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圆形弹性薄板非轴对称大挠度问题的一个解法

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摘 要

本文建议一个求解圆形弹性薄板非轴对称大挠度问题的方法 本文以周边固定受非轴对称载荷作用下圆形薄板的大挠度问题为例阐述所述方法的原理和解题步骤 文中所述方法可以用以求解其他边界及载荷作用下圆形薄板的非轴对称大挠度问题

关键词 圆形弹性薄板 非轴对称大挠度问题 求解方法

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1 引 言

薄板大挠度问题, 由于其非线性, 求解十分困难 目前, 一般采用近似法进行求解, 其中以摄动法比较有效 对于圆形薄板, 钱伟长[1]采用摄动法求解了轴对称变形下圆形薄板的大挠度问题; 王新志[2]讨论了圆薄板非轴对称大变形的位移解; 秦圣立、黄家寅[3]用 两变形法 研究了圆柱型正交各向异性圆形薄板的非线性非对称弯曲问题

本文建议一个非轴对称变形下圆形薄板大挠度问题的求解方法 首先给出按位移求解圆形薄板大挠度问题的新的控制方程; 然后应用摄动法, 由控制方程导出求解各级近似值的控制方程; 最后以周边固定在式(3 1) 非轴对称载荷作用下圆形薄板的非轴对称大挠度问题为例, 阐述本文所述方法的原理和求解步骤

由于本文在控制方程中将体积应变函数作为独立的待求函数, 所得到的控制方程具有如下特点: 当采用摄动法时, 所得求解各级近似解的控制方程均为独立方程, 对于求解十分有利

2 控制方程及摄动法

设有半径为 a 的圆形薄板, 受分布载荷 q 取用圆柱坐标系 r z, 坐标原点置于圆形薄板中心, r 及 为薄板中面内之极坐标, z 轴向下并垂直于薄板中面 再采用以下无因次量:

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$$
\left.
\begin{aligned}
&= r/a,\quad (u, v) = \frac{(1- \quad)}{h^2}(u, v),\quad w = \sqrt{\frac{1-}{2}}\ \frac{w}{h}\\
&e = \frac{a^2}{h^2}e,\quad (\overline{N}, \overline{N}, \overline{N}) = \frac{(1-\ ^2)a^2}{Eh^3}(N_r, N, N_r)\\
&(\overline{M}, \overline{M}, \overline{M}) = \frac{12a^2}{Eh^4}\sqrt{\frac{(1- \quad)^3}{2}}(M_r, M, M_r)\\
&(\overline{Q}, \overline{Q}) = \frac{a^3}{Dh}\sqrt{\frac{1-}{2}}(Q_r, Q),\quad q(\ ,\ ) = \frac{a^4}{Dh}\sqrt{\frac{1-}{2}}q(r,\ )
\end{aligned}
\right\} \tag{2 1}
$$

板

式中，e 为体积应变，它与 u, v 及 w 存在如下关系：

$$
e\text{小联} \frac{u}{r} + \frac{u}{r} + \frac{1}{r}\ \frac{v}{} + \frac{1}{2}\left[\left(\frac{w}{r}\right)^2 + \left(\frac{1}{r}\ \frac{w}{}\right)^2\right] \tag{2 2}
$$

按位移求解，将体积应变取为独立未知函数时，我们有如下确定 u, v, w 及 e 的控制方程：

$$
\left.
\begin{aligned}
&{}^2e + 2\ \left[\left(\frac{1}{}\ \frac{w}{} + \frac{1}{2}\ \frac{^2w}{2}\right)\frac{^2w}{2} - \left(\frac{1}{}\ \frac{^2w}{}x - \frac{1}{2}\ \frac{w}{}\right)^2 = 0\right.\\
&{}^2U + \frac{1+}{}\ \frac{e}{} - \frac{2(1- \quad)}{}e + 2\ \ {}^2w\ \frac{w}{}! + 2\left[\left(\frac{w}{}\right)^2 + \left(\frac{1}{}\ \frac{w}{}\right)^2\right. = 0\\
&\frac{v}{} = \frac{1-}{}\ e - \frac{U}{} - \ \left[\left(\frac{w}{}\right)^2 + \left(\frac{1}{}\ \frac{w}{}\right)^2\right.\\
&{}^2\ {}^2w - 12\left\{e\ {}^2w + \left[\frac{1}{}\ \frac{U}{} - \frac{U}{ma2} + \left(\frac{w}{}\right)^2\ \frac{^2w}{2} + \left[\frac{1}{}\ \frac{v}{}\right.\right.\right.\\
&\quad + \frac{U}{2} + \left(\frac{1}{}\ \frac{w}{}\right)^2\Bigg]\left(\frac{1}{}\ \frac{w}{} + \frac{1}{2}\ \frac{^2w}{2}\right) + \left[\frac{1}{\S}\ \frac{U}{} + \frac{v}{} - \frac{v}{}\right.\\
&\quad + \frac{2}{}\ \frac{w}{}\ \frac{w}{}\Bigg] \frac{}{} \left(\frac{1}{}\ \frac{w}{}\right\} = q(\ ,\ )
\end{aligned}
\right\} \tag{2 3}
$$

