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一类拟线性抛物型方程广义解的Hölder连续性

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摘　要

考虑如下拟线性抛物型方程

$$u_t - \operatorname{div}\boldsymbol{A}(x, t, u, \nabla u) + B(x, t, u, \nabla u) = 0$$

在 $\boldsymbol{A}, B$ 满足很一般的结构条件下证明了它的广义解在 $Q = G \times (0, T)$ 上的局部Hölder连续性.

关键词　抛物型方程　自然增长条件　广义解　局部Hölder连续性
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§1. 引　　言

设 G 是 n 维欧氏空间 E^n 中的有界区域, $W_2^1(G)$ 和 $\overset{\circ}{W}{}_2^1(G)$ 是通常的Sobolev空间. 设 $0 < T < +\infty$, $p \geqslant 1$, $L_p(0, T, L_2(G)) = \{u;\ u: (0, T) \to L_2(G)\}$ 为Banach空间, 范数为

$$\| u \|_{L_p(0, T, L_2(G))} = \left(\int_0^T \| u \|_{L_2(G)}^p dt \right)^{1/p}$$

$$\| u \|_{L_\infty(0, T, L_2(G))} = \underset{t \in (0, T)}{\operatorname{vraimax}} \| u \|_{L_2(G)}$$

$W_2^1(0, T, L_2(G))$ 等类似地定义, 记 $Q = G \times (0, T)$, 在 Q 考虑方程

$$u_t - \operatorname{div}\boldsymbol{A}(x, t, u, \nabla u) + B(x, t, u, \nabla u) = 0 \tag{1.1}$$

其中 $u_t = \partial u / \partial t$, $\nabla u = \{\partial u / \partial x^\alpha, \alpha = 1, 2, \cdots, n\}$, $\boldsymbol{A}(x, t, u, \xi)$ 和 $B(x, t, u, \xi)$ 分别在 $Q \times E^1 \times E^n$ 上定义, 对固定的 x, t 关于 u, ξ 为连续, 对固定的 u, ξ 关于 x, t 可测, 且满足如下结构条件:

$$\left.\begin{aligned} &\nabla u \cdot \boldsymbol{A}(x, t, u, \nabla u) \geqslant |\nabla u|^2 - \kappa |u|^l - f_0(x, t) \\ &|\boldsymbol{A}(x, t, u, \nabla u)| \leqslant \kappa_1 |\nabla u| + \kappa |u|^{l/2} + f_1(x, t) \\ &|B(x, t, u, \nabla u)| \leqslant C(x, t) |\nabla u|^\gamma + \kappa |u|^{l-1} + f_2(x, t) \end{aligned}\right\} \tag{1.2}$$

其中 $\kappa \geqslant 0$, $\kappa_1 \geqslant 1$, $l = 2\left(1 + 2/n\right)$, $1 \leqslant \gamma \leqslant 2$ 为常数,

$$C(x, t) \in L_r(Q) \tag{1.3}$$

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$$\left.\begin{array}{ll} f_i(x,t)\in L_{S_i}(Q) & (i=1,2) \\ S_0, S_2>(n+2)/2, & S_1>n+2 \end{array}\right\} \tag{1.4}$$

称 $u\in L_\infty(0,T,L_2(G))\cap L_2(0,T,W_2^1(G))$ 为(1.1)的广义解，则对 $\forall t\in(0,T)$，$v\in W_2^1(0,T,L_2(G))\cap L_2(0,T,\overset{\circ}{W}{}_2^1(G))$，满足

$$\int_0^t\int_G\left\{-v_tu+\nabla v\cdot\boldsymbol{A}(x,t,u,\nabla u)+vB(x,t,u,\nabla u)\right\}dxdt$$

$$+\int_G v(x,t)u(x,t)\Big|_{t=0}^{t=t}dx=0 \tag{1.1$'$}$$

$$(\forall t\in(0,T),\ v\in W_2^1(0,T,L_2(G))\cap L_2(0,T,\overset{\circ}{W}{}_2^1(G)))$$

当 B 满足自然增长条件时，为使(1.1)有意义，必须增加对 u 的适当可积性限制．在 $\gamma=2$ 时，对 u 的补充限制是 $u\in L_\infty(Q)$ 并在[1]中有详尽的研究，若假设

$$r>(n+2)/(2-\gamma) \tag{1.5}$$

和

$$t^*=\begin{cases}\dfrac{r(n+2)(\gamma-1)}{r(2-\gamma)-(n+2)}, & \text{当}\dfrac{n+2}{2-\gamma}<r<+\infty \\ (n+2)(\gamma-1)/(2-\gamma), & \text{当}\ r=+\infty\end{cases}$$

在增加要求 $u\in L_{t^*}(Q)$ 的条件下，[2]中证明了方程(1.1)的广义解在 Q 内局部有界．本文我们将进一步证明(1.1)的广义解在 Q 内的局部Hölder连续性．即我们证明了如下结果：

定理 设条件(1.2)～(1.5)满足，并且出现在条件(1.2)中的 $\gamma\in(1+2/(n+2),2)$，设 $u\in L_\infty(0,T,L_2(G))\cap L_2(0,T,W_2^1(G))\cap L_\infty(Q)$ 是(1.1)的广义解，则 u 在 Q 内局部Hölder连续．

