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压电材料圆锥顶端作用集中载荷的解*

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摘要: 从压电材料三维问题的通解出发, 用凑合法求解了圆锥顶端作用集中力、点电荷和集中力矩的解, 此解形式简单便于应用. 当圆锥顶角 $2\alpha = \pi$ 时, 集中力、点电荷和扭矩的解可退化得到半空间问题的解.

关键词: 压电材料圆锥; 压缩; 扭转; 弯曲; 点电荷
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引 言

圆锥顶端受集中载荷是弹性力学中的经典问题, 有不少学者研究过这个问题. 对于各向同性体, Love 报导了圆锥顶端作用集中力问题的解[1], Lur' e 从 Papkovich_Neuber 通解出发系统地研究了这类问题[2]. 对于横观各向同性材料, Lekhniskii 和胡海昌利用各自通解分别研究了圆锥顶端受轴向集中力压缩和受横向集中力弯曲的问题[3,4], Chen 也研究了圆锥和锥壳顶端作用集中力的弯曲问题[5]. 丁皓江等研究过球面各向同性圆锥顶端作用集中力和力矩的压缩、弯曲和扭转问题[6].

对于压电材料, Sosa 和 Castro 给出了半平面边界作用集中力和点电荷的解[7], 丁皓江等得到了压电楔形体, 受集中力和点电荷作用时的解, 退化得到半无限平面直线边界受集中力和点电荷作用时的解[8]. 丁皓江等给出了半空间边界作用集中力和点电荷的解[9], 此解用 Fourier 变换进行推导, 最终表达比较冗长和复杂, 不便应用. 本文利用丁皓江等的通解[9], 以凑合解法给出了圆锥顶端作用集中力和点电荷以及力矩的解, 形式简单实用, 此解可退化得到半空间作用集中力和点电荷以及扭矩的解.

1 压电材料的通解和边界条件

丁皓江等给出横观各向同性压电材料的通解[9], 对于特征根 $s_1 \neq s_2 \neq s_3 \neq s_1$ 的情形, 在柱坐标中为

$$\left.\begin{aligned} u_r &= \sum_{i=1}^{3} \frac{\partial \psi_i}{\partial r} - \frac{1}{r}\frac{\partial \psi_0}{\partial \theta}, & w &= \sum_{i=1}^{3} s_i k_{1i} \frac{\partial \psi_i}{\partial x_i}, \\ u_\theta &= \sum_{i=1}^{3} \frac{1}{r}\frac{\partial \psi_i}{\partial \theta} + \frac{\partial \psi_0}{\partial r}, & \phi &= \sum_{i=1}^{3} s_i k_{2i} \frac{\partial \psi_i}{\partial z_i}, \end{aligned}\right\} \tag{1.1}$$

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式中，k_{1i}，k_{2i} 是与材料常数有关的常数，位移函数 ϕ_i，满足下列方程：

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}+\frac{\partial^2}{\partial z_i^2}\right)\phi_i = 0 \quad (i = 0, 1, 2, 3). \tag{1.2}$$