式中，$U = \ \ u,\quad {}^2 = \frac{^2}{2} + \frac{1}{}\ \frac{}{} + \frac{1}{2}\ \frac{^2}{2}$

按位移求解时，有如下用位移表示的内力公式：

薄膜内力

$$
\left.
\begin{aligned}
&\overline{N} = e + \frac{1}{}\ \frac{U}{} - \frac{U}{2} + \left(\frac{w}{}\right)^2 li\\
&\overline{N} = e + \frac{1}{}\ \frac{v}{} + \frac{U}{2} + \left(\frac{1}{}\ \frac{w}{}\right)^2\\
&\overline{N}\ ^{v} \overset{1}{=})\ \frac{1}{2}\left[\frac{1}{2}\ \frac{U}{} + \frac{v}{} - \frac{v}{} + \frac{2}{}\ \frac{w}{}\ \frac{w}{}\right.
\end{aligned}
\right\} \tag{2 4}
$$

弯曲内力

$$\left.\begin{aligned}
\overline{M} &= -\frac{1}{1+}\left[\frac{^2 w}{^2}+\left(\frac{1}{}\frac{w}{}+\frac{1}{^2}\frac{^2 w}{^2}\right)\right] \\
\overline{M}(&= -\frac{1}{1+}\left[\frac{1}{}\frac{w}{}+\frac{1}{^2}\frac{^2 w}{^2}+\frac{^2 w}{^2}\right] \\
\overline{M} &= -\frac{1-}{1+}\left[\frac{1}{}\frac{^2 w}{}-\frac{1}{^2}\frac{w}{}\right] \\
\overline{Q} &= -\frac{}{}(^2 w), \quad \overline{Q} = -\frac{1}{}\frac{}{}(^2 w)
\end{aligned}\right\} \tag{2 5}$$

采用摄动法求解非线性控制方程(2 3), 取 $=\sqrt{\frac{1-}{2}}\frac{w_{max}}{h}$ 为摄动参数, 其中 w_{max} 为薄板最大挠度 根据摄动法原理, 将 e, U, v, w 及 q 分别表示为如下 之级数形式:

$$\left.\begin{aligned}
e &= \overline{e}_2(\ ,\ )^2+\overline{e}_4(\ ,\ )^4+ \\
U &= \overline{U}_2(\ ,\ )^2+\overline{U}_4(\ ,\ )^4+ \\
v &= \overline{v}_2(\ ,\ )^2+\overline{v}_4(\ ,\ )^4+ \\
w &= \overline{w}_1(\ ,\ )+\overline{w}_3(\ ,\ )^3+ \\
q &= \overline{q}_1(\ ,\ )+\overline{q}_3(\ ,\ )^3+
\end{aligned}\right\} \tag{2 6}$$

式中, $\overline{e}_i(\ ,\ )$, $\overline{U}_i(\ ,\ )$, $\overline{v}_i(\ ,\ )$, $\overline{w}_i(\ ,\ )$ 及 $\overline{q}_i(\ ,\ )$ 分别为 $e(\ ,\ )$, $U(\ ,\ )$, $v(\ ,\ )$, $w(\ ,\ )$ 及 $q(\ ,\ )$ 之第 i 级近似值, 均为待确定之函数 为简便计算, 以下分别将它们记为 $\overline{e}_i$, $\overline{U}_i$, $\overline{v}_i$, $\overline{w}_i$ 及 $\overline{q}_i$

将式(2 6) 代入式(2 3) 中比较 F 各次幂之系数, 得如下求解各级近似值的控制方程:

(1) 第一级近似解控制方程

$$)_{2})^2\overline{w}_1=\overline{q}_1 \tag{217}$$

(2) 第二级近似解控制方程

$$\left.\begin{aligned}
)_{2}\mathrm{rd}\,\overline{e}_2 &= 1\ 2L\left[\left(\frac{1}{Q}\frac{5\overline{w}_1}{5Q}+\frac{1}{Q^2}\frac{5^2\overline{w}_1}{5H^2}\ \frac{5^2\overline{w}_1}{5Q^2}-\left(\frac{1}{Q}\frac{5^2\overline{w}_1}{5Q5H}-\frac{1}{Q^2}\frac{5\overline{w}_1}{5H}\right)^2\right]\right] \quad u \\
)_{2}\overline{U}_2 &= \frac{2(1-L)}{L}\overline{e}_2-1\frac{1+L}{L}Q\frac{5\overline{e}_2}{5Q}-2Q)^2\overline{w}_1\frac{5\overline{w}_1}{5Q}-2\left[\left(\frac{5\overline{w}_1}{5Q}\right)^2+\left(\frac{1}{Q}\frac{5\overline{w}_1}{5H}\right)^2\right] \\
\frac{5\overline{v}_2}{5H} &= \frac{1-L}{L}Q\overline{e}_2-\frac{5\overline{U}_2}{5Q}-Q\left[\left(\frac{5\overline{w}_1}{5Q}\right)^2+\left(\frac{1}{Q}\frac{5\overline{w}_1}{5H}\right)^2\right]
\end{aligned}\right\} \tag{218}$$

(3) 第三级近似解控制方程

$$\begin{aligned}
)_{2})^2\overline{w}_3 = 12L\Bigg\{&\overline{e}_2)^2\overline{w}_1+\left[\frac{1}{Q}\frac{5\overline{U}_2}{5Q}-\frac{\overline{U}_2}{Q^2}+\left(\frac{5\overline{w}_1}{5Q}\right)^2\right]\frac{5^2\overline{w}_1}{5Q^2}+\left[\frac{1}{Q}\frac{5\overline{v}_2}{5H}\right. \\
&\left.+\frac{\overline{U}_2}{Q^2}+\left(\frac{1}{Q}\frac{5\overline{w}_1}{5H}\right)^2\right]\left(\frac{1}{Q}\frac{5\overline{w}_1}{5Q}+\frac{1}{Q^2}\frac{5^2\overline{w}_1}{5H^2}\right)+\left[\frac{1}{Q^2}\frac{5\overline{U}_2}{5H}+\frac{5\overline{v}_2}{5Q}-\frac{\overline{v}_2}{Q}\right. \\
&\left.+\frac{2}{Q}\frac{5\overline{w}_1}{5Q}\frac{5\overline{w}_1}{5H}\right]\frac{5}{5Q}\left(\frac{1}{Q}\frac{5\overline{w}_1}{5H}\right)\Bigg\}+\overline{q}_3
\end{aligned} \tag{219}$$

由式(217)、(218)、(219) 等可以看出, 所有控制方程中的方程式均为独立的# 这就是前面提到的本文给出的求解圆形薄板大挠度问题的控制方程(213) 的优点# 无疑, 该优点将给问题的求解带来极大的方便#

31 周边固定圆形薄板非轴对称变形的大挠度问题

设圆形薄板受非轴对称载荷 $q(r, H)$：

$$q(r, H) = \frac{1}{a} pr\cos H \tag{311}$$

其中, p 为常数# 在式(211) 无因次量情况下, 式(311)

$$q(Q, H) = pQ\cos H \tag{312}$$

式中,

$$p = \frac{a^4}{Dh}\sqrt{\frac{1-L}{2L}}p$$

并将其表示为如下级数形式:

$$\overline{p}_i = \overline{p}_1 F + \overline{p}_3 F^3 + , \tag{313}$$

其中, $\overline{p_i}$ 为待定常系数#

对周边固定圆形薄板, 有如下边界条件:

$$Q = 1, \quad w = 0, \quad \frac{5w}{5Q} = 0, \quad U = 0, \quad v = 0 \tag{314}$$

另外, 设

$$Q = 1, \quad e = f(H) \tag{315}$$

其中, $f(H)$ 为待确定函数, 并将它表示为如下关于 F 之级数:

$$f_i(H) = f_2(H)F^2 + f_4(H)F^4 + , \tag{316}$$

式中, $f_i(H)$ 当然也是待确定之函数#

将式(216) 及(316) 代入边界条件(314) 及(315), 并与控制方程(217)、(218)、(219) 等相结合, 且注意到式(312) 及(313), 得到如下求解各级近似值的边值问题:

(1) 求解第一级近似值的边值问题

$$\left.\begin{aligned} &)^2)^2\overline{w}_1 = \overline{p}_1 Q\cos H \\ &Q = 1, \quad \overline{w}_1 = 0, \quad \frac{5\overline{w}_1}{5Q} = 0 \end{aligned}\right\} \tag{317}$$