记 $B(x_0,\rho)=\left\{x\in E^n;\ |x-x_0|<\rho\right\}$，$B(\rho)=B(0,\rho)$．为证明定理成立，我们要用到如下引理．

引理[1] 设 $u\in W_1^1(B(\rho))$，在 $B(\rho)$ 的某个正测度集 S 上，$u=0$．设 $\eta(x)=\eta(|x|)$ 是 $|x|$ 的非增连续函数，满足 $0\leqslant\eta(x)\leqslant 1$ 并在 S 上取值为 $\eta(x)=1$．那么对任何可测子集 $e\subset B(\rho)$，成立

$$\int_e|u(x)|\eta(x)dx\leqslant\frac{C(n)\rho^n}{\mathrm{mes}S}\mathrm{mes}^{1/n}e\int_{B(\rho)}|\nabla u(x)|\eta(x)dx$$

§2. 定理的证明

由于假设 u 在 Q 为有界，这相等于可取 $\kappa=0$ 来作证明即可．不妨设 $B(\rho)\times(0,\rho^2)\subset Q$，令

$$\zeta(x)=\zeta(|x|)=\begin{cases}1, & \text{当}\ |x|\leqslant(1-\lambda)\rho \\ (\lambda\rho)^{-1}(\rho-|x|), & \text{当}\ (1-\lambda)\rho<|x|<\rho \\ 0, & \text{当}\ |x|\geqslant\rho\end{cases}$$

其中 $\lambda\in(0,1)$ 为常数，由[1]又可知，只须对 $u_t\in L_2(Q)$ 作出证明．为此对 $k\in(-M,M)$，取 $v=\zeta^2(x)(u-k)^+$ 作试验函数，将它代入(1.1)$'$ 并利用(1.2)可得(已设 $\kappa=0$)：

$$0=\int_0^t\int_{B(\rho)}\left\{\zeta^2(u-k)^+u_t+\zeta^2\nabla(u-k)^+\cdot\boldsymbol{A}+\zeta^2(u-k)^+B\right.$$

$$+\ 2\zeta(u-k)^+\ \nabla\zeta\cdot \boldsymbol{A}\Big\}dxdt$$

$$\begin{aligned}\frac{1}{2}\int_{B(\rho)}\zeta^2|(u-k)^+|^2_{t=t}dx+\int_0^t\int_{B(\rho)}\zeta^2|\nabla(u-k)^+|^2dxdt\\ \leqslant\frac{1}{2}\int_{B(\rho)}\zeta^2|(u-k)^+|^2_{t=0}dx\\ +\int_0^t\int_{B(\rho)}\Big\{\zeta^2 f_0(x,t)+\zeta^2(u-k)^+\big(C(x,t)|\nabla u|^\gamma+f_2(x,t)\big)\\ +2(u-k)^+|\nabla\zeta|\big(\kappa_1\zeta|\nabla u|+f_1(x,t)\big)\Big\}dxdt\end{aligned}\qquad(2.1)$$

设 $\tau\in(0,\rho^2)$, 由(2.1) 继续得

$$\int_{B(\rho)}\zeta^2|(u-k)^+|^2_t dx\leqslant\int_{B(\rho)}|(u-k)^+|^2_{t=0}dx+2I(\tau),\qquad t\in(0,\tau)\qquad(2.2)$$

$$\begin{aligned}\frac{1}{2}\left\{\underset{0<t<\tau}{\mathrm{vraimax}}\int_{B(\rho)}\zeta^2|(u-k)^+|^2dx+\int_0^\tau\int_{B(\rho)}\zeta^2|\nabla(u-k)^+|^2dxdt\right.\\ \leqslant\int_{B(\rho)}|(u-k)^+|^2_{t=0}dx+2I(\tau)\end{aligned}\qquad(2.3)$$

其中

$$\begin{aligned}I(\tau)=\int_0^\tau\int_{B(\rho)}\Big\{\zeta^2 f_0+\zeta^2(u-k)^+\big(C(x,t)|\nabla u|^\gamma+f_2\big)\\ +2(u-k)^+|\nabla\zeta|\big(\kappa_1\zeta|\nabla u|+f_1\big)\Big\}dxdt\\ \leqslant\varepsilon\int_0^\tau\int_{B(\rho)}\zeta^2|\nabla(u-k)^+|^2dxdt+C(\varepsilon,\\ M)\Big\{\frac{1}{(\lambda\rho)^2}\int_0^\tau\int_{B(\rho)}|(u-k)^+|^2dxdt\\ +\|C(x,t)\|_{\frac{2}{r(2-\gamma)}}(\tau\rho^n)^{\frac{2}{l}+\frac{2}{n+2}-\frac{2}{r(2-\gamma)}}\\ +\|f_2\|_{L_{S_2}(G)}(\tau\rho^n)^{\frac{2}{l}+\frac{2}{n+2}-\frac{1}{S_2}}+\frac{1}{\lambda\rho}\|f_1\|_{L_{S_1}(Q)}(\tau\rho^n)^{\frac{2}{l}+\frac{2}{n+2}-\frac{1}{S_1}}\\ +\|f_0\|_{L_{S_0}(G)}(\tau\rho^n)^{\frac{2}{l}+\frac{2}{n+2}-\frac{1}{S_0}}\Big\}\end{aligned}\qquad(2.4)$$

