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叠层圆柱厚壳混合状态方程的弱形式及其边值问题*

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摘要: 从 Hellinger_Reissner 变分原理出发, 在柱坐标系中, 导出圆柱壳的弱形式混合状态方程和边界条件, 联用状态空间法给出叠层柱壳的解析解, 此法使得求解该类问题的形式得以扩大和统一•

关键词: 混合状态方程; 状态空间; 圆柱壳; 弱形式; 边值问题

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引 言

基于三维弹性力学理论, 文[1]~[6]分别先后讨论了各向同性、正交各向异性板壳的精确解, 但这些解法由于都采用了强形式的平衡方程和边界条件, 使得在求解时面对不同的边值问题必须采用不同的处理方法, 且过多的依赖于技巧性, 因此难以得到推广• 文[7]阐明了弹性力学混合状态方程的重要性并首次给出与之对应的从 Hellinger_Reissner 变分以及修正后导出的 Hamilton 正则方程; 同时也指出求解混合状态方程组是一个有广泛应用但现在还没有很好开拓的领域, 这是因为在连续介质中的 Hamilton 方程要满足复杂的边界条件, 这也就是它的重点和难点• 本文正是基于解决这个问题, 在文[5~7]的基础上, 从两类变量的 Hellinger_Reissner 变分原理出发, 导出了圆柱壳混合状态方程和边界条件的弱形式, 并联用状态空间法给出了强厚度叠层圆柱壳的解析解, 这使得求解该类问题的形式得以扩大和统一•

1 Hellinger_Reissner 变分原理和 Hamilton 正则方程

在柱坐标系下, 三维复合材料具有任意边界条件以 σ_x, σ_θ, σ_r, $\tau_{r\theta}$, τ_{rx}, $\tau_{x\theta}$, u, v, w 为独立变量的泛函如下:

$$U = \iiint_V \Big[\sigma_x \frac{\partial u}{\partial x} + \sigma_\theta \left(\frac{w}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) + \sigma_r \text{际} \frac{\partial w}{\partial r} + \tau_{x\theta} \left(\frac{\partial v}{\partial x} + \frac{1}{r} \frac{\partial u}{\partial \theta} + \quad te$$

$$((1-\lambda_\theta)v-v)p_\theta+((1-\lambda_r)w-w)p_r]\,\mathrm{d}S, \tag{1.1}$$

把所有带 $\partial(\quad)/\partial r$ 的项放在一起, 有

$$U=\iiint_V\left\{\sigma_r\frac{\partial w}{\partial r}+\tau_{r\theta}\frac{\partial v}{\partial r}+\tau_{rx}\frac{\partial u}{\partial r}-H\right\}r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}x+\Gamma, \tag{1.2}$$

其中

$$-H=\sigma_x\frac{\partial u}{\partial x}+\sigma_\theta\left(\frac{w}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\right)+\tau_{x\theta}\left(\frac{\partial v}{\partial x}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right)+\tau_{r\theta}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{v}{r}\right)+$$
$$\_R\,\tau_{rx}\frac{\partial w}{\partial x}-\frac{1}{2}\{\sigma\}^{\mathrm{T}}[c]^{-1}\{\sigma\} \tag{1.3}$$

是 Hamilton 函数的一个二次式• 以上符号都是常规的• $[c]$ 为本构关系的刚度矩阵, 对于各向异性材料

$$[c]=\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ & & c_{33} & c_{34} & c_{35} & c_{36}\\ & & & c_{44} & c_{45} & c_{46}\\ & \text{对称} & & & c_{55} & c_{56}\\ & & & & & c_{66}\end{bmatrix},$$

这里 c_{ij} 为弹性常数•

$$\Gamma=\iint_S[\lambda_x(p_x-p_x)u+\lambda_\theta(p_\theta-p_\theta)v+\lambda_r(p_r-p_r)w+((1-\lambda_x)u-u)p_x+$$
$$((1-\lambda_\theta)v-v)p_\theta+((1-\lambda_r)w-w)p_r]\,\mathrm{d}S, \tag{1.4}$$