(2) 求解第二级近似值的边值问题

$$\left.\begin{aligned} &)^2\overline{e}_2 = (-2H\left[\left(\frac{1}{Q}\frac{5\overline{w}_1}{5Q} + \frac{1}{Q^2}\frac{5^2\overline{w}_1}{5H^2}\right)\frac{5^2\overline{w}_1}{5Q^2} - \left(\frac{1}{Q}\frac{5^2\overline{w}_1}{5Q5H} - \frac{1}{Q^2}\frac{5\overline{w}_1}{5H}\right)^2\right] \\ &Q = 1, \quad \overline{e}_2 = f_2(H) \end{aligned}\right\} \tag{318}$$

将

$$\left.\begin{aligned} &)^2\overline{U}_2 = \frac{2(1-L)}{L}\overline{e}_2 - \frac{1+L}{L}Q\frac{5\overline{e}_2}{5Q} - 2Q)^2\overline{w}_1\frac{5\overline{w}_1}{5Q} - 2\left[\left(\frac{5\overline{w}_1}{5Q}\right)^2 + \left(\frac{1}{Q}\frac{5\overline{w}_1}{5H}\right)^2\right] \\ &Q\text{级 } 1, \quad \overline{U}_2 = 0 \end{aligned}\right\} \tag{319}$$

$$\left.\begin{aligned} &\frac{5\overline{v}_2}{5H} = \frac{1-L}{L}Q\overline{e}_2 - \frac{5\overline{U}_1}{5Q} - Q\left[\left(\frac{5\overline{w}_1}{5Q}\right)^2 2\left(\frac{1}{Q}\frac{5\overline{w}_1}{5H}\right)^2\right] \\ &Q = 1, \quad \overline{v}_2 = 0 \end{aligned}\right\} \tag{3110}$$

1

求解第三级近似值的边值问题

$$\nabla^2\nabla^2 \overline{w_3} = 12L\left\{\nabla^2\overline{e_2}\,\nabla^2\overline{w_1} + \left[\frac{1}{Q}\frac{5\overline{U_2}}{5Q} - \frac{\overline{U_2}}{Q^2} + \left(\frac{5\overline{w_1}}{5Q}\right)^2 \frac{5^2\overline{w_1}}{5Q^2} + \left[\frac{1}{Q}\frac{5\overline{v_2}}{5H} + \frac{\overline{U_2}}{Q^2} + \left(\frac{1}{Q}\frac{5\overline{w_1}}{5H}\right)^2\right]\left(\frac{1}{Q}\frac{5\overline{w_1}}{5Q} + \frac{1}{Q^2}\frac{5^2\overline{w_1}}{5H^2}\right) + \left[\frac{1}{Q^2}\frac{5^2\overline{U_2}}{5H^2} + \frac{5\overline{v_2}}{5Q} - \frac{\overline{v_2}}{Q} + \frac{2}{Q}\frac{5\overline{w_1}}{5Q}\frac{5\overline{w_1}}{5H}\right]\frac{5}{5Q}\left(\frac{1}{Q}\frac{5\overline{w_1}}{5H}\right)\right\} + p_3 Q\cos H$$

$$Q = 1,\quad \overline{w_3} = 0,\quad \frac{5\overline{w_3}}{5Q} = 0 \tag{3.11}$$

下面由边值问题(3.7)～(3.11)求解各级近似值#

(1)

由式(3.7)有

$$\left(\frac{5^2}{5Q^2} + \frac{1}{Q}\frac{5}{5Q} + \frac{1}{Q^2}\frac{5^2}{5H^2}\right)\left(\frac{5^2\overline{w_1}}{5Q^2} + \frac{1}{Q}\frac{5\overline{w_1}}{5Q} + \frac{1}{Q^2}\frac{5^2\overline{w_1}}{5H^2}\right) = p_1 Q\cos H$$

$$Q = 1,\quad w_1 = 0,\quad \frac{5\overline{w_1}}{5Q} = 0 \tag{3.12}$$

边值问题(3.12)为圆形薄板线弹性弯曲问题，其解为：

$$\overline{w_1} = \left(A_1 Q + B_1 Q^3 + \frac{1}{192}\overline{p_1}Q^5\right)\cos H \tag{3.13}$$

式中，$A_1 = \frac{1}{192}\overline{p_1}$，$B_1 = -\frac{1}{96}\overline{p_1}$，$\overline{p_1} = 300\sqrt{5}$

(2)

