厚壁圆柱壳开孔应力集中问题的复变函数解法*

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摘　　要

本文基于考虑横向剪切变形影响的厚壳理论建立了求解圆柱壳开孔应力集中问题的复变函数方法, 得到了此种问题的一般解和满足任意形开孔边界条件的表达式• 该应力集中问题可以简化为求解无穷代数方程组的问题• 用复变函数方法可以规范地求解应力集中问题• 文中给出了圆柱壳开小圆孔和椭圆孔时应力集中系数的数值结果•

关键词　厚圆柱壳　开孔　应力集中问题　复变函数方法与保角映射
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§1. 引　　言

薄壁圆柱壳的开孔应力集中问题人们曾经做了大量的研究[1~4], 文献[5, 6]对此问题进行了评述和讨论• 当壳厚 h 与开孔尺寸 r_0 相比不是小量时, 应采用考虑横向剪切变形影响的 Reissner 扁壳控制方程分析问题[7~8]• 厚壳理论弥补了经典薄壳理论的不足, 在圆柱壳开孔应力集中问题的分析中开始得到应用•

本文采用复变函数方法与保角映射技术, 基于 Reissner 扁壳理论对圆柱壳开任意形小孔的应力集中问题进行了分析研究• 利用复应力函数, 使简化的厚壁圆柱壳的控制方程[9~10]变成了与弹性动力学波动方程相似的互不耦合的 Helmholtz 型方程• 因此, 可利用 D. K. Liu 在文献[11]提出的复变函数方法求解厚壁圆柱壳的开孔应力集中问题• 文中给出了此应力集中问题的一般解• 应用正交函数展开技术, 该应力集中问题可以归结为求解无穷代数方程组的问题• 用此复变函数方法可以规范地求解圆柱壳开小孔的应力集中问题• 文中给出了圆柱壳开小圆孔和椭圆孔时应力集中系数的数值结果并对其进行了讨论•

§2. 控制方程及其一般解

设应力函数, 法向位移函数及广义位移函数分别为 φ, w 和 f, 则圆柱壳开孔应力集中问

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$$+\ \Phi^{(1)}_{n+2}\Big]\Big\}-\frac{D}{4}\sum_{-\infty}^{+\infty}(C_n\mathrm{Re}-D_n\mathrm{Im})\Big\{2\alpha^2(1+\nu)\big[\Phi^{(2)}_{n-1}+2\Phi^{(2)}_{n}+\Phi^{(2)}_{n+1}\big]$$

$$+\ \alpha^2(1-\nu)\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\big[\Phi^{(2)}_{n-2}+2\Phi^{(2)}_{n-1}+\Phi^{(2)}_{n}\big]+\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}$$

$$\cdot\big[\Phi^{(2)}_{n}+2\Phi^{(2)}_{n+1}+\Phi^{(2)}_{n+2}\big]\Big\}+\frac{AD}{4}\sum_{-\infty}^{+\infty}E_n\mathrm{Im}\Big\{\beta^2(1-\nu)\Big[\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\Phi^{(3)}_{n-2}$$

$$-\ \frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\Phi^{(3)}_{n+2}\Big]\Big\}+M^0_\rho=F_3 \tag{3.3c}$$

$$M_{\rho\theta}=\frac{D}{4}\sum_{-\infty}^{+\infty}(A_n\mathrm{Im}+B_n\mathrm{Re})\alpha^2(1-\nu)\Big\{\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\big[\Phi^{(1)}_{n-2}-2\Phi^{(1)}_{n-1}+\Phi^{(1)}_{n}\big]$$

$$-\ \frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\big[\Phi^{(1)}_{n}-2\Phi^{(1)}_{n+1}+\Phi^{(1)}_{n+2}\big]\Big\}+\frac{D}{4}\sum_{-\infty}^{+\infty}(C_n\mathrm{Im}+D_n\mathrm{Re})\alpha^2(1-\nu)$$

$$\cdot\Big\{\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\big[\Phi^{(2)}_{n-2}+2\Phi^{(2)}_{n-1}+\Phi^{(2)}_{n}\big]-\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\big[\Phi^{(2)}_{n}+2\Phi^{(2)}_{n+1}+\Phi^{(2)}_{n+2}\big]$$

$$+\ \frac{AD}{4}\sum_{-\infty}^{+\infty}E_n\mathrm{Re}\Big\{\beta^2(1-\nu)\Big[\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\Phi^{(3)}_{n-2}+\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\Phi^{(3)}_{n+2}\Big]\Big\}+M^0_{\rho\theta}=F_4 \tag{3.3d}$$

$$Q_\rho=-\frac{D}{2}\sum_{-\infty}^{+\infty}(A_n\mathrm{Re}-B_n\mathrm{Im})\alpha^3\Big\{\frac{\eta\omega'(\eta)}{|\omega'(\eta)|}\big[\Phi^{(1)}_{n-2}-3\Phi^{(1)}_{n-1}+3\Phi^{(1)}_{n}-\Phi^{(1)}_{n+1}\big]$$

$$-\ \frac{\eta\overline{\omega'(\eta)}}{|\omega'(\eta)|}\big[\Phi^{(1)}_{n-1}-3\Phi^{(1)}_{n}+3\Phi^{(1)}_{n+1}-\Phi^{(1)}_{n+2}\big]\Big\}+\frac{D}{2}\sum_{-\infty}^{+\infty}(C_n\mathrm{Re}-D_n\mathrm{Im})\alpha^3$$