根据对 r, S_i 的假定,

$$3\sigma=\min\left\{\frac{2}{n+2}-\frac{2}{r(2-\gamma)},\ \frac{2}{n+2}-\frac{1}{S_0},\ \frac{1}{n+2}-\frac{1}{S_1},\ \frac{2}{n+2}-\frac{1}{S_2}\right.>0\qquad(2.5)$$

不妨设 $\rho\leqslant 1$, 考虑到 $\tau\in(0,\rho^2)$, 由(2.4) 继续得

$$\begin{aligned}I(\tau)\leqslant\varepsilon\int_0^\tau\int_{B(\rho)}\zeta^2|\nabla(u-k)^+|^2dxdt\\ +C(\varepsilon,M)\left\{\frac{1}{(\lambda\rho)^2}\int_0^\tau\int_{B(\rho)}|(u-k)^+|^2dxdt+\frac{1}{\lambda}(\tau\rho^n)^{2/l+3\sigma}\right\}\end{aligned}\qquad(2.6)$$

其中我们把 $\|C\|_{L_r(Q)}$, $\|f_i\|_{L_{S_i}(Q)}$ 吸收到 $C(\varepsilon,M)$ 中, 后者与 k, λ, ρ 无关. 联合(2.3)、(2.6), 取 $\varepsilon>0$ 足够小, 即得

$$\begin{aligned}\underset{0<t<\tau}{\mathrm{vraimax}}\int_{B(\rho)}\zeta^2|(u-k)^+|^2dx+\int_0^\tau\int_{B(\rho)}\zeta^2|\nabla(u-k)^+|^2dxdt\\ \leqslant C\left\{\int_{B(\rho)}|(u-k)^+|^2_{t=0}dx+\frac{1}{(\lambda\rho)^2}\int_0^\tau\int_{B(\rho)}|(u-k)^+|^2dxdt\right.\\ \left.+\frac{1}{\lambda}(\tau\rho^n)^{2/l+3\sigma}\right\}\end{aligned}\qquad(2.7)$$

将(2.7)代入(2.6),又得

$$I(\tau) \leqslant \varepsilon\int_{B(\rho)} |(u-k)^+|_{t=0}dx + C(\varepsilon, M)\left\{\frac{1}{(\lambda\rho)^2}\int_0^\tau\int_{B(\rho)} |(u-k)^+|^2 dx dt + \frac{1}{\lambda}(\tau\rho^n)^{2/l+3\sigma}\right\} \tag{2.8}$$

再联合(2.2),继续得

$$\int_{B(\rho-\lambda\rho)} |(u-k)^+|_t^2 dx \leqslant (1+\varepsilon)\int_{B(\rho)} |(u-k)^+|_{t=0}^2 dx + C(\varepsilon, M)\left\{\frac{1}{(\lambda\rho)^2}\int_0^\tau\int_{B(\rho)} |(u-k)^+|^2 dx dt + \frac{1}{\lambda}(\tau\rho^n)^{2/l+3\sigma}\right\} \tag{2.9}$$

设 $\theta \in (0, 1)$ 待定, k_0 满足

$$\operatorname{mes}(B(\rho) \cap \{u(x, 0) > k_0\}) \leqslant \frac{1}{2}\operatorname{mes}B(\rho) \tag{2.10}$$

$$\operatorname*{vraimax}_{B(\rho)\times(0,\theta\rho^2)} u(x, t) - k_0 \leqslant H \text{ 并且 } H \geqslant \rho^{(n+2)\sigma} \tag{2.11}$$

取 $k = k_0$, $\tau = \theta\rho^2$, 则对 $\forall t \in (0, \theta\rho^2)$, 据(2.9)~(2.11)得

$$\left(\frac{3}{4}H\right)^2 \operatorname{mes}\left(B(\rho-\lambda\rho) \cap \left\{x u(x, t) > k_0 + \frac{3}{4}H\right\}\right) \leqslant H^2\operatorname{mes}B(\rho)\left\{\frac{1+\varepsilon}{2} + C(\varepsilon, M)\left[\frac{\theta}{\lambda^2} + \frac{1}{\lambda}\theta^{2/l+3\sigma}\right]\right\} \tag{2.12}$$

从而易得

$$\operatorname{mes}(B(\rho) \cap \{u(x, t) > k_0 + 3H/4\}) \leqslant \operatorname{mes}B(\rho)\left\{[1-(1-\lambda)^n] + \frac{8}{9}(1+\varepsilon) + C(\varepsilon, M)\left[\frac{\theta}{\lambda^2} + \frac{1}{\lambda}\theta^{2/l+3\sigma}\right]\right\} \tag{2.13}$$

根据(2.13),只要取 $\varepsilon > 0$, $\lambda > 0$ 足够小,然后再取 θ 足够小,可使

$$\operatorname{mes}\left(B(\rho) \cap \left\{u(x, t) > k_0 + \frac{3}{4}H\right\}\right) \leqslant (1-\theta_1)\operatorname{mes}B(\rho) \tag{2.14}$$

其中 $\theta_1 > 0$ 是常数且与 ρ, k_0, H 无关•(2.14)等价于

$$\operatorname{mes}\left(B(\rho) \cap \left\{u(x, t) \leqslant k_0 + \frac{3}{4}H\right\}\right) \geqslant \theta_1\operatorname{mes}B(\rho), \quad \forall t \in (0, \theta\rho^2) \tag{2.15}$$

