \alpha(t)\quad (t\in J)$. 同理. $A(\beta,\alpha)(t) \leqslant \beta(t)\quad (t\in J)$.

(b) $\forall x = (x_1, x_2), y = (y_1, y_2) \in Z, x \leqslant y, (\forall t\in J)$, 由引理 1 之 ii) 得

$$e^t A_1(x,z)(t) = a + \int_0^t e^s[x_1(s) + x_2(s)]ds \leqslant a + \int_0^t e^s[y_1(s) + y_2(s)]ds = e^t A_1(y,z)(t).$$

$$e^{H(t)}A_2(x,z)(t) = b + \int_0^t e^{H(s)}[h(s)x_2(s) + G(s,x(s),z(s))]ds \leqslant b + \int_0^t e^{H(s)}[h(s)y_2(s) + G(s,y(s),z(s))]ds = e^{H(t)}A_2(y,z)(t).$$

所以 $A(\cdot, z)$ 单调增, 同理可得 $A(z,\cdot)$ 单调减.

(c) 由(a) 及(b) 直接得到.

引理 4 设 $D = (D_1, D_2) \subset E\times E$ 有界, 那么

i) 当 D 在 J 上等度连续时, 有

$$\gamma(D) = \max_{t\in J}\gamma(D(t)) = \gamma(D(J)).$$

ii) D 可数时有 $\gamma\left(\int_0^t D_j(s)ds\right) \leqslant 2\int_0^t \gamma(D_j(s))ds \qquad (j = 1, 2)$.

iii) $\gamma(D) = \gamma(D_1) + \gamma(D_2)$.

证 i), ii) 见[9]. 下证 iii). 设 $B_1, B_2, \cdots, B_n \subset E\times E, D \subset \bigcup_{i=1}^n B_i, \mathrm{diam}B_i \leqslant \gamma(D) + \varepsilon(\varepsilon > 0)$. 令 $B_i = (B_i^1, B_i^2)$, 使 $D_j \subset \bigcup_{i=1}^n B_i^j (j = 1, 2)$. 不难验证, $\mathrm{diam}B_i^1 + \mathrm{diam}B_i^2 = \mathrm{diam}B_i (i = 1, 2, \cdots, n)$, 故 $\mathrm{diam}B_i^1 + \mathrm{diam}B_i^2 \leqslant \gamma(D) + \varepsilon$. 这推出 $\gamma(D_1) + \gamma(D_2) \leqslant \gamma(D) + \varepsilon$. 令 $\varepsilon\to 0$ 得 $\gamma(D_1) + \gamma(D_2) \leqslant \gamma(D)$.

反过来, 设 $B_i^j \subset E (i = 1, 2, \cdots, n, j = 1, 2)$, 使 $D_j \subset \bigcup_{i=1}^n B_i^j$ 且 $\mathrm{diam}B_i^j \leqslant \gamma(D_j) + \varepsilon(\varepsilon > 0)$, 从而 $D \subset \bigcup_{i=1}^n (B_i^1, B_i^2)$, $\mathrm{diam}(B_i^1, B_i^2) = \mathrm{diam}B_i^1 + \mathrm{diam}B_i^2 \leqslant \gamma(D_1) + \gamma(D_2) + 2\varepsilon$. 即得 $\gamma(D) \leqslant \gamma(D_1) + \gamma(D_2) + 2\varepsilon$. 令 $\varepsilon\to 0$ 得 $\gamma(D) \leqslant \gamma(D_1) + \gamma(D_2)$.

引理 5 设 $D = (D_1, D_2), C = (C_1, C_2) \subset Z$ 等度连续且可数, 条件 $(S_1)\sim(S_4)$ 成立, 又设

$$\begin{cases} e^T < \dfrac{4}{3} \\ 1 - e^{-H(T)} < \dfrac{1}{2(3 + 2M)} \end{cases} \tag{13}$$