(1.4) 式中的 λ_x, λ_θ, λ_r 的取值规则为:

$S\in S_\sigma$(应力边界) $\lambda_x=\lambda_\theta=\lambda_r=1$,

$S\in S_u$(位移边界) $\lambda_x=\lambda_\theta=\lambda_r=0$,

$S\in S_{\sigma-u}$(应力_位移边界), 在某一方向上(如 x) 为力的边界条件, 则

$\lambda_x=1$, 否则 $\lambda_x=0$•

$\delta U=0$ 对应着弹性体的真解状态•

对(1.2)式变分运算, 得

$$\left.\begin{aligned}&\iiint_V\left(\frac{\partial\tau_{rx}}{\partial r}+\frac{\partial H}{\partial u}\right)\delta u\,\mathrm{d}V+\iint_{S_\sigma}(p_x-p_x)\delta u\,\mathrm{d}S=0,\\&\iiint_V\left(\frac{\partial\tau_{r\theta}}{\partial r}+\frac{\partial H}{\partial v}\right)\delta v\,\mathrm{d}V+\iint_{S_\sigma}(p_\theta-p_\theta)\delta v\,\mathrm{d}S=0,\\&\iiint_V\left(\frac{\partial\sigma_r}{\partial r}+\frac{\partial H}{\partial w}\right)\delta w\,\mathrm{d}V+\iint_{S_\sigma}(p_r-p_r)\delta w\,\mathrm{d}S=0;\end{aligned}\right\} \tag{1.5a}$$

$$\left.\begin{aligned}&\iiint_V\left(\frac{\partial u}{\partial r}-\frac{\partial H}{\partial\tau_{rx}}\right)\delta\tau_{rx}\,\mathrm{d}V+\iint_{S_u}[(w-w)n_x+(u-u)n_r]\delta\tau_{rx}\,\mathrm{d}S=0,\\&\iiint_V\left(\frac{\partial v}{\partial r}-\frac{\partial H}{\partial\tau_{r\theta}}\right)\delta\tau_{r\theta}\,\mathrm{d}V+\iint_{S_u}[(w-w)n_\theta+(v-v)n_r]\delta\tau_{r\theta}\,\mathrm{d}S=0,\\&\iiint_V\left(\frac{\partial w}{\partial r}-\frac{\partial H}{\partial\sigma_r}\right)\delta\sigma_r\,\mathrm{d}V+\iint_{S_u}(w-w)n_r\delta\sigma_r\,\mathrm{d}S=0;\end{aligned}\right\} \tag{1.5b}$$

$$
\text{d}\quad \left.\begin{aligned}
&\iiint_V \left(\frac{\partial u}{\partial x} - \frac{\partial H}{\partial \sigma_x}\right)\delta\sigma_x \, \mathrm{d}V + \iint_{S_u} (u - u)\, n_x \delta\sigma_x \, \mathrm{d}S = 0, \\
&\iiint_V \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{\partial H}{\partial \sigma_\theta}\right)\delta\sigma_\theta \, \mathrm{d}V + \iint_{S_u} (v - v)\, n_\theta \delta\sigma_\theta \, \mathrm{d}S = 0, \\
&\iiint_V \left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} - \frac{\partial H}{\partial \tau_{x\theta}}\right)\delta\tau_{x\theta} \, \mathrm{d}V + \iint_{S_u} [(v - v)\, n_x + (u - u)\, n_\theta]\, \delta\tau_{x\theta} \, \mathrm{d}S = 0 \bullet
\end{aligned}\right\}
\qquad (1.5\text{c})
$$

采用记号:

$$
q = (u, v, w)^{\mathrm{T}}, \quad p = (\tau_{rx}, \tau_{r\theta}, \sigma_r)^{\mathrm{T}}, \quad p_1 = (\sigma_x, \sigma_\theta, \tau_{x\theta})^{\mathrm{T}}, \qquad (1.6)
$$