$$\cdot\Big\{\frac{\eta\omega'(\eta)}{|\omega'(\eta)|}\big[\Phi^{(2)}_{n-2}+3\Phi^{(2)}_{n-1}+3\Phi^{(2)}_{n}+\Phi^{(2)}_{n-1}\big]-\frac{\eta\overline{\omega'(\eta)}}{|\omega'(\eta)|}\big[\Phi^{(2)}_{n-1}+3\Phi^{(2)}_{n}$$

$$+\ 3\Phi^{(2)}_{n+1}+\Phi^{(2)}_{n+2}\big]\Big\}-\frac{1}{2}\sum_{-\infty}^{+\infty}E_n\mathrm{Im}\Big\{\beta\Big[\frac{\eta\omega'(\eta)}{|\omega'(\eta)|}\Phi^{(3)}_{n-1}-\frac{\eta\overline{\omega'(\eta)}}{|\omega'(\eta)|}\Phi^{(3)}_{n+1}\Big]\Big\}$$

$$+\ Q^0_\rho=F_5 \tag{3.3e}$$

式中, $F_i(i=1,2,3,4,5)$ 是开孔相贯线边界上的限制力; $N^0_\rho, N^0_{\rho\theta}, M^0_\rho, M^0_{\rho\theta}, Q^0_\rho$ 表示没有开孔时的基本应力状态, 也就是圆柱壳膜应力状态时的广义内力•

§4. 开孔附近的应力集中

不失一般性, 研究圆柱壳开孔的应力集中问题• 当圆柱壳没有开孔时, 由内压产生的基本应力状态可以描述为

$$\varphi_0=\frac{N_0}{2}\Big\{x^2+\frac{1}{2}y^2\qquad {}_0=\frac{N_0R}{2Eh}(2-\nu),\quad f_0=0$$

当 $\eta=\exp[i\theta]$ 时, 该基本应力状态对应如下广义内力

$$\left.\begin{aligned}&N^0_\rho=\frac{N_0}{8}\Big[6-\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}-\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\Big],\ N^0_{\rho\theta}=-\frac{N_0}{8}i\Big[\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}-\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\Big]\\&M^0_\rho=M^0_{\rho\theta}=Q^0_\rho=0,\quad w\end{aligned}\right\} \tag{4.1}$$

式中, N_0 为圆柱壳中的环向内力, $N_0=PR$; P 是内压•

设圆柱壳开孔的边界条件为自由边界, 则边界条件的表达式为

和 $X_n^1 = A_n,\ X_n^2 = B_n,\ X_n^3 = C_n,\ X_n^4 = D_n,\ X_n^5 = E_n$

用 $\exp[-is\theta]$ 乘以式(4.3)的两端，并在区间$(-\pi, \pi)$利用正交性进行积分，可得到如下方程组

$$\sum_{j=1}^{5}\left\{\sum_{n=-\infty}^{+\infty}\epsilon_{ns}^{ij}X_n^j\right\} = \epsilon_{is} \qquad (i = 1, 2, 3, 4, 5;\ n = 0, \pm 1, \cdots) \tag{4.4}$$

其中

$$\epsilon_{ns}^{ij} = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\epsilon_n^{ij}\exp[-is\theta]\,d\theta,\quad \epsilon_{is} = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\epsilon_i\exp[-is\theta]\,d\theta$$

式(4.4)就是确定未知系数 A_n, B_n, C_n, D_n, E_n 的无穷代数方程组•

圆柱壳承受内压时开孔附近的应力集中系数是问题的研究热点• 应力集中系数可定义为开孔边界上任意点的应力与圆柱壳切向应力的比值• 开孔周界上只存在环向膜应力和环向弯矩•

环向膜应力和环向弯曲应力集中系数可表示为

$$N_\theta^* = \frac{N_\theta}{N_0},\quad M_\theta^* = \frac{6M_\theta}{hN_0} \tag{4.5}$$

式中，N_0 是无开孔时圆柱壳中的最大环向膜应力；N_θ 和 M_θ 分别是开孔边界上的环向膜应力和环向弯矩• 由方程(3.1)和考虑边界自由的条件，可得环向膜应力和环向弯曲应力集中系数的计算式

$$\begin{aligned}N_\theta^* = {} & \frac{3}{2} - \frac{\sqrt{DEh}}{N_0}\sum_{-\infty}^{+\infty}(A_n\mathrm{Im} + B_n\mathrm{Re})\Bigg\{\alpha^2\Bigg[K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} \\ & - 2K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n + K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg] \\ & \cdot\exp\left\{\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\} + \sqrt{DEh}\sum_{-\infty}^{+\infty}(C_n\mathrm{Im} + D_n\mathrm{Re}) \\ & \cdot\Bigg\{\alpha^2\Bigg[K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + 2K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\} \\ & + K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg]\exp\left\{-\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\}\end{aligned} \tag{4.6a}$$

$$\begin{aligned}M_\theta^* = {} & \frac{6D(1+\nu)}{hN_\theta}\sum_{-\infty}^{+\infty}(A_n\mathrm{Re} - B_n\mathrm{Im})\Bigg\{\alpha^2\Bigg[K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} \\ & - 2K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n + K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg] \\ & \cdot\exp\left\{\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\} - \frac{6D(1+\nu)}{hN_0}\sum_{-\infty}^{+\infty}(C_n\mathrm{Re} - D_n\mathrm{Im}) \\ & \cdot\Bigg\{\alpha^2\Bigg[K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} - 2K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n \\ & + K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg]\exp\left\{-\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\}\end{aligned} \tag{4.6b}$$