其中，$M = \max\{|\mathscr{T}_1|, |\mathscr{T}_2|\}$• 如果 $\max\{\gamma(D), \gamma(C)\} > 0$, 则

$$e\quad \gamma(A(D, C)) < \max\{\gamma(D), \gamma(C)\} a \tag{14}$$

证 首先，由于 K 为正规锥，所以存在正数 N，使当 $x \in Z$ 时，$\|x(t)\| = |x_1(t)| + |x_2(t)| \leqslant N\ (t \in J)$• 引理 1 之 iii) 说明存在 $m \in L^1(J, R_+)$ 使得

$$|G(t, x(t), y(t))| \leqslant m(t) \quad (t \in J; x, y \in Z)•$$

其次，$\forall x = (x_1, x_2) \in D, y = (y_1, y_2) \in C, t_1, t_2 \in J$, 有

$$|A_1(x, y)(t_1) - A_1(x, y)(t_2)| = \left| e^{-t_1}\left[a + \int_0^{t_1} e^s (x_1(s) + x_2(s)) ds\right] - e^{-t_2}\left[a + \int_0^{t_2} e^s (x_1(s) + x_2(s)) ds\right] \right| \leqslant$$

$$|a|\,|e^{-t_1} - e^{-t_2}| + |e^{-t_1} - e^{-t_2}| \left|\int_0^{t_1} e^s (x_1(s) + x_2(s)) ds\right| + e^{-t_2}\left|\int_{t_1}^{t_2} e^s (x_1(s) + x_2(s)) ds\right| \leqslant$$

$$|a|\,|e^{-t_1} - e^{-t_2}| + |e^{-t_1} - e^{-t_2}| \int_0^{t_2} e^s \cdot 2N ds + e^{-t_2} 2N \left|\int_{t_1}^{t_2} e^s ds\right| \leqslant$$

$$|a|\,|e^{-t_1} - e^{-t_2}| + 2N(e^T - 1)|e^{-t_1} - e^{-t_2}| + 2N|e^{t_2} - e^{t_1}|•$$

$$|A_2(x, y)(t_1) - A_2(x, y)(t_2)| =$$

$$\left| e^{-H(t_1)}\left[b + \int_0^{t_1} e^{H(s)} (h(s)x_2(s) + G(s, x(s), y(s))) ds\right] - e^{-H(t_2)}\left[b + \int_0^{t_2} e^{H(s)} (h(s)x_2(s) + G(s, x(s), y(s))) ds\right] \right| \leqslant$$

$$|b|\,|e^{-H(t_1)} - e^{-H(t_2)}| + |e^{-H(t_1)} - e^{-H(t_2)}| \left|\int_0^{t_1} e^{H(s)} (h(s)x_2(s) + G(s, x(s), y(s))) ds\right| + e^{-H(t_2)} \left|\int_{t_1}^{t_2} e^{H(s)} (h(s)x_2(s) + G(s, x(s), y(s))) ds\right| \leqslant$$

$$|b|\,|e^{-H(t_1)} - e^{-H(t_2)}| + |e^{-H(t_1)} - e^{-H(t_2)}| \left|\int_0^{T} e^{H(s)} [Nh(s) + m(s)] ds\right| + \left|\int_{t_1}^{t_2} e^{H(s)} [h(s)N + m(s)] ds\right|•$$

故 $A(D, C)$ 在 J 上等度连续•

再次, 对任意 $t \in J$, 由 (S_4) 得

$$\gamma(G(t, D, C)) = \frac{1}{2}\gamma(f(t, D_1, D_2) + f(t, C_1, C_2) + h(t)[\mathscr{T}_1(D_1(t) - C_1(t)) + \mathscr{T}_2(D_2(t) - C_2(t))]) \leqslant$$

$$\frac{1}{2}\gamma(f(t, D_1, D_2)) + \frac{1}{2}\gamma(f(t, C_1, C_2)) + \frac{|\mathscr{T}_1|}{2} h(t)(\gamma(D_1) + \gamma(C_1)) + \frac{|\mathscr{T}_2|}{2} h(t)(\gamma(D_2) + \gamma(C_2)) \leqslant$$

$$\frac{1}{2} h(t)(1 + 2M)\gamma(D) + \frac{1}{2} h(t)(1 + 2M)\gamma(C)•$$

由此并结合引理 4 之 ii) 得

$$\gamma(A_1(D, C)(t)) = \gamma\left(\left\{ e^{-t}\left[a + \int_0^t e^s (x_1(s) + x_2(s)) ds\right] : x_j \in D_j, j = 1, 2 \right\}\right) \leqslant$$