则(1.5)式可以写成以下形式:

$$
\left.\begin{aligned}
&\iiint_V \left(\frac{\partial p}{\partial r} + \frac{\partial H}{\partial q}\right)\delta q \, \mathrm{d}V 2 \; \Gamma_1 = 0, \\
&\iiint_V \left(\frac{\partial q}{\partial r} - \frac{\partial H}{\partial p}\right)\delta p \, \mathrm{d}V + \Gamma_2 = 0, \\
\text{j}\quad &\iiint_V \left(Dq - \frac{\partial H}{\partial p_1}\right)\delta p_1 \, \mathrm{d}V + \Gamma_3 = 0,
\end{aligned}\right\}
\qquad (1.7)
$$

其中:

$$
\Gamma_1 = \begin{bmatrix}
\iint_{S_\sigma} (p_x - p_x)\, \delta u \, \mathrm{d}S = 0 \\
\iint_{S_\sigma} (p_\theta - p_\theta)\, \delta v \, \mathrm{d}S = 0 \\
\iint_{S_\sigma} (p_r - p_r)\, \delta w \, \mathrm{d}S = 0
\end{bmatrix}, \qquad \text{S}
$$

$$
\Gamma_2 = \begin{bmatrix}
\iint_{S_u} [(w - w)\, n_x + (u - u)\, n_r]\, \delta\tau_{rx} \, \mathrm{d}S = 0 \\
\iint_{S_u} [(w - w)\, n_\theta\theta + (v - v)\, n_r]\, \delta\tau_{r\theta} \, \mathrm{d}S = 0 \\
\iint_{S_u} (w - w)\, n_r \delta\sigma_r \, \mathrm{d}S = 0
\end{bmatrix},
$$

$$
\Gamma_3 = \begin{bmatrix}
\iint_{S_u} (u - u)\, n_x \delta\sigma_x \, \mathrm{d}S = 0 \\
\partial \quad \iint_{S_u} (v - v)\, n_\theta \delta\sigma_\theta \, \mathrm{d}S = 0 \\
\iint_{S_u} [(v - v)\, n_x + (u - u)\, n_\theta]\, \delta\tau_{x\theta} \, \mathrm{d}S = 0
\end{bmatrix},
$$

$$
D = \begin{bmatrix}
-\frac{\partial}{\partial x} & 0 & 0 \\
0 \quad \text{w} & -\frac{1}{r}\frac{\partial}{\partial \theta} & -\frac{1}{r} \\
-\frac{1}{r}\frac{\partial}{\partial \theta} & -\frac{\partial}{\partial x} & 0
\end{bmatrix} \bullet
$$

从(1.7)式可以看出, 有

$$\frac{\partial p}{\partial r}=-\frac{\partial H}{\partial q},\quad \frac{\partial q}{\partial r}=\frac{\partial H}{\partial p},\tag{1.8}$$

这就是以坐标 r 为展开方向的 Hamilton 正则方程•

2　混合状态方程和边界条件的弱形式及其求解

选择 $\boldsymbol{F}=(p,q)^{\mathrm{T}}$ 组成状态向量, 将(1.7) 写成如下的混合状态微分方程组, 即

$$\iiint_V \boldsymbol{G}_1\frac{\partial}{\partial r}\boldsymbol{F}\mathrm{d}V=\iiint_V \boldsymbol{G}_1\boldsymbol{HF}\mathrm{d}V+\iiint_V \boldsymbol{G}_1\boldsymbol{D}_1\boldsymbol{p}_1\mathrm{d}V+\iint_S \boldsymbol{G}_1\boldsymbol{S}_1\mathrm{d}S,\tag{2.1}$$

$$\iiint_V \boldsymbol{G}_2\boldsymbol{D}_2\boldsymbol{F}\mathrm{d}V+\iiint_V \boldsymbol{G}_2\boldsymbol{B}\boldsymbol{p}_1\mathrm{d}V+\iint_S \boldsymbol{G}_2\boldsymbol{S}_2\mathrm{d}S=0,\tag{2.2}$$