简记

$$Q_0(\rho) = B(2\rho) \times (0, \theta\rho^2), \quad Q_1(\rho) = B(\rho) \times \left(\frac{3}{4}\theta\rho^2, \theta\rho^2\right)$$

设 $\rho \leqslant 1$, $Q_0(\rho) \subset Q$, 我们要证明

$$\operatorname*{osc}_{Q_1(\rho)} u = \operatorname*{vraimax}_{Q_1(\rho)} u - \operatorname*{vraimin}_{Q_1(\rho)} u \leqslant \eta \operatorname*{osc}_{Q_0(\rho)} u + C\rho^{(n+2)\sigma} \tag{2.16}$$

其中 $\eta \in (0, 1)$, $C > 0$ 是与 ρ 无关的常数• 为此,命

$$\mu_1 = \operatorname*{vraimax}_{Q_0(\rho)} u, \quad \mu_2 = \operatorname*{vraimin}_{Q_0(\rho)} u$$

那么集合 $B(\rho) \cap \{u(x, 0) > (\mu_1+\mu_2)/2\}$ 或 $B(\rho) \cap \{u(x, 0) < (\mu_1+\mu_2)/2\}$ 中必有一个满足(不妨设是):

$$\operatorname{mes}\left(B(\rho) \cap \left\{u(x, 0) > \frac{1}{2}(\mu_1+\mu_2)\right\}\right) \leqslant \frac{1}{2}\operatorname{mes}B(\rho) \tag{2.17}$$

故若取 $k_0=(\mu_1+\mu_2)/2$, 那么(2.10)满足, 不妨设

$$\mu_1-\mu_2\geqslant 2\rho^{(n+2)\sigma} \tag{2.18}$$

(否则(2.16)成立), 据(2.18), 取 $H=(\mu_1+\mu_2)/2$, 则(2.11)成立. 从而对 $\forall t\in(0,\theta\rho^2)$, 由(2.15)得

$$\mathrm{mes}\left(B(\rho)\cap\left\{u(x,t)\leqslant\frac{7}{8}\mu_1+\frac{1}{8}\mu_2\right\}\right)\geqslant\theta_1\mathrm{mes}B(\rho) \tag{2.19}$$

下记

$$F=F_0\rho^{(n+2)\sigma},\quad W=\ln\frac{(\mu_1-\mu_2)/8}{\mu_1-u+F}$$

而

$$F_0=\|f_0\|^{1/2}_{L_{s_0}(Q)}+\|f_0\|_{L_{s_0}(Q)}+\|f_1\|_{L_{s_1}(Q)}+\|f_2\|_{L_{s_2}(Q)}+1$$

则由 μ_1,μ_2 的定义和(2.18), 知在 $Q_0(\rho)$ 上成立

$$\left.\begin{aligned}&0<F\leqslant\mu_1-u+F\leqslant(\mu_1-\mu_0)(1+2F_0)\\&W\geqslant\ln\frac{1}{8(1+2F_0)}=-\ln 8(1+2F_0)\end{aligned}\right\} \tag{2.20}$$

设 $\zeta(x)=\zeta(|x|)$ 和 $\phi(t)$ 分别满足

$$\zeta(x)=\begin{cases}1, & |x|\leqslant 3\rho/2\\ 0, & |x|\geqslant 2\rho,\end{cases}\qquad |\nabla\zeta(x)|\leqslant 2/\rho$$

$$\phi(t)=\begin{cases}0, & t\leqslant 0\\ 1, & t\geqslant\theta\rho^2/2,\end{cases}\qquad 0\leqslant\phi'(t)\leqslant 2/\theta\rho^2$$

取

$$v=\frac{\zeta^2(x)\phi^2(t)}{\mu_1-u+F}$$

作试验函数代入(1.1)′可得

$$\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}=0,\qquad\forall t\in(0,\theta\rho^2) \tag{2.21}$$

其中

$$\mathrm{I}=\int_0^t\int_{B(2\rho)}\frac{\zeta^2\phi^2u_t}{\mu_1-u+F}dxdt=\int_{B(2\rho)}\zeta^2\phi^2W(x,t)dx-\int_0^t\int_{B(2\rho)}2\phi\phi'\zeta^2Wdxdt$$

$$\mathrm{II}=\int_0^t\int_{B(2\rho)}\frac{\zeta^2\phi^2}{\mu_1-u+F}\nabla u\cdot\boldsymbol{A}dxdt\geqslant\int_0^t\int_{B(2\rho)}\zeta^2\phi^2\left(|\nabla W|^2-\frac{f_0}{F}\right)dxdt$$

$$\mathrm{III}=\int_0^t\int_{B(2\rho)}\frac{2\zeta\phi^2}{\mu_1-u+F}\nabla\zeta\cdot\boldsymbol{A}dxdt\leqslant\int_0^t\int_{B(2\rho)}2\zeta\phi^2|\nabla\zeta|\left(\kappa_1|\nabla W|+\frac{f_1}{F}\right)dxdt$$

$$\mathrm{IV}=\int_0^t\int_{B(2\rho)}\frac{\zeta^2\phi^2B}{\mu_1-u+F}dxdt\leqslant\int_0^t\int_{B(2\rho)}\zeta^2\phi^2\left(\frac{C(x,t)|\nabla u|^\gamma}{\mu_1-u+F}+\frac{f_2}{F}\right)dxdt$$