将(1.1)代入本构关系，即得用位移函数 ϕ_i 表示应力和电位移的表达式：

$$\left.\begin{aligned}
\sigma_r &= 2c_{66}\sum_{i=1}^{3}\frac{\partial^2\phi_i}{\partial r^2}+\sum_{i=1}^{3}m_i\frac{\partial^2\phi_i}{\partial z_i^2}-2c_{66}\frac{\partial}{\partial r}\left(\frac{\partial\phi_0}{r\partial\theta}\right),\\
\sigma_\theta &= 2c_{66}\sum_{i=1}^{3}\left(\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}\right)\phi_i+\sum_{i=1}^{3}m_i\frac{\partial^2\phi_i}{\partial z_i^2}+2c_{66}\frac{\partial}{\partial r}\left(\frac{\partial\phi_0}{r\partial\theta}\right),\\
\sigma_z &= \sum_{i=1}^{3}a_i\frac{\partial^2\phi_i}{\partial z_i^2},\quad D_z = \sum_{i=1}^{3}c_i\frac{\partial^2\phi_i}{\partial z_i^2},\\
\tau_{\theta z} &= \sum_{i=1}^{3}s_ia_i\frac{\partial^2\phi_i}{r\partial\theta\partial z_i}+s_0c_{44}\frac{\partial^2\phi_0}{\partial r\partial z_0},\\
D_\theta &= \sum_{i=1}^{3}s_ic_i\frac{\partial^2\phi_i}{r\partial\theta\partial z_i}+s_0e_{15}\frac{\partial^2\phi_0}{\partial r\partial z_0},\\
\tau_{zr} &= \sum_{i=1}^{3}s_ia_i\frac{\partial^2\phi_i}{\partial r\partial z_i}-s_0c_{44}\frac{\partial^2\phi_0}{r\partial\theta\partial z_0},\\
D_r &= \sum_{i=1}^{3}s_ic_i\frac{\partial^2\phi_i}{\partial r\partial z_i}-s_0e_{15}\frac{\partial^2\phi_0}{r\partial\theta\partial z_0},\\
\tau_{r\theta} &= 2c_{66}\sum_{i=1}^{3}\frac{\partial}{\partial r}\left(\frac{\partial\phi_i}{r\partial\theta}\right)+c_{66}\left(2\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z_0^2}\right)\phi_0,
\end{aligned}\right\} \tag{1.3}$$

式中

$$\left.\begin{aligned}
a_i &= s_i^2(c_{33}k_{1i}+e_{33}k_{2i})-c_{13},\quad m_i = n_i+2c_{66},\\
c_i &= e_{15}(1+k_{1i})-\varepsilon_{11}k_{2i},\quad n_i = -a_is_i^2.
\end{aligned}\quad (i = 1, 2, 3)\right\} \tag{1.4}$$

设圆锥顶角为 2α，将锥顶取为原点，z 轴为圆锥中心轴，指向体内，xy 平面平行于各向同性面，在锥顶作用集中力 $\boldsymbol{P} = P_x\boldsymbol{i}+P_y\boldsymbol{j}+P_z\boldsymbol{k}$，集中力矩 $\boldsymbol{M} = M_x\boldsymbol{i}+M_y\boldsymbol{j}+M_z\boldsymbol{k}$ 和点电荷 Q，这里 $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ 为笛卡尔坐标的三个单位矢量. 在其表面作用有面力 X_r^α，X_θ^α，X_z^α 及指定电位移 D_n^α.

在柱坐标中，圆锥表面的边界条件为

$$z/r = \cot\alpha:\quad \left.\begin{aligned}
&\sigma_r\cos\alpha-\tau_{rz}\sin\alpha = X_r^\alpha,\quad \tau_{r\theta}\cos\alpha-\tau_{\theta z}\sin\alpha = X_\theta^\alpha,\\
&\tau_{rz}\cos\alpha-\sigma_z\sin\alpha = X_z^\alpha,\quad D_r\cos\alpha-D_z\sin\alpha = D_n^\alpha.
\end{aligned}\right\} \tag{1.5}$$

用 $z = b$（常数）截出一段圆锥，此锥的整体平衡方程为：

$$\boldsymbol{P}+\int_0^{2\pi}\int_0^{b\tan\alpha}(\tau_{rz}\boldsymbol{e}_r+\tau_{\theta z}\boldsymbol{e}_\theta+\sigma_z\boldsymbol{e}_z)\,r\,\mathrm{d}r\,\mathrm{d}\alpha$$
$$+\int_0^{2\pi}\int_0^{b}(X_r^\alpha\boldsymbol{e}_r+X_\theta^\alpha\boldsymbol{e}_\theta+X_z^\alpha\boldsymbol{e}_z)\,z\,\mathrm{d}z\,\mathrm{d}\theta\tan\alpha/\cos\alpha = 0, \tag{1.6}$$