这就是弱形式的混合状态方程和边界条件的统一方程•

其中

$$\boldsymbol{G}_1=\begin{bmatrix}\delta u & & & & & \\ & \delta v & & & [0] & \\ & & \delta w & & & \\ & & & \delta\tau_{rx} & & \\ & [0] & & & \delta\tau_{r\theta} & \\ & & & & & \delta\sigma_r\end{bmatrix},\quad \boldsymbol{S}_1=\begin{bmatrix}p_x-p_x\\ p_\theta-p_\theta\\ p_r-p_r\\ (w-w)n_x+(u-u)n_r\\ (w-w)n_\theta+(v-v)n_r\\ (w-w)n_r\end{bmatrix},$$

$$\boldsymbol{H}=\begin{bmatrix}-\frac{1}{r} & 0 & 0 & 0 & 0 & 0\\ 0 & -\frac{2}{r} & 0 & 0 & 0 & 0\\ -\frac{\partial}{\partial x} & -\frac{1}{r}\frac{\partial}{\partial\theta} & -\frac{1}{r} & 0 & 0 & 0\\ S_{55} & S_{54} & S_{53} & 0 & 0 & -\frac{\partial}{\partial x}\\ S_{45} & S_{44} & S_{43} & 0 & \frac{1}{r} & -\frac{1}{r}\frac{\partial}{\partial\theta}\\ S_{35} & S_{34} & S_{33} & 0 & 0 & 0\end{bmatrix},\quad \boldsymbol{D}_1=\begin{bmatrix}-\frac{\partial}{\partial x} & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta}\\ 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} & -\frac{\partial}{\partial x}\\ 0 & \frac{1}{r} & 0\\ S_{51} & S_{52} & S_{56}\\ S_{41} & S_{42} & S_{46}\\ S_{31} & S_{32} & S_{36}\end{bmatrix},$$

$$\boldsymbol{G}_2=\begin{bmatrix}\delta\sigma_x & 0 & 0\\ 0 & \delta\sigma_\theta & 0\\ 0 & 0 & \delta\tau_{x\theta}\end{bmatrix},\quad \boldsymbol{D}_2=\begin{bmatrix}S_{15} & S_{14} & S_{13} & -\frac{\partial}{\partial x} & 0 & 0\\ S_{25} & S_{24} & S_{23} & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} & -\frac{1}{r}\\ S_{65} & S_{64} & S_{63} & -\frac{1}{r}\frac{\partial}{\partial\theta} & -\frac{\partial}{\partial x} & 0\end{bmatrix},$$

$$\boldsymbol{B}=\begin{bmatrix}S_{11} & S_{12} & S_{16}\\ S_{21} & S_{22} & S_{26}\\ S_{61} & S_{62} & S_{66}\end{bmatrix},\quad \boldsymbol{S}_2=\begin{bmatrix}(u-u)n_x\\ (v-v)n_\theta\\ (u-u)n_\theta+(v-v)n_x\end{bmatrix},$$

S_{ij} 为本构关系的柔度系数•

为了求解(2.1) 和(2.2) 式, 设

$$\boldsymbol{F}=\boldsymbol{F}(r)\boldsymbol{N}_1(x,\theta),\quad \boldsymbol{p}_1=\boldsymbol{p}(r)\boldsymbol{N}_2(x,\theta)\text{•}\tag{2.3}$$

将(2.3) 代入(2.1) 、(2.2) 式, 在 x, θ 坐标面内完成二重积分后, 经过化简, 可得

$$\frac{\mathrm{d}}{\mathrm{d}r}\boldsymbol{F}(r) = \boldsymbol{AF}(r) + \boldsymbol{S}(r) \cdot \qquad (2.4)$$

式(2.4)的解为:

$$\boldsymbol{F}(r) = \mathrm{e}^{Ar}\boldsymbol{F}(0) + \int_0^r \mathrm{e}^{A(r-\tau)}\boldsymbol{S}(\tau)\mathrm{d}\tau \cdot \qquad (2.5)$$