下面证明存在常数 $C>0$, 使

$$\int_{B(2\rho)}\zeta^2\phi^2W(x,t)|_{t=\theta\rho^2}dx+\int_0^{\theta\rho^2}\int_{B(2\rho)}\zeta^2\phi^2|\nabla W|^2dxdt\leqslant C\rho^n \tag{2.22}$$

事实上, 根据 F 的定义我们有

$$\mu_1-u+F\leqslant 2M+F_0\rho^{(n+2)\sigma}\leqslant 2M+F_0$$

考虑到 σ 满足(2.5), 于是易得:

$$\int_0^{\theta\rho^2}\int_{B(2\rho)}\zeta^2\phi^2\frac{C(x,t)|\nabla u|^\gamma}{\mu-u+F}dxdt$$

$$\leqslant (2M+F_0)\varepsilon\int_0^{\theta\rho^2}\int_{B(2\rho)}\zeta^2\psi^2|\nabla W|^2dxdt+C(\varepsilon)(\theta\rho^{n+2})^{2/l+2\sigma} \tag{2.23}$$

$$\int_0^{\theta\rho^2}\int_{B(2\rho)}\frac{f_0}{F}dxdt\leqslant C(\theta\rho^{n+2})^{2/l+2\sigma}\leqslant C\rho^n$$

$$\int_0^{\theta\rho^2}\int_{B(2\rho)}\left(|\nabla\zeta|\frac{f_1}{F}+\frac{f_2}{F}\right)dxdt\leqslant C(\theta\rho^{n+2})^{2/l+\sigma}\leqslant C\rho^n$$

取(2.21)中的 $t=\theta\rho^2$，(2.23)中的 ε 足够小，综合以上结果得

$$\int_{B(2\rho)}\zeta^2\psi^2W(x,t)|_{t=\theta\rho^2}dx+\int_0^{\theta\rho^2}\int_{B(2\rho)}\zeta^2\psi^2|\nabla W|^2dxdt$$

$$\leqslant\varepsilon\int_0^{\theta\rho^2}\int_{B(2\rho)}\zeta^2\psi^2|\nabla W|^2dxdt+C(\varepsilon)\rho^n+\int_0^{\theta\rho^2}\int_{B(2\rho)}2\psi\psi'\zeta^2Wdxdt \tag{2.24}$$

根据(2.19)和 W 的定义，我们有

$$\mathrm{mes}\left(B(\rho)\cap\{W(x,t)\leqslant 0\}\right)\geqslant\mathrm{mes}\left(B(\rho)\cap\{u(x,t)\leqslant 7\mu_1/8+\mu_2/8\}\right)$$

$$\geqslant\theta_1\mathrm{mes}B(\rho)\geqslant\frac{\theta_1}{2^n}\mathrm{mes}B(2\rho),\qquad\forall t\in(0,\theta\rho^n) \tag{2.25}$$

在 $B(2\rho)$ 上对函数 $W(\cdot,t)^+$ 应用引理并结合(2.24)、(2.25)即得(2.22)•

表示 W 为 $W=W^+-W^-$ ($W^\pm=\max(\pm W,0)$)，那么(2.20)隐含了 $0\leqslant W^-\leqslant\ln 8(1+2F_0)$• 由(2.22)继续得

$$\int_{\theta\rho^2/2}^{\theta\rho^2}\int_{B(2\rho)}\zeta^2|\nabla W|^2dxdt\leqslant\int_0^{\theta\rho^2}\int_{B(2\rho)}\zeta^2\psi^2|\nabla W|^2dxdt$$

$$\leqslant C\rho^n+\int_{B(2\rho)}\zeta^2(W^-)_{t=\theta\rho^2}dx\leqslant C\rho^n \tag{2.26}$$

根据(2.25)，在 $B(2\rho)$ 上对 $(W^+)^2$ 应用引理并借助 Hölder 不等式得

$$\int_{\theta\rho^2/2}^{\theta\rho^2}\int_{B(2\rho)}\zeta^2(W^+)^2dxdt\leqslant C(n,\theta_1)\rho^2\int_{\theta\rho^2/2}^{\theta\rho^2}\int_{B(2\rho)}\zeta^2|\nabla W^+|^2dxdt \tag{2.27}$$

联合(2.26)、(2.27)得

$$\int_{\theta\rho^2/2}^{\theta\rho^2}\int_{B(3\rho/2)}|W^+|^2dxdt\leqslant\int_{\theta\rho^2/2}^{\theta\rho^2}\int_{B(2\rho)}\zeta^2|W^+|^2dxdt\leqslant\Lambda\rho^{n+2} \tag{2.28}$$

其中的常数 $\Lambda>0$ 与 ρ 无关•

设 $H>0$ 待定，对 $\nu=0,1,2,\cdots$，置

$$k_\nu=H-\frac{H}{2^\nu},\quad\rho_\nu=\rho+\frac{\rho}{2^{\nu+1}},\quad\tau_\nu=\frac{\theta\rho^2}{2}+\left(1-\frac{1}{2^\nu}\right)\frac{\theta\rho^2}{4} \tag{2.29}$$