$$Q = \int_0^{2\pi}\int_0^{b\tan\alpha}D_zr\,\mathrm{d}r\,\mathrm{d}\theta+\int_0^{2\pi}\int_0^{b}D_n^\alpha z\,\mathrm{d}z\,\mathrm{d}\theta\tan\alpha/\cos\alpha, \tag{1.7}$$

$$\boldsymbol{M}+\int_0^{2\pi}\int_0^{b\tan\alpha}[-b\tau_{\theta z}\boldsymbol{e}_r+(b\tau_{rz}-r\sigma_z)\boldsymbol{e}_\theta+r\tau_{\theta z}\boldsymbol{e}_z]\,r\,\mathrm{d}r\,\mathrm{d}\theta+$$
$$\int_0^{2\pi}\int_0^{b}[-X_\theta^\alpha\boldsymbol{e}_r+(X_r^\alpha-X_z^\alpha\tan\alpha)\boldsymbol{e}_\theta+X_\theta^\alpha\tan\alpha\boldsymbol{e}_z]\,z^2\mathrm{d}z\tan\alpha/\cos\alpha = 0, \tag{1.8}$$

式中, $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$ 为柱坐标的三个单位向量, 与 $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ 的关系如下:

$$\boldsymbol{e}_r = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \quad \boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}, \quad \boldsymbol{e}_z = \boldsymbol{k} \tag{1.9}$$

本文只研究圆锥侧面为自由的情形•

2 作用 M_z 的扭转问题的解

此为圆锥自由扭转问题, 取

$$\phi_i = 0\ (i = 1, 2, 3) \ \text{和}\ \phi_0 = \frac{A_0}{R_0}, \tag{2.1}$$

式中, $R_0 = (r^2 + s_0^2 z^2)^{\frac{1}{2}}$, $s_0^2 = c_{66}/c_{44}$

将(2.1)代入(1.1)和(1.3)得到位移、电势、应力和电位移的各分量• 常数 A_0 可由总体平衡条件确定, 即

$$M_z + \int_0^{2\pi}\int_0^{b\tan\alpha} r\tau_{r\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = 0 \tag{2.2}$$

将 $\tau_{r\theta}$ 的表达式代入(2.2)得

$$A_0 = M_z \Big\backslash \left[2\pi c_{66}\left(\frac{3\tan^2\alpha + 2s_0^2}{(\tan^2\alpha + s_0^2)^{3/2}} - \frac{2}{s_0}\right.\right. \cdot (\quad \partial\ \phi \tag{2.3}$$

当退化到半空间即 $\alpha = \pi/2$ 时, $A_0 = -M_z s_0/4\pi c_{66}$

3 作用 P_z 和点电荷 Q 的解

这是一个轴对称问题, 取

$$\phi_0 = 0, \quad \phi_i = A_i \ln(R_i + z_i) \quad (i = 1, 2, 3) \cdot \tag{3.1}$$

将(3.1)代入(1.1)和(1.3)可得位移、电势、应力和电位移的各分量的表达式•

锥面的边界条件(1.5)的第二式已经满足, 第三式和第四式可由总体平衡条件推出, 因此需要满足边界条件(1.5)的第一式和下列总体平衡条件:

$$P_z + \int_0^{2\pi}\int_0^{b\tan\alpha} \sigma_z r\,\mathrm{d}r\,\mathrm{d}\theta = 0, \quad Q = \int_0^{2\pi}\int_0^{b\tan\alpha} D_z r\,\mathrm{d}r\,\mathrm{d}\theta \cdot \tag{3.2}$$