$$2e^{-t}\int_0^t e^s \gamma(D_1(s)+D_2(s))ds \leqslant 2(1-e^{-T})\gamma(D)\cdot$$

$$2\ \gamma(A_2(D,C)(t)) = \gamma\left(\left\{e^{-H(t)}\left[b+\int_0^t e^{H(s)}(h(s)x_2(s)+G(s,x(s),y(s)))ds\right]: x\in D, y\in C\right\}\right) \leqslant$$
$$2e^{-H(t)}\int_0^t e^{H(s)}\gamma(h(s)D_2(s)+G(s,D,C))ds \leqslant$$
$$2e^{-H(t)}\gamma(D_2)(e^{H(t)}-1)+e^{-H(t)}\cdot\frac{1}{2}(1+2M)\cdot(e^{H(t)}-1)(\gamma(D)+\gamma(C)) \leqslant$$
$$2(1-e^{-H(T)})\gamma(D_2)+\frac{1}{2}(1+2M)(1-e^{-H(T)})(\gamma(D)+\gamma(C))\cdot$$

令 $\tau = \max\{\gamma(D),\gamma(C)\}$，利用引理 4 之 iii) 及(13) 式得

$$\gamma(A(D,C)(t)) = \gamma(A_1(D,C)(t))+\gamma(A_2(D,C)(t)) \leqslant$$
$$2(1-e^{-T})\gamma(D)+2(1-e^{-H(T)})\gamma(D_2)+\frac{1}{2}(1+2M)(1-e^{-H(T)})[\gamma(D)+\gamma(C)] \leqslant$$
$$2(1-e^{-T})\tau+(3+2M)(1-e^{-H(T)})\tau < \frac{1}{2}\tau+\frac{1}{2}\tau = \tau$$

最后根据引理 4 之 i) 得 $\gamma(A(D,C)) < \tau$• 引理获证•

3 主 要 结 果

鉴于引理 3, 可定义序列如下:

$$\alpha^{n+1} = A(\alpha^n, \beta^n), \quad \beta^{n+1} = A(\beta^n, \alpha^n) \quad (n = 0, 1, 2, \cdots) \tag{15}$$

其中, $\alpha^0 = (\alpha_1^0, \alpha_2^0) = \alpha$, $\beta^0 = (\beta_1^0, \beta_2^0) = \beta$• 不难根据引理 3 之(a)(b)得到

$$\alpha = \alpha^0 \leqslant \alpha^1 \leqslant \cdots \alpha^n \leqslant \cdots \leqslant \beta^n \leqslant \cdots \leqslant \beta^1 \leqslant \beta^0 = \beta \tag{16}$$

本文的主要结果是

定理 1 假设(S_1)~(S_4)及(13)式成立, 那么 IVP(Ⅰ)存在唯一解•

证 定义序列 $\{\alpha^n\}_{n=0}^{\infty} \triangleq D$, $\{\beta^n\}_{n=0}^{\infty} \triangleq C$ 如(15)式, 则由 K 的正规性及(16)式得 D, C 均有界可数, 类似于引理 5 的证明可得 D, C 等度连续• (16)式又说明 $D, C \subset Z$, 故由引理 5 知, 若 $\max\{\gamma(D), \gamma(C)\} > 0$, 那么 D, C 满足(14)式•

另一方面, $D \subset A(D,C) \cup \{\alpha^0\}$, $C \subset A(C,D) \cup \{\beta^0\}$• 所以 $\gamma(D) \leqslant \gamma(A(D,C))$, $\gamma(C) \leqslant \gamma(A(C,D))$• 若 $\max\{\gamma(D), \gamma(C)\} > 0$, 则由(14)式得

$$\gamma(D) < \max\{\gamma(D), \gamma(C)\}, \quad \gamma(C) < \max\{\gamma(D), \gamma(C)\},$$