再设

$$\left.\begin{array}{ll}\zeta(x)=\begin{cases}1, & |x|\leqslant\rho_{\nu+1}\\ 0, & |x|\geqslant\rho_\nu\end{cases} & |\nabla\zeta(x)|\leqslant\dfrac{1}{\rho_\nu-\rho_{\nu+1}}=\dfrac{2^{\nu+2}}{\rho}\\ \psi(t)=\begin{cases}0, & t\leqslant\tau_\nu\\ 1, & t\geqslant\tau_{\nu+1}\end{cases} & 0\leqslant\psi'(t)\leqslant\dfrac{1}{\tau_{\nu+1}-\tau_\nu}=\dfrac{2^{\nu+3}}{\theta\rho^2}\end{array}\right\} \tag{2.30}$$

则可取

$$v=\frac{\zeta^2\psi^2(W-k)^+}{\mu_1-u+F}\qquad(k\geqslant 0)$$

作试验函数，代入(1.1)′易得

$$0=\int_0^t\int_{B(\rho_\nu)}\left\{\frac{\zeta^2\psi^2(W-k)^+}{\mu_1-u+F}u_t+\frac{\zeta^2\psi^2\nabla(W-k)^+\cdot A}{\mu_1-u+F}\right.$$
$$+\frac{\zeta^2\psi^2(W-k)^+}{(\mu_1-u+F)^2}\nabla u\cdot A+\frac{2\zeta\psi^2(W-k)^+}{\mu_1-u+F}\nabla\zeta\cdot A$$
$$\left.+\frac{\zeta^2\psi^2(W-k)^+}{\mu_1-u+F}B\right\}dx\,dt=\mathrm{I}'+\mathrm{II}'+\mathrm{III}'+\mathrm{IV}'+\mathrm{V}' \tag{2.31}$$

利用结构条件(1.2),分别估计 $\mathrm{I}'\sim\mathrm{V}'$ 如下

$$\mathrm{I}'=\frac{1}{2}\int_{B(\rho_\nu)}\zeta^2\psi^2|(W(x,t)-k)^+|^2dx-\int_0^t\int_{B(\rho_\nu)}\zeta^2\psi\psi'|(W-k)^+|^2dx\,dt$$

$$\mathrm{II}'\geqslant\int_0^t\int_{B(\rho_\nu)\cap\{W(x,t)>k\}}\frac{\zeta^2\psi^2}{W}\left(|\nabla W|^2-\frac{f_0}{F^2}\right)dx\,dt$$

$$\mathrm{III}'\geqslant\int_0^t\int_{B(\rho_\nu)\cap\{W(x,t)>k\}}\zeta^2\psi^2(W-k)^+\left(|\nabla W|^2-\frac{f_0}{F^2}\right)dx\,dt$$

$$\mathrm{IV}'\leqslant\int_0^t\int_{B(\rho_\nu)}2\zeta\psi^2(W-k)^+|\nabla\zeta|\left(\kappa_1|\nabla W|+\frac{f_1}{F}\right)dx\,dt$$

$$\mathrm{V}'\leqslant\int_0^t\int_{B(\rho_\nu)}\zeta^2\psi^2(W-k)^+\left[\varepsilon(\mu_1-u+F)|\nabla W|^2\right.$$
$$\left.+\frac{C(\varepsilon)C(x,t)^{2/(2-\gamma)}+f_2}{\mu_1-u+F}\right]dx\,dt$$

于是由(2.31)得

$$\operatorname*{vraimax}_{0<t<\theta\rho^2}\int_{B(\rho_\nu)}\zeta^2\psi^2|(W-k)^+|^2dx+\int_0^{\theta\rho^2}\int_{B(\rho_\nu)}[\zeta^2\psi^2|\nabla(W-k)^+|^2$$
$$+\zeta^2\psi^2(W-k)^+|\nabla W|^2]dx\,dt$$
$$\leqslant C\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W(x,t)>k\}}\left\{\zeta^2\psi\psi'(W-k)^2+\frac{f_0}{F^2}\zeta^2\psi^2+\zeta^2\psi^2(W-k)\frac{f_0}{F^2}\right.$$
$$+\zeta\psi^2(W-k)|\nabla\zeta|\left(|\nabla W|+\frac{f_1}{F}\right)+\varepsilon(2M+F_0)\zeta^2\psi^2(W-k)|\nabla W|^2$$
$$\left.+\zeta^2\psi^2(W-k)\frac{C(\varepsilon)C(x,t)^{2/(2-\gamma)}+f_2}{F}\right\}dx\,dt \tag{2.32}$$

根据(2.25)并对函数 $|(W-k)^+|^2$ 应用引理,得

$$\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W(x,t)>k\}}\zeta^2\psi\psi'|(W-k)^+|^2dx\,dt$$
$$\leqslant\varepsilon\int_0^{\theta\rho^2}\int_{B(\rho_\nu)}\zeta^2\psi^2|\nabla(W-k)^+|^2dx\,dt+C(\varepsilon)\frac{4^\nu}{\rho^2}\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)}|(W-k)^+|^2dx\,dt \tag{2.33}$$