§5. 数值结果与讨论

采用上述分析方法可以计算厚壁圆柱壳开椭圆孔(或圆孔)时的应力集中系数• 椭圆的短轴与长轴分别为 a 和 b，其短轴置于 x 轴上• 对于椭圆孔映射函数可取如下形式

题可以归结为求解如下简化的厚壁扁壳控制方程[9~10]

$$\left.\begin{aligned} & D\nabla^2\nabla^2 w + \frac{1}{R}\frac{\partial^2\varphi}{\partial x^2} = 0,\quad \frac{1}{Eh}\nabla^2\nabla^2\varphi - \frac{1}{R}\frac{\partial^2 w}{\partial x^2} = 0 \\ & \frac{1}{2}(1-\nu)AD\nabla^2 f - f = 0 \end{aligned}\right\} \tag{2.1}$$

式中，D 是圆柱壳的抗弯刚度，$D = \dfrac{Eh^3}{12(1-\nu^2)}$；$A = \dfrac{6}{5Gh} = \dfrac{12(1+\nu)}{5Eh}$；$E$ 和 ν 分别为杨氏弹性模量和泊松比，(x, y) 是 x 轴置于圆柱壳母线方向的直角坐标系；∇^2 是 Laplace 算子 $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$．

在直角坐标系 (x, y) 中，利用应力函数 φ，法向位移函数 w 及广义位移函数 f，圆柱壳中的广义内力可以表示为

$$\left.\begin{aligned} & N_x = \frac{\partial^2\varphi}{\partial y^2},\quad N_y = \frac{\partial^2\varphi}{\partial x^2},\quad N_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} \\ & M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + (1-\nu)AD\frac{\partial^2 f}{\partial x\partial y} \\ & M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) - (1-\nu)AD\frac{\partial^2 f}{\partial x\partial y} \\ & M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x\partial y} - \frac{1}{2}(1-\nu)AD\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) \\ & Q_x = -D(1-\nu)\frac{\partial\nabla^2 w}{\partial x} + \frac{\partial f}{\partial y},\quad Q_y = -D(1-\nu)\frac{\partial\nabla^2 w}{\partial y} - \frac{\partial f}{\partial x} \end{aligned}\right\} \tag{2.2}$$

利用复变应力函数[5] $\sigma(x, y) = w + \dfrac{i}{\sqrt{DEh}}\varphi$，式(2.1)中的前两个方程变为如下一个方程

$$\nabla^2\nabla^2\sigma - 4\alpha^2\frac{\partial^2\sigma}{\partial x^2} = 0 \tag{2.3}$$

式中，α 是圆柱壳的曲率参数，$\alpha = \dfrac{[12(1-\nu^2)]^{\frac{1}{4}}}{2\sqrt{Rh}}\exp[i\pi/4]$

方程(2.3)的解可以描述为

$$\sigma = \sigma_1 + \sigma_2 \tag{2.4}$$

这里，函数 σ_1 和 σ_2 应分别满足如下方程

$$\nabla^2\sigma_1 - 2\alpha\frac{\partial\sigma_1}{\partial x} = 0,\quad \nabla^2\sigma_2 + 2\alpha\frac{\partial\sigma_2}{\partial x} = 0 \tag{2.5}$$

设 $\sigma_1 = \exp[\alpha x]u_1(x, y)$，$\sigma_2 = \exp[-\alpha x]u_2(x, y)$，函数 u_1 和 u_2 应满足如下形式的方程

$$\nabla^2 u - \alpha^2 u = 0 \tag{2.6}$$

采用复变量方法[11]，设 $\zeta = x + iy$；$\bar{\zeta} = x - iy$，式(2.6)和式(2.1)的最后一个方程可变成如下形式

$$\frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}} - \left(\frac{\alpha}{2}\right)^2 u = 0,\quad \frac{\partial^2 f}{\partial\zeta\partial\bar{\zeta}} - \left(\frac{\beta}{2}\right)^2 f = 0 \tag{2.7}$$

式中，$\beta = \left[\dfrac{2}{(1-\nu)AD}\right]^{\frac{1}{2}} = \dfrac{\sqrt{10}}{h}$

求解厚壁圆柱壳中任意形开孔附近的应力集中问题时, 为满足开孔的边界条件可使用保角映射方法. 将 ζ 平面上非圆开孔边界线 L 的外域(或内域) 映射为 η 平面上边界为 S 的单位圆的外域(或内域). 令双方外域无穷远点相对应, 则映射函数应具有如下形式

$$\zeta = \omega(\eta) = c\eta + \text{全纯函数} \tag{2.8}$$

式中, c 为与开孔的尺寸 a 有关的实常数; 所谓函数在无限域内全纯是指在域内的任意有限部分全纯. 同时当 $|\eta|$ 充分大时, 全纯函数可描述成如下形式

$$m_0 + \frac{m_1}{\eta} + \cdots + \frac{m_k}{\eta^k} + \cdots$$

这里, k, m_k 分别是自然数和常数; 为保证映射函数的单值性, 在 S 域内 $\omega'(\eta)$ 不能为零.