将应力和电位移的有关表达式代入(3.2)和(1.5)的第一式得

$$\left.\begin{aligned}
&\sum_{i=1}^{3}\left(\frac{s_i}{H_i\tan\alpha} - 1\right. a_i A_i = -P_z/2\pi, \\
&\sum_{i=1}^{3}\left(\frac{s_i}{H_i\tan\alpha} - 1\right) c_i A_i = Q/2\pi, \\
&\sum_{i=1}^{3}\left(\frac{s_i a_i \tan\alpha}{H_i^3} - \frac{2c_{66}}{H_i N_i} - \frac{n_i s_i}{\tan\alpha H_i^3}\right. A_i = 0,
\end{aligned}\right\} \tag{3.3}$$

式中, $H_i = \sqrt{1 + s_i^2\tan^2\alpha}$, $\quad N_i = (H_i + s_i/\tan\alpha) \quad (i = 0, 1, 2, 3)$ $\quad (3.4)$

求解(3.3)可得 $A_i \quad (i = 1, 2, 3)$• 当退化到半空间, 即 $\alpha = \pi/2$ 时

$$\left.\begin{aligned}
A_1 &= [P_z(s_2a_2c_3 - s_3a_3c_2) + Q(s_2a_2a_3 - s_3a_3a_2)]/\Delta, \\
A_2 &= [P_z(s_3a_3c_1 - s_1a_1c_3) + Q(s_3a_3a_1 - s_1a_1a_3)]/\Delta, \\
A_3 &= [P_z(s_1a_1c_2 - s_2a_2c_1) + Q(s_1a_1a_2 - s_2a_2a_1)]/\Delta,
\end{aligned}\right\} \tag{3.5}$$

式中

$$\Delta = 2\pi\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ s_1a_1 & s_2a_2 & s_3a_3 \end{vmatrix}. \tag{3.6}$$

4 作用 P_x 的弯曲问题

取 $\phi_0 = \frac{A_0 r\sin\theta}{R_0+z_0}$, $\phi_i = \frac{A_i r\cos\theta}{R_i+z_i}$ $(i=1,2,3)$ (4.1)

将(4.1)代入(1.1)和(1.3)得到位移、电势、应力和电位移的各分量.

在锥面自由的情况下,如果在(1.6)中要求下式成立,则不难证明边界条件(1.5)中的第一式和第二式只需保留其一即可.

$$P_x + \int_0^{2\pi}\int_0^{b\tan\alpha}(\tau_{rz}\cos\theta - \tau_{\theta z}\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta = 0 \tag{4.2}$$

将应力和电位移的有关表达式代入(4.2)和(1.5)的第一、第三以及第四式得

$$\left.\begin{aligned}
&\sum_{i=1}^{3}s_ia_i\left(1-\frac{s_i}{H_i\tan\alpha}\right)A_i - c_{44}s_0\left(1-\frac{s_0}{H_0\tan\alpha}\right)A_0 = \frac{P_x}{\pi},\\
&\sum_{i=1}^{3}\left[\frac{2c_{66}}{H_iN_i^2}+\frac{n_i}{H_i^3}-s_ia_i\left(1-\frac{\tan\alpha}{H_iN_i}-\frac{s_i}{H_i^3}\right)\right]A_i + \left(\frac{2c_{66}}{H_0N_0^2}-\frac{c_{44}s_0\tan\alpha}{H_0N_0}\right)A_0 = 0,\\
&\sum_{i=1}^{3}\left(\frac{s_ia_1}{H_iN_i}-\frac{s_i^2a_i}{\tan\alpha H_i^3}-\frac{a_i\tan\alpha}{H_i^3}\right)A_i + \frac{c_{44}s_0}{H_0N_0}A_0 = 0,\\
&\sum_{i=1}^{3}\left(\frac{s_ic_1}{H_iN_i}-\frac{s_i^2c_i}{\tan\alpha H_i^3}-\frac{c_i\tan\alpha}{H_i^3}\right)A_i + \frac{e_{15}s_0}{H_0N_0}A_0 = 0,
\end{aligned}\right\} \tag{4.3}$$

于是由(4.3)可解得 $A_i(i=0,1,2,3)$.