又根据 Moser[3] 的结果,成立

$$\left(\int_0^{\theta\rho^2}\int_{B(\rho_\nu)}|\zeta\psi(W-k)^+|^l dx\,dt\right)^{2/l}$$
$$\leqslant C(n)\left\{\operatorname*{vraimax}_{0<t<\theta\rho^2}\int_{B(\rho_\nu)}\zeta^2\psi^2|(W-k)^+|^2dx+\int_0^{\theta\rho^2}\int_{B(\rho_\nu)}\zeta^2\psi^2|\nabla(W-k)^+|^2dx\,dt\right.$$
$$\left.+\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)}|\nabla\zeta|^2|(W-k)^+|^2dx\,dt\right\} \tag{2.34}$$

简写

$$|A_\nu(k)| = \int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W(x,t)>k\}} dx\,dt$$

由于 $\rho_\nu \leqslant 2\rho$, $\tau_\nu \in (0, \theta\rho^2)$, 因而 $|A_2(k)| \leqslant C\theta\rho^{n+2}$, 借助 Hölder 不等式, 得

$$\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W(x,t)>k\}}\left[\zeta^2\psi^2\frac{f_0}{F^2} + \zeta^2\psi^2(W-k)\frac{f_0}{F^2}\right]dx\,dt$$

$$\leqslant\left\{\left[\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W(x,t)>k\}}|\zeta\psi(W-k)|^l dx\,dt\right]^{1/l}|A_\nu(k)|^{1-1/l-1/S_0}\right.$$

$$\left.+|A_\nu(k)|^{1-1/S_0}\right\}\frac{\|f_0\|_{L_{S_0}(G)}}{F^2}$$

$$\leqslant C\left\{\left[\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}}|\zeta\psi(W-k)|^l dx\,dt\right]^{1/l}|A_\nu(k)|^{1/l+2/(n+2)-1/S_0-2\sigma}\right.$$

$$\left.+|A_\nu(k)|^{2/l+3/(n+2)-1/S_0-2\sigma}\right\}$$

$$\leqslant \varepsilon\left[\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}}|\zeta\psi(W-k)|^l dx\,dt\right]^{2/l} + C(\varepsilon)|A_\nu(k)|^{2/l+\sigma} \tag{2.35}$$

$$\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}}\zeta\psi^2(W-k)|\nabla\zeta|\left(|\nabla W|+\frac{f_1}{F}\right)dx\,dt$$

$$\leqslant \varepsilon\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}}\zeta^2\psi^2|\nabla(W-k)|^2 dx\,dt$$

$$+C(\varepsilon)\left\{\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}}|\nabla\zeta|^2(W-k)^2 dx\,dt + |A_\nu(k)|^{2/l+\sigma}\right. \tag{2.36}$$

$$\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}}\zeta^2\psi^2(W-k)\frac{C(\varepsilon)C(x,t)^{2/(2-\gamma)}+f_2}{F}dx\,dt$$

$$\leqslant \varepsilon\left[\int_0^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}}|\zeta\psi(W-k)|^l dx\,dt\right]^{2/l} + C(\varepsilon)|A_\nu(k)|^{2/l+\sigma} \tag{2.37}$$

联合(2.32)~(2.37), 取 $\varepsilon>0$ 足够小, 得

$$\operatorname*{vraimax}_{0<t<\theta\rho^2}\int_{B(\rho_\nu)}\zeta^2\psi^2|(W-k)^+|^2 dx + \int_0^{\theta\rho^2}\int_{B(\rho_\nu)}\zeta^2\psi^2|\nabla(W-k)^+|^2 dx\,dt$$

$$\leqslant C\left\{\frac{4^\nu}{\rho^2}\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}}(W-k)^2 dx\,dt + |A_\nu(k)|^{2/l+\sigma}\right. \tag{2.38}$$

其中常数 C 与 k, ν, ρ 无关. 联合(2.34)、(2.38) 得

据 M

$$\int_{\tau_{\nu+1}}^{\theta\rho^2}\int_{B(\rho_{\nu+1})\cap\{W>k\}}(W-k)^2 dx\,dt$$

$$\leqslant C|A_\nu(k)|^{-2/l}\left\{\frac{4^\nu}{\rho^2}\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}}(W-k)^2 dx\,dt + |A_\nu(k)|^{2/l+\sigma}\right. \tag{2.39}$$

对 $\forall h > k \geqslant 0$, 显然有

$$(h-k)^2 A_{\nu+1}(h) \leqslant \int_{\tau_{\nu+1}}^{\theta\rho^2}\int_{B(\rho_{\nu+1})\cap\{w>k\}}(W-k)^2 dx\,dt$$

$$\leqslant \int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{w>k_\nu\}} (W-k)^2 dx dt \tag{2.40}$$

$$\int_{\tau_{\nu+1}}^{\theta\rho^2}\int_{B(\rho_{\nu+1})\cap\{W>h\}} (W-h)^2 dx dt \leqslant \int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{w>k\}} (W-k)^2 dx dt \tag{2.41}$$

用 $h>k$ 取代(2.39)中的 k, 然后再联合(2.40)、(2.41), 得

$$\int_{\tau_{\nu+1}}^{\theta\rho^2}\int_{B(\rho_{\nu+1})\cap\{W>h\}} (W-h)^2 dx dt$$

$$\leqslant C\left[\frac{1}{(h-k)^2}\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}} (W-k)^2 dx dt\right]^{1-2/l}$$

$$\cdot\left\{\frac{4^\nu}{\rho^2}\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}} (W-k)^2 dx dt\right.$$

$$\left.+\left[\frac{1}{(h-k)^2}\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k\}} (W-k)^2 dx dt\right]^{2/l+\sigma}\right\}$$