在 η 平面上, 方程式(2.7) 可变成如下形式

$$\frac{\partial^2 u}{\partial\eta\partial\bar\eta} - \left(\frac{\alpha}{2}\right)^2 \omega'(\eta)\overline{\omega'(\eta)}u = 0,\quad \frac{\partial^2 f}{\partial\eta\partial\bar\eta} - \left(\frac{\beta}{2}\right)^2 \omega'(\eta)\overline{\omega'(\eta)}f = 0 \tag{2.9}$$

根据文献[11], 方程式(2.9), 解为如下表达式

$$u = \sum_{-\infty}^{+\infty} E'_n K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n,\ f = \sum_{-\infty}^{+\infty} E''_n K_n[\beta|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n \tag{2.10}$$

式中, E'_n 和 E''_n 为任意复常数; $K_n(\cdot)$ 是修正 Bessel 函数.

将式(2.10) 的第一个方程代入(2.4) 中, 则应力集中问题的解可表示为

$$\begin{aligned}\sigma = {} & \exp\left\{\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\}\sum_{-\infty}^{+\infty} F_n K_n(\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n \\ & + \exp\left\{-\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\}\sum_{-\infty}^{+\infty} G_n K_n(\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n\end{aligned} \tag{2.11}$$

式中, F_n 和 G_n 为任意复常数, 设 $F_n = A_n + iB_n$, $G_n = C_n + iD_n$

$$\begin{aligned}\Phi_n^{(1)} &= \exp\left\{\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\}K_n(\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n = u_n^{(1)} + iu_n^{(2)} \\ \Phi_n^{(2)} &= \exp\left\{-\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\}K_n(\alpha|\omega(\eta)|)\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n = u_n^{(3)} + iu_n^{(4)} \\ \Phi_n^{(3)} &= K_n[\beta|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n = u_n^{(5)} + iu_n^{(6)}\end{aligned}$$

由方程(2.11) 和方程(2.10), 可得方程(2.1) 的一般解如下

$$\left.\begin{aligned} w &= \sum_{-\infty}^{+\infty} w_n = \sum_{-\infty}^{+\infty}[A_n u_n^{(1)} - B_n u_n^{(2)} + C_n u_n^{(3)} - D_n u_n^{(4)}] \\ \varphi &= \sqrt{DEh}\sum_{-\infty}^{+\infty}\varphi_n = \sqrt{DEh}\sum_{-\infty}^{+\infty}[A_n u_n^{(2)} + B_n u_n^{(1)} + C_n u_n^{(4)} + D_n u_n^{(3)}] \\ f &= \sum_{-\infty}^{+\infty} E_n K_n[\beta|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n \end{aligned}\right\} \tag{2.12}$$

式中, A_n, B_n, C_n, D_n, E_n 是 5 个由开孔边界条件决定的未知实常数; $u_n^{(i)}$ $(i = 1, 2, 3, 4, 5, 6)$ 为 6 个实函数.

§3. 内力, 内力矩及边界条件

利用内力与应力函数, 内力矩与法向位移和广义位移之间的关系, 可以得到在 η 平面上

的内力和内力矩和复合表达式

$$\left.\begin{aligned}
&N_\rho+N_\theta=\frac{4}{\omega'(\eta)\overline{\omega'(\eta)}}\frac{\partial^2\varphi}{\partial\eta\partial\bar\eta}=\nabla^2\varphi,\ N_\theta+N_\rho+2iN_{\rho\theta}=\frac{4\eta^2}{\rho^2\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial\varphi}{\partial\eta}\right]\\
&M_\rho+M_\theta=-\frac{4D(1+\nu)}{\omega'(\eta)\overline{\omega'(\eta)}}\frac{\partial^2 w}{\partial\eta\partial\bar\eta}=-D(1+\nu)\nabla^2 w\\
&M_\theta-M_\rho+2iM_{\rho\theta}=\frac{4D(1-\nu)\eta^2}{\rho^2\overline{\omega'(\eta)}}\left\{\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial w}{\partial\eta}\right]+iA\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial f}{\partial\eta}\right]\right\}\\
&Q_\rho-iQ_\theta=-\frac{2\eta}{\rho|\omega'(\eta)|}\left[D\frac{\partial\nabla^2 w}{\partial\eta}+i\frac{\partial f}{\partial\eta}\right]
\end{aligned}\right\}\tag{3.1}$$

由方程(3.1)可以得到如下表达式

$$\left.\begin{aligned}
&N_\rho-iN_{\rho\theta}=\frac{1}{2}\nabla^2\varphi-\frac{2\eta^2}{\rho^2\overline{\omega'(\eta)}}\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial\varphi}{\partial\eta}\right]\\
&M_\rho-iM_{\rho\theta}=-\frac{1}{2}D(1+\nu)\nabla^2 w-\frac{2D(1-\nu)\eta^2}{\rho^2\overline{\omega'(\eta)}}\left\{\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial w}{\partial\eta}\right]\right.\\
&\qquad\left.+iA\frac{\partial}{\partial\eta}\left[\frac{1}{\omega'(\eta)}\frac{\partial f}{\partial\eta}\right]\right\}\\
&Q_\rho-iQ_\theta=-\frac{2\eta}{\rho|\omega'(\eta)|}\left[D\frac{\partial\nabla^2 w}{\partial\eta}+i\frac{\partial f}{\partial\eta}\right]
\end{aligned}\right\}\tag{3.2}$$