这样 $\max\{\gamma(D), \gamma(C)\} < \max\{\gamma(D), \gamma(C)\}$, 矛盾• 所以 $\gamma(D) = \gamma(C) = 0$, 即$\{\alpha^n\}$, $\{\beta^n\}$都相对紧, 亦即存在子列 $\{\alpha^{n_k}\}_{k=0}^{\infty} \subset \{\alpha^n\}_{n=0}^{\infty}$, $\{\beta^{n_k}\}_{k=0}^{\infty} \subset \{\beta^n\}_{n=0}^{\infty}$ 及 $x, y \in Z$, 使 $\alpha^{n_k} \to x$, $\beta^{n_k} \to y$ $(k \to \infty)$• 注意到(16)式, 从而不难验证

$$\lim_{n\to\infty}\alpha^n = \lim_{n\to\infty}A(\alpha^{n-1}, \beta^{n-1}) = x, \quad \lim_{n\to\infty}\beta^n = \lim_{n\to\infty}A(\beta^{n-1}, \alpha^{n-1}) = y,$$

由假设(S_3)易证 A 关于 $x, y \in Z$ 连续,所以由上式可得 $A(x, y) = x, A(y, x) = y$• 由引理 2 得 IVP(Ⅱ) 存在偶合解 v, w 满足 $x = (x_1, x_2) = (v, v'), y = (y_1, y_2) = (w, w')$

令 $\rho = w - v$, 那么 $\rho(0) = \rho'(0) = 0$, 并由(4),(5) 及(3):

$$\rho''(t) = G(t, w, v) - G(t, v, w) = h(t)[\mathscr{T}_1(w(t) - v(t)) + \mathscr{T}_2(w'(t) - v'(t))] = h(t)[\mathscr{T}_1\rho(t) + \mathscr{T}_2\rho'(t)]•$$

根据 $\mathscr{T}_1, \mathscr{T}_2$ 的假设得 $\rho(t) \equiv 0$, 即 $v(t) \equiv w(t) \triangleq u(t) \quad (t \in J)$• (1) 式说明 $G(t, u, u) = f(t, u, u')$, 因此 u 是 IVP(Ⅰ) 的解•

利用(2)式, 对任何 $\alpha \leqslant v \leqslant u \leqslant \beta$, 有

$$-h(t)[\mathscr{T}_1(u(t) - v(t)) + \mathscr{T}_2(u'(t) - v'(t))] \leqslant f(t, u, u') - f(t, v, v') \leqslant h(t)[\mathscr{T}_1(u(t) - v(t)) - \mathscr{T}_2(u'(t) - v'(t))]•$$

由此立即看出 u 是 IVP(Ⅰ)的唯一解• 证毕•

定理 2 假设(S_1)~(S_3)成立, 如果 K 是正则锥, 即么 IVP(Ⅰ)有唯一解•

证 设 $\{\alpha^n\}_{n=0}^{\infty}$, $\{\beta^n\}_{n=0}^{\infty}$ 如(15)式给出, 那么引理 3 说明(16)式成立, 由于 K 为正则锥, 所以 $\{\alpha^n\}$, $\{\beta^n\}$ 均收敛• 余下同定理 1 证明•

注 1 本文结果容易推广到如下 n 阶问题:

$$\begin{cases} x^{(n)} = f(t, x, x', \cdots, x^{(n-1)}), \\ x^{(i-1)}(0) = a_i \quad (i = 1, 2, \cdots, n)• \end{cases} \qquad (\mathrm{I}_n)$$

事实上, 只要将假设(S_1)~(S_4)作相应的改变, 而引理 2 中的算子 A 为:

$$A_i(x, y)(t) = e^{-t}\left[a_i + \int_0^t e^s(x_i(s) + x_{i+1}(s))ds\right] \quad (i = 1, 2, \cdots, n-1),$$

$$A_n(x, y)(t) = e^{-H(t)}\left[a_n + \int_0^t e^{H(s)}(h(s)x_n(s) + G(s, x(s), y(s)))ds\right]•$$

其中 $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \underbrace{E \times E \times \cdots \times E}_{n} \triangleq E^n$ 那么引理 3 的结果同样成立(这就是避免使用微分不等式的优点所在), 从而其他结果成立•

注 2 如果 E 是弱序列完备的 Banach 空间, 那么假设(S_4)可以去掉•

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Existence and Uniqueness of Solutions for Initial Value Problems of Second Order Ordinary Differential Equations in Banach Space

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Abstract: In this paper, the initial value problems of second order ordinary differential equations in Banach spaces are discussed. By using the monotone iterative technique, some existence and uniqueness theorems for solutions are obtained.

Key words: ordered Banach spaces; measures of noncompactness; mixed monotone operators