) 求解$\overline{e_2}$

将式(3.8)中之待定函数$f_2(H)$表示为

$$f_2(H) = \sum_{n=0}^{]} A_n\cos nH \tag{3.14}$$

其中，A_n为待定系数#

将式(3.13)代入式(3.8)，有

$$\frac{5^2\overline{e_2}}{5Q^2} + \frac{1}{Q}\frac{5\overline{e_2}}{5Q} + \frac{1}{Q^2}\frac{5^2\overline{e_2}}{5H^2} = -L\left[8B_1^2Q^2 + \frac{1}{4}B_1\overline{p_1}Q^4 + \frac{1}{576}\overline{p_1}^2Q^6 + \left(16B_1^2Q^2 + \frac{5}{12}B_1\overline{p_1}Q^4 + \frac{1}{384}\overline{p_1}^2Q^6\right)\cos 2H\right]$$

$$Q = 1,\quad Le_2 = \sum_{n=0}^{]} A_n\cos nH \tag{3.15}$$

边值问题(3.15)之解为

$$\overline{e_2} = A_0 + L\left(B_0 - \frac{1}{2}B_1^2Q^4 - \frac{1}{144}B_1\overline{p_1}Q^6 - \frac{1}{36864}\overline{p_1}^2Q^8\right) + \left[A_2Q^2 + L\left(B_2Q^2 - \frac{4}{3}B_1^2Q^4 - \frac{5}{384}B_1\overline{p_1}Q^6 - \frac{1}{23040}\overline{p_1}^2Q^8\right)\right]\cos 2H \tag{3.16}$$

式中，$B_0 = \frac{1}{2}B_1^2 + \frac{1}{144}B_1\overline{p_1} + \frac{1}{36864}\overline{p_1}^2$，$B_2 = \frac{4}{3}B_1^2 + \frac{5}{384}B_1\overline{p_1} + \frac{1}{23040}\overline{p_1}^2$，$A_0$及$A_1$尚是未知的，将在面下确定#

) 求解$\overline{U_2}$

将式(3113)及(3116)代入式(319),有

$$\left.\begin{aligned}&\frac{5^2\overline{U_2}}{5Q^2}+\frac{1}{Q}\frac{5\overline{U_2}}{5Q}+\frac{1}{Q^2}\frac{5^2\overline{U_2}}{5H^2}=\frac{2(1-L)}{L}A_0+2(1-L)B_0-2A_1^2-16A_1B_1Q^2\\&-\left[3(11-L)B_1^2+\frac{3}{16}A_1\overline{p_1}\right]Q^4-\frac{13-L}{18}B_1\overline{p_1}Q^6-\frac{70-5L}{18432}\overline{p_1}^2Q^8\\&-\left[(4A_2+12A_1B_1+4LB_2)Q^2+\left(\frac{1}{6}A_1\overline{p_1}+\frac{88-24L}{3}B_1^2\right)Q^4\right.\\&\left.+\frac{130-20L}{192}B_1\overline{p_1}Q^6+\frac{42-5L}{11520}\overline{p_1}^2Q^8\right]\cos 2H\\&Q=1,\ \overline{U_2}=0\end{aligned}\right\}\tag{3117}$$

边值问题(3117)之解为:

$$\begin{aligned}\overline{U_2}=&\left(A_3+\frac{1-L}{2}B_0-\frac{1}{2}A_1^2\right)Q^2-A_1B_1Q^4-\left(\frac{11-L}{12}B_1^2+\frac{1}{192}A_1\overline{p_1}\right)Q^6\\&-\frac{13-L}{1152}B_1\overline{p_1}Q^8-\frac{14-L}{368640}\overline{p_1}^2Q^{10}+\left[\left(\frac{1}{3}A_2+B_3\right)Q^2-\left(A_1B_1+\frac{1}{3}A_2+\frac{L}{3}B_2\right)Q^4\right.\\&\left.-\left(\frac{11-3L}{12}B_1^2+\frac{1}{192}A_1\overline{p_1}\right)Q^6-\frac{13-2L}{1152}B_1\overline{p_1}Q^8-\frac{42-5L}{1105920}\overline{p_1}^2Q^{10}\right]\cos 2H\end{aligned}\tag{3118}$$