现在我们讨论 η 平面上的边界条件表达式· 圆柱壳承受外部荷载作用时, 可以给出开孔边界上广义内力的限制条件· 当 $\eta=\exp[i\theta]$ 时, 广义内力边值问题的限制条件可以表示为

$$\begin{aligned}
N_\rho=&-\frac{\sqrt{DEh}}{4}\sum_{-\infty}^{+\infty}(A_n\mathrm{Im}+B_n\mathrm{Re})\left\{2\alpha^2[\Phi_{n-1}^{(1)}-2\Phi_n^{(1)}+\Phi_{n+1}^{(1)}]+\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\right.\\
&\left.\cdot[\Phi_{n-2}^{(1)}-2\Phi_{n-1}^{(1)}+\Phi_n^{(1)}]+\alpha^2\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(1)}-2\Phi_{n+1}^{(1)}+\Phi_{n+2}^{(1)}]\right\}\\
&+\frac{\sqrt{DEh}}{4}\sum_{-\infty}^{+\infty}(C_n\mathrm{Im}+D_n\mathrm{Re})\left\{2\alpha^2[\Phi_{n-1}^{(2)}+2\Phi_n^{(2)}+\Phi_{n+1}^{(2)}]-\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\right.\\
&\left.\cdot[\Phi_{n-2}^{(2)}+2\Phi_{n-1}^{(2)}+\Phi_n^{(2)}]-\alpha^2\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(2)}+2\Phi_{n+1}^{(2)}+\Phi_{n+2}^{(2)}]\right\}+N_\rho^0=F_1
\end{aligned}\tag{3.3a}$$

$$\begin{aligned}
N_{\rho\theta}=&-\frac{\sqrt{DEh}}{4}\sum_{-\infty}^{+\infty}(A_n\mathrm{Re}-B_n\mathrm{Im})\left\{\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}[\Phi_{n-2}^{(1)}-2\Phi_{n-1}^{(1)}+\Phi_n^{(1)}]\right.\\
&\left.-\alpha^2\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(1)}-2\Phi_{n+1}^{(1)}+\Phi_{n+2}^{(1)}]\right\}-\frac{\sqrt{DEh}}{4}\sum_{-\infty}^{+\infty}(C_n\mathrm{Re}-D_n\mathrm{Im})\\
&\cdot\left\{\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}[\Phi_{n-2}^{(2)}+2\Phi_{n-1}^{(2)}+\Phi_n^{(2)}]-\alpha^2\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(2)}+2\Phi_{n+1}^{(2)}\right.\\
&\left.+\Phi_{n+2}^{(2)}]\right\}+N_{\rho\theta}^0=F_2
\end{aligned}\tag{3.3b}$$

$$\begin{aligned}
M_\rho=&\frac{D}{4}\sum_{-\infty}^{+\infty}(A_n\mathrm{Re}-B_n\mathrm{Im})\left\{2\alpha_2(1+\nu)[\Phi_{n-1}^{(1)}-2\Phi_n^{(1)}+\Phi_{n+1}^{(1)}]-\alpha^2(1-\nu)\right.\\
&\cdot\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}[\Phi_{n-2}^{(1)}-2\Phi_{n-1}^{(1)}+\Phi_n^{(1)}]-\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}[\Phi_n^{(1)}-2\Phi_{n+1}^{(1)}
\end{aligned}$$

$$\left.\begin{aligned} &N_\rho = N_\rho^1 + N_\rho^0 = 0,\quad M_{\rho\theta} = N_{\rho\theta}^1 + N_{\rho\theta}^0 = 0,\quad M_\rho = M_\rho^1 + M_\rho^0 = 0\\ &M_{\rho\theta} = M_{\rho\theta}^1 + M_{\rho\theta}^0 = 0,\quad Q_\rho = Q_\rho^1 + Q_\rho^0 = 0\end{aligned}\right\}\qquad(4.2)$$

式中，$N_\rho^1, N_{\rho\theta}^1, M_\rho^1, M_{\rho\theta}^1, Q_\rho^1$ 代表与开孔扰动应力状态相对应的广义内力.

将有关式子代入(4.2)中，可以得到边值问题的如下表达式

$$\sum_{j=1}^{5}\left\{\sum_{-\infty}^{+\infty}\epsilon_n^{ij}X_n^j\right\} = \epsilon_i\qquad(i = 1, 2, 3, 4, 5)\qquad(4.3)$$

其中

$$\begin{aligned}
\epsilon_n^{11} = &-\mathrm{Im}\Bigg\{\Bigg\{2\alpha^2\Bigg[K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} - 2K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\\
&+ K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg] + \alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\Bigg[K_{n-2}[\alpha|\omega(\eta)|]\\
&\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2} - 2K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + K_n[\alpha|\omega(\eta)|]\\
&\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\Bigg] + \alpha^2\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\Bigg[K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\\
&- 2K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1} + K_{n+2}[\alpha|\omega(\eta)|]\\
&\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\Bigg]\Bigg\}\exp\left\{\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\}\Bigg\} = -\mathrm{Im}\,\delta_1,\quad \epsilon_n^{12} = -\mathrm{Re}\,\delta_1,
\end{aligned}$$

$$\begin{aligned}
\epsilon_n^{13} = &\ \mathrm{Im}\Bigg\{\Bigg\{2\alpha^2\Bigg[K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + 2K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\\
&+ K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg] - \alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\Bigg[K_{n-2}[\alpha|\omega(\eta)|]\\
&\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2} + 2K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + K_n[\alpha|\omega(\eta)|]\\
&\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\Bigg] - \alpha^2\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\Bigg[K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n} + 2K_{n+1}\\
&\cdot[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1} + K_{n+2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\Bigg]\Bigg\}\\
&\cdot\exp\left\{-\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\}\Bigg\} = \mathrm{Im}\,\delta_2,\quad \epsilon_n^{14} = \mathrm{Re}\,\delta_2,\quad \epsilon_n^{15} = 0,
\end{aligned}$$