分别用 $k_{\nu+1}, k_\nu$ 取代 h, k, 由上式得

$$J_{\nu+1} \leqslant C\left[\frac{J_\nu}{(k_{\nu+1}-k_\nu)^2}\right]^{1-2/l}\left\{\frac{4^\nu}{\rho^2}J_\nu+\left[\frac{J_\nu}{(k_{\nu+1}-k_\nu)^2}\right]^{2/l+\sigma}\right\}$$

$$\leqslant C\left\{\left(\frac{4^{\nu+1}}{H^2}\right)^{1-2/l}\frac{4^\nu}{\rho^2}J^{1+2/(n+2)}+\left(\frac{4^{\nu+1}J_\nu}{H^2}\right)^{1+\sigma}\right\} \tag{2.42}$$

其中 $\nu=0,1,2,\cdots$, 常数 $C>0$ 与 ρ,ν 无关. 而

$$J_\nu=\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)\cap\{W>k_\nu\}} (W-k_\nu)^2 dx dt=\int_{\tau_\nu}^{\theta\rho^2}\int_{B(\rho_\nu)} |(W-k_\nu)^+|^2 dx dt$$

由数学归纳法, 并注意(2.28), 则易证对一切正整数 ν, 有

$$J_\nu \leqslant \delta^\nu \Lambda\rho^{n+2} \qquad (\nu=0,1,2,\cdots) \tag{2.43}$$

于是, 有

$$0=\lim_{\nu\to\infty}J_\nu=\int_{3\theta\rho^2/4}^{\theta\rho^2}\int_{B(\rho)} |(W-H)^+|^2 dx dt$$

即

$$\operatorname*{vraimax}_{Q_1(\rho)} W=\operatorname*{vraimax}_{B(\rho)\times(3\theta\rho^2/4,\,\theta\rho^2)} W \leqslant H$$

根据 W 的定义, 由上式立即得

$$\operatorname*{vraimax}_{Q_1(\rho)} u \leqslant \mu_1-\frac{1}{8}e^{-H}(\mu_1-\mu_2)+F$$

$$\operatorname*{osc}_{Q_1(\rho)} u \leqslant\left(1-\frac{1}{8}e^{-H}\right)\operatorname*{osc}_{Q_0(\rho)} u+F_0\rho^{(n+2)\sigma}$$

于是(2.16)得证, 并且其中的 $\eta=1-e^{-H}/8\in(0,1)$, $C=F_0$ 与 ρ 无关.

设 $(x_0,t_0)\in Q$ 为任意, $\rho\leqslant 1$ 使 $B(x_0,2\rho)\times(t_0-\theta\rho^2,t_0)\subset Q$. 记

$$\omega(\rho)=\operatorname*{osc}_{B(x_0,2\rho)\times(t_0-\theta\rho^2,\,t_0)} u$$

那么根据前面作过的证明, 有

$$\omega(\rho/2)\leqslant\eta(\omega(\rho))+(C+2)\rho^{(n+2)\sigma}$$

令 $\rho_\nu=\rho_0/2^\nu$, 经过迭代, 得

$$\omega(\rho_\nu) \leqslant \eta^\nu \omega(\rho_0) + (C+2)\rho_0^{(n+2)\sigma}\eta^{\nu-1}\sum_{i=1}^{\nu-1}(\eta 2^{(n+2)\sigma})^{-i} \tag{2.44}$$

取 $\eta < 1$ 使满足 $\eta 2^{(n+2)\sigma} > 1$, 则由(2.44) 继续得

$$\omega(\rho_\nu) \leqslant \eta^\nu\left\{\omega(\rho_0) + \frac{1}{\eta}(C+2)\eta_0^{(n+2)\sigma}\left[1-\frac{1}{\eta 2^{(n+2)\sigma}}\right]^{-1}\right\} \tag{2.45}$$

对任何 $\rho \leqslant \rho_0 \leqslant 1$, 可以找到整数 ν, 使

$$\rho_{\nu+1} < \rho \leqslant \rho_\nu, \text{ 即 } \nu \leqslant \log_2(\rho_0/\rho) < \nu+1$$

由(2.45) 得

$$\omega(\rho) \leqslant \frac{1}{\eta}\left(\frac{\rho}{\rho_0}\right)^{\log_2(1/\eta)}\left\{\omega(\rho_0) + \frac{1}{\eta}(C+2)\eta_0^{(n+2)\sigma}\left[1-\frac{1}{\eta 2^{(n+2)\sigma}}\right]^{-1}\right\}$$

由于 $\eta \in (0,1)$, $\log_2(1/\eta) > 0$, 故 u 在 Q 内局部 Hölder 连续. 于是定理得证.

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The Hölder Continuity of Generalized Solutions of a Class Quasilinear Parabolic Equations

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Abstract

Let A and B satisfy the structural conditions (2), the local Hölder continuity interior to $Q = G \times (0, T)$ is proved for the generalized solutions of quasilinear parabolic equations as follows:

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) + B(x, t, u, \nabla u) = 0$$

Key words parabolic equation, natural growth condition, generalized solution, local Hölder continuity