式中

$$A_3=A_1B_1+\frac{1}{2}A_1^2-\frac{1-L}{2}B_0+\frac{1}{192}A_1\overline{p_1}+\frac{11-L}{12}B_1^2+\frac{13-L}{1152}B_1\overline{p_1}+\frac{14-L}{368640}\overline{p_1}^2$$

$$B_3=A_1B_1+\frac{L}{3}B_2+\frac{1}{192}A_1\overline{p_1}+\frac{11-3L}{12}B_1^2+\frac{13-2L}{1152}B_1\overline{p_1}+\frac{42-5L}{1105920}\overline{p_1}^2$$

$$A_0=\frac{2L}{1-L}A_3$$

) 求解$\overline{v_2}$

将式(3113)、(3116)及(3118)代入(3110),有

$$\left.\begin{aligned}&\frac{5\overline{v_2}}{5H}=2\left[-\left(\frac{2}{3}A_2+2B_3\right)Q+\left(2A_1B_1+\frac{3+L}{3L}A_2+\frac{3+L}{3}B_2\right)Q^3+\left(\frac{1}{96}A_1\overline{p_1}\right.\right.\\&\left.\left.+\frac{1-L}{6}B_1^2\right)Q^5+\frac{5-L}{1152}B_1\overline{p_1}Q^7+\frac{6-L}{552960}\overline{p_1}^2Q^9\right]\cos 2H\\&Q=1,\ v_2=0\end{aligned}\right\}\tag{3119}$$

将式(3119)第一式对 H 进行积分,并利用条件: $H=0,\overline{v_2}=0$ 确定由于积分出现的任意函数$f(Q)$,有

$$\left.\begin{aligned}\overline{v_2}=&\left[-\left(\frac{1}{3}A_2+B_2\right)Q+\left(A_1B_1+\frac{3+L}{6L}A_2+\frac{3+L}{6}B_2\right)Q^3+\left(\frac{1}{192}A_1\overline{p_1}\right.\right.\\&\left.\left.+\frac{1-L}{12}B_1^2\right)Q^5+\frac{5-L}{2304}B_1\overline{p_1}Q^7+\frac{6-L}{1105920}\overline{p_1}^2Q^9\right]\sin 2H\end{aligned}\right\}\tag{3120}$$

然后,利用式(3119)中之边界条件: $Q=1,\overline{v_2}=0$ 确定常数 A_2,有

$$A_2=\frac{6L}{3-L}\left[B_3-A_1B_1-\frac{1-L}{12}B_1^2-\frac{3+L}{6}B_2-\frac{1}{192}A_1\overline{p_1}-\frac{5-L}{2304}B_1\overline{p_1}-\frac{6-L}{1105920}\overline{p_1}^2\right.$$

(3)

将式(3113)、(3116)、(3118)及(3120)代入式(3111),有

$$\left\{\begin{aligned} &\left(\nabla^2+\frac{1}{Q}\frac{5}{5Q}+\frac{1}{Q^2}\frac{5^2}{5H^2}\right)\left(\frac{5^2\overline{w}_3}{5Q^2}+\frac{1}{Q}\frac{5\overline{w}_3}{5Q}+\frac{1}{Q^2}\frac{5^2\overline{w}_3}{5H^2}\right)=12L\Bigg\{2R_1(Q)\\ &\quad+R_2(Q)\left(4B_1Q+\frac{1}{16}\overline{p}_1Q^3\right)+[2R_3(Q)+R_4(Q)]\left(3B_1Q+\frac{5}{96}\overline{p}_1Q^3\right.\\ &\quad+[2R_5(Q)+R_6(Q)-R_7(Q)]\left(B_1Q+\frac{1}{96}\overline{p}_1Q^3\right)+\frac{1}{12L}\overline{p}_3Q\Bigg\}\cos H\\ &\quad+12L\Bigg\{R_2(Q)\left(4B_1Q+\frac{1}{16}\overline{p}_1Q^3\right)+R_4(Q)\left(3B_1Q+\frac{5}{96}\overline{p}_1Q^3+[R_6(Q)\right.\\ &\quad+R_7(Q)]\left(B_1Q+\frac{1}{96}\overline{p}_1Q^3\right)\Bigg\}\cos 3H\\ &Q=1,\quad \overline{w}_3=0,\quad \frac{5\overline{w}_3}{5Q}=0 \end{aligned}\right. \tag{3.21}$$

式中

$$R_1(Q)=A_0+LB_0-\frac{L}{2}B_1^2Q^4-\frac{L}{144}B_1\overline{p}_1Q^6-\frac{L}{36864}\overline{p}_1^2Q^8$$

$$R_2(Q)=(A_2+LB_2)Q^2-\frac{4L}{3}B_1^2Q^4-\frac{5L}{384}B_1\overline{p}_1Q^6-\frac{L}{23040}\overline{p}_1^2Q^8$$

$$R_3(Q)=\frac{1-L}{2L}A_0+\frac{1-L}{2}B_0-\frac{1-5L}{12}B_1^2Q^4-\frac{1-7L}{1152}B_1\overline{p}_1Q^6-\frac{1-9L}{368640}\overline{p}_1^2Q^8$$