$$\begin{aligned}
\epsilon_n^{21} = &-\mathrm{Re}\Bigg\{\Bigg\{\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\Bigg[K_{n-2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2} - 2K_{n-1}[\alpha|\omega(\eta)|]\\
&\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\Bigg] - \alpha^2\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\\
&\cdot\Bigg[K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n} - 2K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\\
&+ K_{n+2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\Bigg]\Bigg\}\exp\left\{\frac{\alpha}{2}[\omega(\eta) + \overline{\omega(\eta)}]\right\}\Bigg\}\\
&= -\mathrm{Re}\,\delta_3,\quad \epsilon_n^{22} = \mathrm{Im}\,\delta_3,
\end{aligned}$$

$$\begin{aligned}
\epsilon_n^{23} = &-\mathrm{Re}\Bigg\{\Bigg\{\alpha^2\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\Bigg[K_{n-2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2} + 2K_{n-1}[\alpha|\omega(\eta)|]\\
&\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\Bigg] - \alpha^2\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}
\end{aligned}$$

$$\cdot\left[K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n+2K_{n+1}[\alpha|\omega(\eta)|]\left\{\rho\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\right.$$

$$\left.+K_{n+2}[\alpha|\omega(\eta)|]\left\{\rho\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\right]\Bigg\}\exp\left\{-\frac{\alpha}{2}[\omega(\eta)+\overline{\omega(\eta)}]\right\}$$

$$=-\rho\,\mathrm{Re}\,\delta_4,\quad \epsilon_n^{24}=\mathrm{Im}\,\delta_4,\quad \epsilon_n^{25}=0,$$

$$\epsilon_n^{31}=\mathrm{Re}\Bigg\{\Bigg\{2\alpha^2(1+\nu)\left[K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}-2K_n[\alpha|\omega(\eta)|]\right.$$

$$\left.\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n+K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\right]-\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}$$

$$\cdot\left[K_{n-2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}-2K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.$$

$$\left.+K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n\right]-\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\left[K_n[\alpha|\omega(\eta)|]\right.$$

$$\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-2K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}+K_{n+2}[\alpha|\omega(\eta)|]$$

$$\left.\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\right]\Bigg\}\exp\frac{\alpha}{2}[\omega(\eta)+\overline{\omega(\eta)}]\Bigg\}=\mathrm{Re}\,\delta_5,\quad \epsilon_n^{32}=-\mathrm{Im}\,\delta_5,$$

$$\epsilon_n^{33}=-\mathrm{Re}\Bigg\{\Bigg\{2\alpha^2(1+\nu)\left[K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}+2K_n[\alpha|\omega(\eta)|]\right.$$

$$\left.\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n+K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\right]+\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}$$

$$\cdot\left[K_{n-2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}+2K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}\right.$$

$$\left.+K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n\right]+\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\left[K_n[\alpha|\omega(\eta)|]\right.$$

$$\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n+2K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}+K_{n+2}[\alpha|\omega(\eta)|]$$

$$\left.\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\right]\Bigg\}\exp-\frac{\alpha}{2}[\omega(\eta)+\overline{\omega(\eta)}]\Bigg\}=-\mathrm{Re}\,\delta_6,\quad \epsilon_n^{34}=\mathrm{Im}\,\delta_6,$$

$$\epsilon_n^{35}=-A\,\mathrm{Im}\left\{\beta^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\left[K_{n-2}[\beta|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}\right]\right.$$

$$\left.-\beta^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}K_{n+2}[\beta|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\right\},$$

$$\epsilon_n^{41}=\mathrm{Im}\Bigg\{\Bigg\{\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\left[K_{n-2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}\right.$$

$$\left.-2K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}+K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n\right]$$

$$-\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\left[K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^n-2K_{n+1}[\alpha|\omega(\eta)|]\right.$$

$$\left.\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}+K_{n+2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\right]\Bigg\}$$

$$\cdot\exp\left\{\frac{\alpha}{2}[\omega(\eta)+\overline{\omega(\eta)}]\right\}\Bigg\}=\mathrm{Im}\,\delta_7,\quad \epsilon_n^{42}=\mathrm{Re}\,\delta_7,$$

$$\epsilon_n^{43}=\mathrm{Im}\Bigg\{\Bigg\{\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\left[K_{n-2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}\right.$$

$$= \quad + 2K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}\Bigg]$$

$$\Bigg(-\alpha^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\Bigg[K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n} + 2K_{n+1}[\alpha|\omega(\eta)|]$$

$$\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1} + K_{n+2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\Bigg]\Bigg\}$$

$$\cdot\exp\left\{-\frac{\alpha}{2}[\omega(\eta)+\overline{\omega(\eta)}]\right\}\Bigg\} = \mathrm{Im}\,\delta_8,\quad \epsilon_n^{44} = \mathrm{Re}\,\delta_8,$$

$$\epsilon_n^{45} = A\,\mathrm{Re}\Bigg\{-\beta^2(1-\nu)\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}\Bigg[K_{n-2}[\beta|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}$$

$$+\beta^2(1-\nu)\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}K_{n+2}[\beta|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\Bigg]\Bigg\},$$

$$\epsilon_n^{51} = -\mathrm{Re}\Bigg\{\Bigg\{\alpha^3\frac{\eta\omega'(\eta)}{|\omega'(\eta)|}\Bigg[K_{n-2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}$$