应用本法求解复合材料叠层壳时,利用状态向量 $\boldsymbol{F}$ 在层间的连续性及状态转移矩阵,问题即可解决•

3 应 用 和 算 例

例 1 两端简支三层壳,内表面受液体压力 $q(\theta)$ 作用• γ 是液体密度,

$$q = -\gamma b(\cos\theta - \cos\varphi), \qquad 当\ \theta < \varphi,$$
$$q = 0, \qquad 当\ \theta \geqslant \varphi,$$

$\varphi = 2\pi/3$• 几何参数为: $h_1 = h_3 = 0.1h$, $h_2 = 0.8h$, $l = 2\pi R_0$, l 为壳长, R_0 为中面半径• 本构关系为正交各向异性,

$$\begin{bmatrix}\sigma_x\\ \sigma_\theta\\ \sigma_r\\ \tau_{r\theta}\\ \tau_{rx}\\ \tau_{x\theta}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0\\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_\theta\\ \varepsilon_r\\ \gamma_{r\theta}\\ \gamma_{rx}\\ \gamma_{x\theta}\end{bmatrix}, \qquad r \qquad ($$

材料特性为:

$$c_{22}/c_{11} = 0.543\,103, \quad c_{12}/c_{11} = 0.246\,269,$$
$$c_{23}/c_{11} = 0.115\,017, \quad c_{13}/c_{11} = 0.083\,172,$$
$$c_{33}/c_{11} = 0.530\,172, \quad c_{44}/c_{11} = 0.266\,810,$$
$$c_{55}/c_{11} = 0.159\,914, \quad c_{66}/c_{11} = 0.262\,931 \cdot$$

令 ρ 为第一层材料与第二层材料 c_{11} 比值,若 $\rho = 1$ 则为单层壳,若 $\rho \neq 1$ 则为三层壳• 取(2.3)式中的状态向量函数如下:

$$u = \sum_m\sum_n u_{mn}(r)\cos\frac{m\pi x}{l}\cos(n\theta), \quad \tau_{rx} = \sum_m\sum_n \tau_{rx,mn}(r)\cos\frac{m\pi x}{l}\cos(n\theta),$$
$$v = \sum_m\sum_n v_{mn}(r)\sin\frac{m\pi x}{l}\sin(n\theta), \quad \tau_{r\theta} = \sum_m\sum_n \tau_{r\theta,mn}(r)\sin\frac{m\pi x}{l}\sin(n\theta),$$
$$w = \sum_m\sum_n w_{mn}(r)\sin\frac{m\pi x}{l}\cos(n\theta), \quad \sigma_r = \sum_m\sum_n \sigma_{r,mn}(r)\sin\frac{m\pi x}{l}\cos(n\theta),$$
$$\sigma_x = \sum_m\sum_n \sigma_{x,mn}(r)\sin\frac{m\pi x}{l}\cos(n\theta), \quad \sigma_\theta = \sum_m\sum_n \sigma_{\theta,mn}(r)\sin\frac{m\pi x}{l}\cos(n\theta),$$
$$\tau_{x\theta} = \sum_m\sum_n \tau_{x\theta,mn}(r)\cos\frac{m\pi x}{l}\sin(n\theta) \cdot$$

根据所选函数可知,壳的边界条件,在 $x = 0, l$ 处, $w = v = \sigma_x = 0$ 已经满足• 计算结果见表 1,对应的级数取项 $m = 1, 3, \cdots, 29$; $n = 0, 1, \cdots, 15$•

例 2 两端固支三层壳• 所受荷载,几何参数,材料特性同例 1• 仍然选用上例的函数,此时边界条件不能全部满足• 将固支边变成简支边,并在简支边加上原固支边的纵向反力(待