$$R_4(Q)=\frac{1}{2}A_1^2+\frac{1}{3}A_2+B_3-(A_2+LB_2)-\frac{1-15L}{12}B_1^2Q^4-\frac{1-14L}{1152}B_1\overline{p}_1Q^6-\frac{1-15L}{368640}\overline{p}_1^2Q^8$$

$$R_5(Q)=\frac{1-L}{2L}A_0+\frac{1-L}{2}B_0-\frac{5-L}{12}B_1Q^4-\frac{7-L}{1152}B_1\overline{p}_1Q^6-\frac{9-L}{368640}\overline{p}_1^2Q^8$$

$$R_6(Q)=-\frac{1}{2}A_1^2-\frac{1}{3}A_2-B_3+\left(\frac{1}{L}A_2+B_2\right)Q^2-\frac{15-L}{12}B_1^2Q^4$$
$$-\frac{14-L}{1152}B_1\overline{p}_1Q^6-\frac{15-L}{368640}\overline{p}_1^2Q^8$$

$$R_7(Q)=-A_1^2-\frac{2}{3}A_2-2B_3+\frac{1+L}{L}(A_2+B_2)Q^2-\frac{5(1+L)}{6}B_1^2Q^4$$
$$-\frac{7(1+L)}{1152}B_1\overline{p}_1Q^6-\frac{9(1+L)}{552960}\overline{p}_1^2Q^8$$

边值问题(3.21)之解为:

$$\begin{aligned} \overline{w}_3=&\Bigg[\left(\frac{D_2-3D_1}{2}+\frac{1}{192}\overline{p}_3\right)Q+\left(\frac{D_1-D_2}{2}-\frac{1}{96}\frac{\overline{p}_3}{6L}\right)Q^3+\frac{C_1}{192}Q^5+\frac{C_2}{1152}Q^7\\ &+\frac{C_3}{3840}Q^9-\frac{L(1+L)}{15360}B_1^2\overline{p}_1Q^{11}-\frac{L(1197+225L)}{20160 @ 92160}B_1\overline{p}_1^2Q^{13}-\frac{L(570-78L)}{37632 @ 5898240}\overline{p}_1^3Q^{15}\\ &+\frac{1}{192}\overline{p}_3^3Q^5\Bigg]\cos H+\Bigg[\frac{1}{2}(D_4-5D_3)Q^3+\frac{1}{2}(3D_3-D_4)Q^5+\frac{C_4}{640}Q^7+\frac{C_5}{2880}Q^9\\ &\frac{L(105-51L)}{387072}B_1^2p_1Q^{11}-\frac{L(178-95L)}{17920 @ 11520}B_1\overline{p}_1Q^{13}-\frac{13L(1+L)}{34560 @ 1474560}\overline{p}_1^3Q^{15}\Bigg]\cos 3H \end{aligned} \tag{3.22}$$

式中

$$C_1=12LB_1\left[\frac{4(1+L)}{L}A_0+4(1+L)B_0+2A_1^2+\frac{4}{3}A_2+4B_3\right]$$

$$C_2=12L\left\{\frac{L^2-1}{L}B_1B_2+\frac{1}{96}\overline{p}_1\left[\frac{6(1+L)}{L}A_0+6(1+L)B_0+3A_1^2+2A_2+6B_3\right]\right.$$

$$C_3 = 12\mathrm{L}\left[\frac{\mathrm{L}^2-1}{96\mathrm{L}}B_2\overline{p_1} - 2(1+L)B_1^2\right]$$

$$C_4 = -12\mathrm{L}\left\{\frac{1+L}{L}B_1[2A_2+(1+L)B_2] + \frac{1}{96}\overline{p_1}\left(A_1^2+\frac{2}{3}A_2+2B_3\right)\right\}$$

$$C_5 = -12\mathrm{L}\left[\frac{1}{96}\overline{p_1}\left(\frac{2}{\mathrm{L}}A_2+\frac{1+2v-L^2}{\mathrm{L}}B_{\mathrm{Q}}\right)+\frac{7(1+L)}{3}B_1^2\right]$$

$$D_1 = \frac{C_1}{192}+\frac{C_2}{1152}+\frac{C_3}{3840}-\frac{L(1+L)}{15360}B_1^2\overline{p_1}-\frac{L(1197+225\mathrm{L})}{20160 @ 92160}B_1\overline{p_1}^2 - \frac{L(570-78\mathrm{L})}{37632 @ 5898240}\overline{p_1}^3$$