当退化到半空间即 $\alpha=\pi/2$ 时

$$\left.\begin{aligned}
&A_0 = -P_x/2\pi,\quad A_1 = [P_x(a_2c_3-a_3c_2)]/\Delta,\\
&A_2 = [P_x(a_3c_1-a_1c_3)]/\Delta,\quad A_3 = [P_x(a_1c_2-a_2c_1)]/\Delta,
\end{aligned}\right\} \tag{4.4}$$

式中

$$\Delta = 2\pi\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ s_1a_1 & s_2a_2 & s_3a_3 \end{vmatrix}. \tag{4.5}$$

5 作用集中力矩 M_y 的弯曲问题

取 $\phi_0 = \frac{A_0 r\sin\theta}{R_0(R_0+z_0)}$, $\phi_i = \frac{A_i r\cos\theta}{R_i(R_i+z_i)}$ $(i=1,2,3)$ (5.1)

将(5.1)代入(1.1)和(1.3)可得位移、电势、应力和电位移各分量的表达式.

锥面的边界条件为: $X_r^\alpha=0$, $X_\theta^\alpha=0$, $X_z^\alpha=0$ 和 $D_n^\alpha=0$. 由整体平衡可以推出 $X_r^\alpha=0$ 与 $X_\theta^\alpha=0$ 不独立. 除了要满足锥面上的边界条件外,还要满足整体平衡条件

$$M_y - \int_0^{2\pi}\int_0^{b\tan\alpha} r^2\sigma_z\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta = 0 \tag{5.2}$$

由(5.2)式及 $X_r^\alpha=0$, $X_z^\alpha=0$ 和 $D_n^\alpha=0$ 可得到 A_0, A_1, A_2 和 A_3 的下列方程组:

$$
\left.\begin{aligned}
&\sum_{i=1}^{3} a_i\left(\frac{s_i^2}{H_i^3\tan^3\alpha}-\frac{3s_i}{H_i\tan\alpha}-2\right)A_i=\frac{M_Y}{\pi},\\
&\sum_{i=1}^{3}\left[2c_{66}\left(\frac{2}{H_iN_i}-\frac{s_i}{\tan\alpha H_i^3}+\frac{3s_in_i}{\tan\alpha H_i^5}-\tan\alpha\left(\frac{3s_ia_i}{H_i^5}-\frac{s_ia_i}{H_i^3}\right)\right)\right]A_i=\\
&\qquad 0+\left[2c_{66}\left(\frac{2}{H_0N_0}-\frac{s_0}{\tan\alpha H_0^3}\right)-\frac{\tan\alpha}{H_0^3}\right]A_i,\\
&\sum_{i=1}^{3}\left(\frac{s_ia_i}{H_i^5}-\frac{s_ia_i}{H_i^3}-\frac{3s_ia_i}{H_i^5}\right)A_i+\frac{s_0c_{44}}{H_0^3}A_0=0,\\
&\sum_{i=1}^{3}\left(\frac{s_ic_i}{H_i^5}-\frac{s_ic_i}{H_i^3}-\frac{3s_ic_i}{H_i^5}\right)A_i+\frac{s_0e_{15}}{H_0^3}A_0=0\cdot
\end{aligned}\right\}\qquad(5.3)
$$

于是由(5.3)可解得 $A_i(i=0,1,2,3)$•

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Elasticity Solutions for a Piezoelectric Cone Under Concentrated Loads at Its Apex

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Abstract: Based on the general solution of the three_dimensional problem for piezoelectric materials, the problem of a piezoelectric cone subjected to concentrated loads at its apex is solved by trial_and_error method. The displacements and stresses are explicitly given for the cases of compression in the presence of point charge, bending and torsion. These solutions are simple in form and convenient for application. When the apex angle 2α equals π, the solutions for concentrated force, point charge and torsion reduce to those of the half_space problem.

Key words: piezoelectric cone; compression; torsion; bending; point charge