求)，该反力通过在 $x = 0, l$ 处由 $\iint_S (u - u)\, n_x \mathrm{d}S = 0$ 得到• 计算结果见表 2•

表 1 挠度和应力 $(h/R_0 = 1.0)$

	$x = l/2, \theta = 0$	$\rho = 1$			$\rho = 5$		
		$wc_{11}/(\gamma bh)$	$\sigma_x/(\gamma b)$	$\sigma_\theta/(\gamma b)$	$wc_{11}^{(2)}/(\gamma bh)$	$\sigma_x/(\gamma b)$	$\sigma_\theta/(\gamma b)$
本文	1+	14.136	6 569 8	14.410	7.466 2	18.126	38 670
	1-	13.847	0 754 0	0.738 8	7.397 7	2.836 0	2 695 4
	2+	13.847	0 754 0	0.738 8	7.397 7	0.468 4	0 402 4
	2-	12.897	2 290 8	0.481 4	6.698 8	1.013 2	0 132 8
	3+	12.897	2 290 8	0.481 4	6.698 8	5.140 5	0 767 4
	3-	12.848	2 631 4	0.727 5	6.675 9	6.154 5	1.695 8
文[5]	3-	12.830	2 629 0	0.727 0	6.668 9	6.149 0	1.695 0

1+ ：内层内表面，1- ：内层外表面；2 表示中层，3 表示外层

表 2 单层壳和三层壳的应力

		$\rho = 1(h/R_0 = 0.4)$				$\rho = 5(h/R_0 = 0.6)$			
		$\sigma_x/\gamma b$	$\sigma_\theta/\gamma b$	$\sigma_x^{(0)}/\gamma b$	$\tau_{rx}/\gamma b$	$\sigma_x/\gamma b$	$\sigma_\theta/\gamma b$	$\sigma_x^{(0)}/\gamma b$	$\tau_{rx}/\gamma b$
本文	1+	1.468	1.701	45.44	0 000	2 623	4 570	54 34	0 000
	1-	1.947	1.951	6.044	2 654	3 570	4 859	- 24 11	2 047
	2+	1.947	1.951	6.044	2 654	0 589	0 798	- 4 822	2 047
	2-	4.953	3.444	- 12.58	2 284	1. 297	0 817	1. 112	1. 180
	3+	4.953	3.444	- 12.58	2 284	6 620	4 270	5 562	1. 180
	3-	5.396	3.636	- 45.13	0 000	7. 468	4 514	- 45 64	0 000
SAP5	1+	1.442	1.279	9.499	4 002	2 752	5 162	9 432	6 798
	1-	2.009	1.811	5.307	3 876	3 454	5 269	- 0 518	6 650
	2+	2.009	1.811	5.307	3 876	0 593	0 974	- 0 104	1. 330
	2-	5.668	4.616	- 13.17	3 692	1. 302	0 878	- 1. 914	1. 037
	3+	5.668	4.616	- 13.17	3 692	6 837	4 657	- 9 571	5 185
	2-	6.116	4.895	- 17.16	3 746	7. 526	4 798	- 18 16	5 252

σ_x, σ_θ 在 $x = l/2, \theta = 0$ 处• $\sigma_x^{(0)}, \tau_{rx}$ 在 $x = 0, \theta = 0$ 处•

1+ ：内层内表面，1- ：内层外表面；2 表示中层，3 表示外层

4 结　论

本文给出的圆柱壳弱形式求解混合状态方程是有效的，并给合状态空间法给出了解析解，物理概念清晰，边值问题处理统一，无需特殊技巧，且使得求解问题的形式得以扩大• 本方法完全可以用于板壳结构的动力计算问题•

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Weak Formulation of Mixed State Equation and Boundary Value Problem of Laminated Cylindrical Shell

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Abstract: Weak formulation of mixed state equations including boundary conditions are presented in a cylindrical coordinate system by introducing Hellinger_Reissner variational principle. Analytical solutions are obtained for laminated cylindrical shell by means of state space method. The present study extends and unifies the solution of laminated shells.

Key words: mixed state equation; state space; cylindrical shell; weak formulation; boundary value problem