$$-3K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + 3K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}$$

$$-K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg] - \alpha^3\frac{\eta\overline{\omega'(\eta)}}{|\omega'(\eta)|}\Bigg[K_{n-1}[\alpha|\omega(\eta)|]$$

$$\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} - 3K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n} + 3K_{n+1}[\alpha|\omega(\eta)|]$$

$$\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1} - K_{n+2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\Bigg]\Bigg\}$$

$$\cdot\exp\left\{-\frac{\alpha}{2}[\omega(\eta)+\overline{\omega(\eta)}]\right\}\Bigg\} = -\mathrm{Re}\,\delta_9,\quad \epsilon^{52} = \mathrm{Im}\,\delta_9,$$

$$\epsilon_n^{53} = \mathrm{Re}\Bigg\{\Bigg\{\alpha^3\frac{\eta\omega'(\eta)}{|\omega'(\eta)|}\Bigg[K_{n-2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-2}$$

$$+3K_{n-1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + 3K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n}$$

$$+K_{n+1}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg] + \alpha^3\frac{\eta\overline{\omega'(\eta)}}{|\omega'(\eta)|}\Bigg[K_{n-1}[\alpha|\omega(\eta)|]$$

$$\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1} + 3K_n[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n} + 3K_{n+1}[\alpha|\omega(\eta)|]$$

$$\cdot\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1} + K_{n+2}[\alpha|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+2}\Bigg]\Bigg\}$$

$$\cdot\exp\left\{-\frac{\alpha}{2}[\omega(\eta)+\overline{\omega(\eta)}]\right\} = \mathrm{Re}\,\delta_{10},$$

$$\epsilon_n^{54} = -\mathrm{Im}\,\delta_{10},$$

$$\epsilon_n^{55} = -\frac{1}{D}\mathrm{Im}\Bigg\{\beta\frac{\eta\omega'(\eta)}{|\omega'(\eta)|}\Bigg[K_{n-1}[\beta|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n-1}$$

$$-\beta\frac{\eta\overline{\omega'(\eta)}}{|\omega'(\eta)|}K_{n+1}[\beta|\omega(\eta)|]\left\{\frac{\omega(\eta)}{|\omega(\eta)|}\right\}^{n+1}\Bigg]\Bigg\},$$

$$\epsilon_1 = -\frac{N_0}{2\sqrt{DEh}}\left[6-\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}-\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\right],\quad \epsilon_2 = -\frac{N_0 i}{2\sqrt{DEh}}\left[\frac{\eta^2\omega'(\eta)}{\overline{\omega'(\eta)}}-\frac{\eta^2\overline{\omega'(\eta)}}{\omega'(\eta)}\right],$$

$$\epsilon_3 = 0,\quad \epsilon_4 = 0,\quad \epsilon_5 = 0,$$

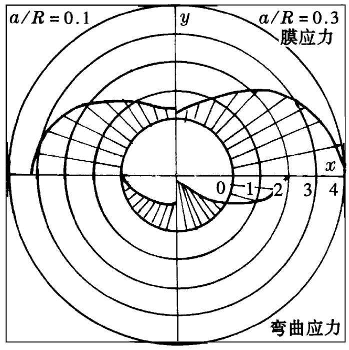

图 1 圆孔沿孔边膜应力与弯矩应力系数
($a/b=1.0$, $a/h=2.0$)

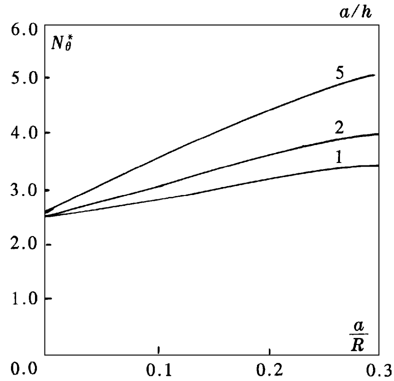

图 2 圆孔膜应力集中系数随开孔率变化规律
($a/b=1.0$, $\theta=0$)

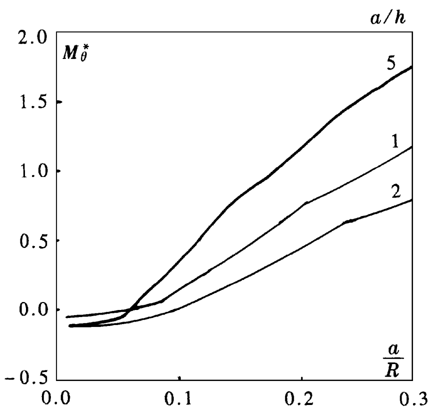

图 3 圆孔弯曲应力集中系数随开孔率变化规律
($a/b=1.0$, $\theta=0$)

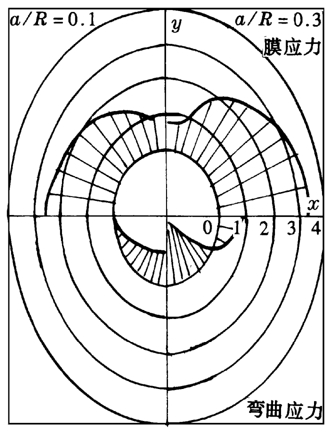

图 4 椭圆孔沿孔边膜应力与弯矩应力系数
($a/b=0.75$, $a/h=2.0$)