$$D_2 = \frac{5C_1}{192}+\frac{7C_2}{1152}+\frac{9C_3}{3840}-\frac{11\mathrm{L}(1+L)}{15360}B_1^2\overline{p_1}-\frac{13\mathrm{L}(1197+225\mathrm{L})}{20160 @ 92160}B_1\overline{p_1}^2 - \frac{L(285-39\mathrm{L})}{18816 @ 393216}\overline{p_1}^3$$

$$D_3 = \frac{C_4}{640}+\frac{C_5}{2880}-\frac{L(105-51\mathrm{L})}{387072}B_1^2\overline{p_1}-\frac{L(178-95\mathrm{L})}{17920 @ 11520}B_1\overline{p_1}^2 - \frac{13\mathrm{L}(1+L)}{34560 @ 1474560}\overline{p_1}^3$$

$$D_4 = \frac{7C_4}{640}+\frac{9C_5}{2880}-\frac{11\mathrm{L}(105-51\mathrm{L})}{387072}B_1^2\overline{p_1}-\frac{13\mathrm{L}(178-95\mathrm{L})}{17920 @ 11520}B_1\overline{p_1}^2 - \frac{13\mathrm{L}(1+L)}{34560 @ 98304}\overline{p_1}^3$$

$$\overline{p_3} = -\frac{192}{(1-\mathrm{Q}_0^2)^2}\left\{\frac{1}{2}(D_2-3D_1)+\frac{1}{2}(D_1-D_2+D_4-5D_3)\mathrm{Q}_0^2+\left[\frac{1}{2}(3D_3-D_4)+\frac{C_1}{192}\right]\mathrm{Q}_0^4+\left(\frac{C_2}{1152}+\frac{C_4}{640}\right)\mathrm{Q}_0^6+\left(\frac{C_3}{3840}+\frac{C_5}{2880}\right)\mathrm{Q}_0^8-\left[\frac{L(1+L)}{15360}+\frac{L(105-51\mathrm{L})}{387072}\right]B_1^2\overline{p_1}\mathrm{Q}_0^{10}-\left[\frac{L(1197+225\mathrm{L})}{20160 @ 92160}+\frac{L(178-95\mathrm{L})}{17920 @ 11520}\right]B_1\overline{p_1}^2\mathrm{Q}_0^{12}-\left[\frac{L(570-78\mathrm{L})}{37632 @ 5898240}+\frac{13\mathrm{L}(1+L)}{34560 @ 1474560}\right]\overline{p_1}^3\mathrm{Q}_0^{14}\right\}$$

$$\mathrm{Q}_0 = \frac{1}{\sqrt{5}}$$

利用摄动法求解时, 一般求解到第三级近似解即可得到足够的精确度# 因此, 我们也将求解过程终止到第三级近似值# 于是, 由式(216)及(313)有

$$\left.\begin{aligned} e &= \overline{e_2}(Q,H)F^2, \quad U = \overline{U_2}(Q,H)F^2, \quad v = \overline{v_2}(Q,H)F^2 \\ w &= \overline{w_1}(Q,H)F + \overline{w_3}(Q,H)F^3, \quad p = \overline{p_1}\mathrm{F} + \overline{p_3}\mathrm{F}^3 \end{aligned}\right\} \qquad (3123)$$

根据给定的载何 p 计算 $p = \frac{a^4}{Dh}\sqrt{\frac{1-L}{2\mathrm{L}}}p$; 然后将 p 代入式(3123)最后一式求解相应的 F 值; 再将所得的 F 值代入式(3123)其余各式, 便得到薄板变形 $e(Q,H)$, $U(Q,H)$, $v(Q,H)$ 及 $w(Q,H)$ 之计算式; 进而将 $e(Q,H)$, $U(Q,H)$, $v(Q,H)$ 及 $w(Q,H)$ 代入式(214)及(215)则得到薄板内力的计算式# 至此, 问题的求解便完结了#

以上虽然讨论的是周边固定在式(311)载荷作用下圆形薄板非轴对称大挠度的求解问题, 但所述方法是一般性的, 可用以求解具有其他边界载荷的圆形薄板非轴对称大挠度问题#

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A New Technique for Solving Non_Axially Symmetrical Large Deflection of Elastic and Thin Circular Plates

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Abstract

A new technique for solving large deflection problem of circular plates flexural non_axisymmetrically is proposed in this paper. The large deflection problem of a circular plate with built_in edge under non_axisymmetrical load is taken as an example to clarify the principle and procedure of the technique mentioned here. The technique given here can also be used to solve large deflection problem of circular plates under other non_axisymmetrical loads and boundary conditions.

Key words thin elastic circular plates, non_axisymmetrical large deflection problem, a new technique