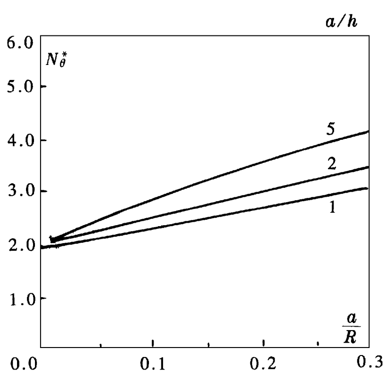

图 5 椭圆孔膜应力集中系数随开孔率变化规律
($a/b=0.75$, $\theta=0$)

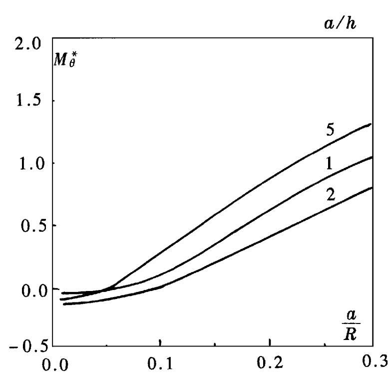

图 6 圆孔弯曲应力集中系数随开孔率变化规律
($a/b=0.75$, $\theta=0$)

$$\zeta = \omega(\eta) = r_0\left(\eta + \frac{m}{\eta}\right) \tag{5.1}$$

式中, $r_0 = (a+b)/2$, $m = (a-b)/(a+b)$•

将式(5.1)代入(4.4)和(4.6),取 $n = s = 11$, $\nu = 0.30$, 可以计算膜力集中系数 N_θ^* 和弯曲力集中系数 M_θ^*• 当 $a/b = 1.0$ 时, N_θ^* 和 M_θ^* 随孔边周向角的变化如图1所示; N_θ^* 和 M_θ^* 随比值 a/R 的变化分别如图2和图3所示• 当 $a/b = 0.75$ 时, N_θ^* 和 M_θ^* 随孔边周向角的变化如图4所示; N_θ^* 和 M_θ^* 随比值 a/R 的变化规律分别如图5和图6所示•

在一定参数下, 对于厚圆柱壳含圆形开孔其计算结果与文献[9, 10]的结果相同, 说明本分析方法及计算程序是可靠的, 可用来分析工程实际问题•

通过计算可以看到, 膜应力集中系数 N_θ^* 随参数 a/h 的增加而增加, 而弯曲应力集中系数 M_θ^* 随参数 a/h 的变化规律比较复杂• 当参数 $a/h > 2$ 时, 弯曲应力集中系数 M_θ^* 随参数 a/h 的增加而增加; 当参数 $a/h \leqslant 2$ 时, 弯曲应力集中系数 M_θ^* 随参数 a/h 的增加反而减少• 这是计及横向剪切变形影响的结果• 本文给出了圆柱壳开孔问题统一规范的解法• 对圆形开孔的分析研究表明方法是成功的• 给出将任意形状开孔边界映射成单位圆的映射函数, 则本文提出的方法可用于求解厚圆柱壳开任意形小孔的应力集中问题•

参 考 文 献

1 . . ÅÂÎ¶, À¿È¶¿ÄÂ±È°Ñ ±ÂÑ¸¶¿°»³ À²½±Ä° ÀÄ³¶ÂÃÄ°Ñ ¿±ÁÀ³¶ÂÇ¿ÀÃÄ° ¼ÂÅ´À³À´À È°½°¿µÂ±, , **10**(1946), 397—406.

2 A. C. Eringen, et al., State of stress in a circular cylindrical shell with a circular hole, *WRC Bullentin*, **102** (1965), 102—108.

3 P. Van Dyke, Stresses about a circular hole in a cylindrical shell, *AIAA*, *J*, **3**(9)(1965).

4 钱令希, 圆柱壳开圆孔问题一单圆孔的基本解, 大连工学院学报, **3**(4) (1965).

5 徐秉汉、裴俊厚、朱邦俊,《壳体开孔的理论与实验》, 国防工业出版社, 北京 (1987).

6 薛明德, 国内外关于圆柱壳开孔接管问题的研究概况, 压力容器, **8**(2) (1991), 9—15.

7 Reissner, E., et al., A note on the linear theory of shallow shear_deformable shell, *J. Appl. Math. Phys*(ZAMP), **33** (1982), 425—427.

8 Reissner, E. et al., On the effect of transverse shear deformability on stress concentration factors for twist and sheared shallow spherical shells, *J. Appl. Mech.*, **53** (1986), 697—601.

9 卢文达、程尧舜, 横向剪切变形对具有一小圆孔的浅壳应力集中系数的影响, 应力数学和力学, **12**(2) (1991), 195—202.

10 程尧舜、卢文达, 开小圆孔圆柱壳在内压作用下的应力集中, 工程力学, **10**(3) (1993), 48—53.

11 刘殿魁、盖秉政、陶贵源, 论孔附近的动应力集中, 力学学报特刊 (1981), 65—67.

On the Stress Concentration in Thick Cylindrical Shells with an Arbitrary Cutout

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Abstract

In this paper, based on the theory of thick shells including effects of transverse shear deformations, a complex variable analytic method to solve stress concentrations in circular cylindrical shells with a small cutout is established. A general solution and expression satisfying the boundary conditions on the edge of arbitrary cutouts are obtained. The stress problem can be reduced to the solution of an infinite algebraic equation series, and can be normalized by means of this method. Numerical results for stress concentration factors of the shell with a small circular and elliptic cutout are presented.

Key words thick cylindrical shell, cutout, stress concentration, complex variable method and conformal mapping